Acta Univ. Sapientiae, Mathematica, **11**, 1 (2019) 40–53

DOI: 10.2478/ausm-2019-0004

Zero forcing number of degree splitting graphs and complete degree splitting graphs

Charles Dominic
Department of Mathematics,
CHRIST (Deemed to be university),
Bangalore, India
email: charlesdominicpu@gmail.com

Abstract. A subset $\mathbb{Z} \subseteq \mathsf{V}(\mathsf{G})$ of initially colored black vertices of a graph G is known as a zero forcing set if we can alter the color of all vertices in G as black by iteratively applying the subsequent color change condition. At each step, any black colored vertex has exactly one white neighbor, then change the color of this white vertex as black. The zero forcing number $\mathbb{Z}(\mathsf{G})$, is the minimum number of vertices in a zero forcing set $\mathbb{Z}$ of G (see [11]). In this paper, we compute the zero forcing number of the degree splitting graph ($\mathcal{DS}$-Graph) and the complete degree splitting graph ($\mathcal{CDS}$-Graph) of a graph. We prove that for any simple graph, $\mathbb{Z}[\mathcal{DS}(\mathsf{G})] \leq \mathsf{k} + \mathsf{t}$, where $\mathbb{Z}(\mathsf{G}) = \mathsf{k}$ and t is the number of newly introduced vertices in $\mathcal{DS}(\mathsf{G})$ to construct it.

1 Introduction

In this article, we consider only simple, finite and undirected graphs. In graph theory, the notion of zero forcing was introduced by the AIM Minimum Rank-Special Graph Work Group (see [11]). For a graph G the zero forcing number $\mathbb{Z}(\mathsf{G})$ can be defined as follows:

2010 Mathematics Subject Classification: 05C50
Key words and phrases: zero forcing number, splitting graph

- Color change rule: Consider a colored graph G in which every vertex is colored as either white or black. If u is a black vertex of G and exactly one neighbor v of u is white, then change the color of v to black.

- For a given a coloring of G, the *derived coloring* is the result of applying the color-change rule until no more changes are possible.

- A primarily colored black vertex set $\mathbb{Z} \subseteq V(G)$ is called a zero forcing set if all vertices's of G changes to black after limited applications of the color-change rule. The zero forcing number $\mathbb{Z}(G)$, is the minimum $|\mathbb{Z}|$ over all zero forcing sets in G (see [11]).

The zero forcing number $\mathbb{Z}(G)$ can be used to bound the minimum rank for numerous families of graphs (see [11]), also it can be use as a tool for logic circuits (see [2]).

We use the following definitions and notations from [3].

- Open neighborhood and closed neighborhood. The set of all vertices adjacent to a vertex v excluding the vertex v is called the open neighborhood of v and is denoted by $N(v)$. The set of all vertices adjacent to a vertex v including the vertex v is called the closed neighborhood of v and is denoted by $N[v]$, i.e, $N[v] = \{v \cup N(v)\}$.

- Cartesian product. The *Cartesian product* $G\Box H$ of two graphs G and H is the graph with vertex set equal to the Cartesian product $V(G) \times V(H)$ and where two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G\Box H$ if and only if, either g_1 is adjacent to g_2 in G or h_1 is adjacent to h_2 in H, that is, if $g_1 = g_2$ and h_1 is adjacent to h_2 or $h_1 = h_2$ and g_1 is adjacent to g_2.

- Tensor product. Let G and H be two distinct graphs. The tensor product $G \oplus H$ has vertex set $V(G \oplus H) = V(G) \times V(H)$, edge set $E(G \oplus H) = \{(u, v)(w, x) \mid uw \in E(G) \text{ and } vx \in E(H)\}$.

- Join of two graphs. Let G and H be two distinct graphs. The graph obtained by joining every vertex of G to every vertex of H is called the join of two graphs G and H and is denoted by $G \vee H$, i.e, $G \vee H$ is the graph union $G \cup H$ together with all the edges xy where $x \in v(G)$ and $y \in V(H)$.

- The circular ladder graph or the prism graphs are the graphs obtained by taking Cartesian product of a cycle graph C_n with a single edge K_2 i.e, $CL_n = C_n \square K_2$.
- When the color change rule is applied to a vertex u to change the color of v, we say u forces v and write $u \rightarrowtail v$.

The Splitting graph $\mathcal{S}(G)$ of G was introduced by E. Sampathkumar and H.B. Walikar [8] and is the graph $\mathcal{S}(G)$ obtained by taking a new vertex v' corresponding to each vertex $v \in G$ and join v' to all vertices of G adjacent to v. The graph thus obtained is the splitting graph (see [8]). It is immediate that $S(G) - E(G) = G \oplus K_2$.
In [5], Premodkumar et al. studied the concept of the zero forcing number of the splitting graph of a graph G and gave the exact values of the zero forcing number of several classes of splitting graphs.

The degree splitting graph was introduced by R. Ponraj and S. Somasundaram [4]. Let G be a graph with $V(G) = D_1 \cup D_2 \cup \ldots \cup D_t \cup B$ where each D_i is a set of vertices of the same degree with minimum two elements and $B = V(G) \setminus \cup_{i=1}^{t} D_i$. The degree splitting graph of G, denoted by $\mathcal{DS}(G)$, is obtained from G by adding vertices $d_1, d_2, ..., d_t$ and joining the vertex d_i to each vertex of D_i for $1 \leq i \leq t$.

For a graph $G = (V, E)$, let A_i denote the set of vertices in G having degree i, $0 \leq i \leq \Delta(G)$, $A_1 \cup A_2 \cup \ldots \cup A_{\Delta(G)} = V(G)$ and $A_1 \cap A_2 \cap \ldots \cap A_{\Delta(G)} = \emptyset$. The complete degree splitting graph of a graph G is the graph $\mathcal{CDS}(G)$ obtained from the graph G by adding new vertices v_i' corresponding to each set A_i in G and joining v_i' to all vertices of A_i.

Example 1 *Consider the tree T depicted in the following figure. The degree splitting graph and the complete degree splitting graph of the tree T are shown in the Figure 1.*

This paper aims to discuss the zero forcing number of the degree splitting graph $\mathcal{DS}(G)$ and the complete degree splitting graph $\mathcal{CDS}(G)$ of a graph G. For more definitions on graphs refer to [3]. For a detailed study of zero forcing refer to [11, 6, 7].

Proposition 1 *The zero forcing number can be easily determined for the following degree and complete degree splitting graphs:*

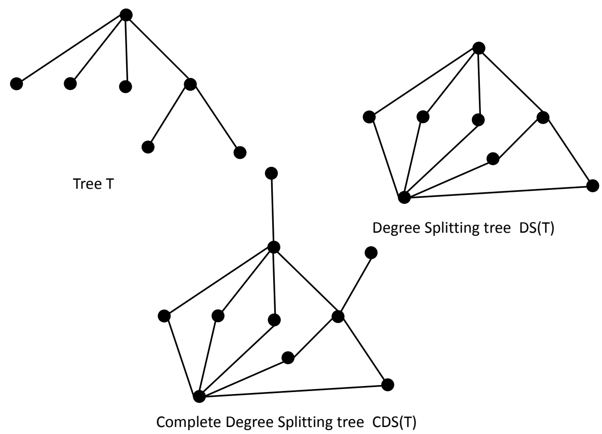

Figure 1:

- *For* P_n*, a path on* $n \geq 5$ *vertices,* $\mathbb{Z}[\mathcal{DS}(P_n)] = \mathbb{Z}[\mathcal{CDS}(P_n)] = 3$*.*
- *For* C_n *a cycle on* $n \geq 3$ *vertices,* $\mathbb{Z}[\mathcal{DS}(C_n)] = \mathbb{Z}[\mathcal{CDS}(C_n)] = 3$*.*

If G is a totally disconnected graph, then the degree splitting graph of G is the star graph. By using this fact we have the following

Proposition 2 *If* G *is a totally disconnected graph with at least two vertices, then* $\mathbb{Z}[\mathcal{DS}(G)] = \mathbb{Z}[\mathcal{CDS}(G)] = n-1$*, where* n *is the number of vertices of the graph* G*.*

Theorem 3 *Let* G *be any simple graph of order* $n \geq 2$ *with* $\mathbb{Z}(G) = k$ *and let* t *be the number of vertices introduced in* G *to construct* $\mathcal{DS}(G)$*. Then* $\mathbb{Z}[\mathcal{DS}(G)] \leq k + t$*.*

Proof. With out loss of generality assume that G is a simple graph of order $n \geq 2$ and let $\mathbb{Z}$ be an optimal zero forcing set of G with vertices $\{v_1, v_2, \ldots, v_k\}$. The degree splitting graph $\mathcal{DS}(G)$ of G is obtained from G by taking new vertices $d_1, d_2, \ldots, d_t$ and joining it to each D_i. Consider the degree splitting graph $\mathcal{DS}(G)$ and color the vertices $d_1, d_2, \ldots, d_t$ black. Since $\mathbb{Z}$ is a zero forcing set of G and $d_1, d_2, \ldots, d_t$ are black vertices, $\{v_1, v_2, \ldots, v_k\} \cup \{d_1, d_2, \ldots, d_t\}$ forms a zero forcing of $\mathcal{DS}(G)$. Hence the result follows. □

The above proof remains valid for the complete degree splitting graph $\mathcal{CDS}(G)$. Therefore we have the following

Theorem 4 *Let* G *be any simple graph of order* $n \geq 2$ *with* $\mathbb{Z}(G) = k$ *and let* t *be the number of vertices introduced in* G *to construct* $\mathcal{CDS}(G)$*. Then* $\mathbb{Z}[\mathcal{CDS}(G)] \leq k + t$.

Corollary 5 *Let* G *be the degree splitting graph of the cartesian product of* P_n *with* P_m, $n \leq m$*. Then* $\mathbb{Z}(\mathcal{DS}(P_n \square P_m)) \leq n + 3$.

We recall the following result from [9] to prove the next result.

Theorem 6 [9] *Let* G_1 *and* G_2 *be two connected graphs. Then* $\mathbb{Z}(G_1 \vee G_2) = min\{|G_2| + \mathbb{Z}(G_1), |G_1| + \mathbb{Z}(G_2)\}$.

Theorem 7 *Let* G *be a regular graph of order* $n > 1$ *and let* $\mathbb{Z}(G) = k$, $k > 1$ *be a positive integer. Then* $\mathbb{Z}(\mathcal{DS}(G)) = k + 1$.

Proof. Assume that G is a regular graph. The graph $\mathcal{DS}(G)$ is obtained from G by taking a new vertex v and joining v to all other vertices in G that is, $\mathcal{DS}(G) = G \vee H$, where H is a graph with a single vertex v. Therefore, $\mathbb{Z}(H) = 1$. We have from theorem 6,

$$\mathbb{Z}(G \vee H) = \min\{|H| + \mathbb{Z}(G), |G| + \mathbb{Z}(H)\} = \min\{1 + k, n + 1\} = 1 + k.$$

□

Now we give special attention to the zero forcing number of the regular graphs considered in [11]. We recall the following results from [11].

Theorem 8 [11]

(i) *For the hypercube* Q_n, $\mathbb{Z}(Q_n) = 2^{n-1}$.

(ii) *If* G *is the prism graph* CL_n*, then* $\mathbb{Z}(G) = 4$.

(iii) *If* G *is the Petersen graph, then* $\mathbb{Z}(G) = 5$.

(iv) *If* G *is the Complete bipartite graph* $K_{m,n}$*, then* $\mathbb{Z}(G) = m + n - 2$.

The following results are the immediate consequence of the above two theorems

Corollary 9 (i) *If* G *is the Petersen graph, then* $\mathbb{Z}(\mathcal{DS}(G)) = \mathbb{Z}(G) + 1 = 6$.

(ii) *If* G *is the complete bipartite graph* $K_{n,n}$, $n \geq 2$, *then* $\mathbb{Z}(\mathcal{DS}(G)) = 2n - 1$.

(iii) *If* G *is the degree splitting graph of the prism graph* CL_n, *then* $\mathbb{Z}(G) = 5$.

(iv) *If* G *is the degree splitting graph of the* n*-regular Hypercube graph* Q_n, *then* $\mathbb{Z}(G) = 2^{n-1} + 1$.

If G is a regular graph, then we have the following:

Corollary 10 *Let* G *be a regular graph and let* $\mathbb{Z}(G) = k$. *Then* $\mathbb{Z}[\mathcal{DS}(G)] = \mathbb{Z}[\mathcal{CDS}(G)] = k + 1$.

We use the following observation from [11] to prove the next proposition.

Observation 11 [11] *For any simple graph* G, $\delta(G) \leq \mathbb{Z}(G)$, *where* $\delta(G)$ *denote the minimum degree of* G.

The degree splitting graph of the cycle C_k, is known as the wheel graph W_n, where $n = k + 1$.

Proposition 12 *If* G *is the wheel graph* W_n *on* n *vertices, then* $\mathbb{Z}[\mathcal{DS}(G)] = \mathbb{Z}[\mathcal{CDS}(G)] = 4$.

Proof. Let G be the wheel graph W_n on n vertices. Then $\delta[\mathcal{DS}(G)] = 4$, and we have from Observation 11

$$4 \leq \mathbb{Z}(\mathcal{DS}(G)). \tag{1}$$

Since $\mathcal{DS}(G)$ is a graph obtained from G by taking a single vertex v and joining v to all vertices of the cycle C_k. From Proposition 1 and Theorem 3 we conclude that

$$\mathbb{Z}(\mathcal{DS}(G)) \leq \mathbb{Z}(W_n) + 1 = 4. \tag{2}$$

Hence from Equatins (1) and (2) the result follows. □

Proposition 13 *If* G *is the star graph* $K_{1,n}$ *on* $n+1$ *vertices, where* $n \geq 2$, *then* $\mathbb{Z}[\mathcal{DS}(G)] = \mathbb{Z}[\mathcal{CDS}(G)] = n$.

Proof. The degree splitting graph of the star graph is the complete bipartite graph $K_{2,n}$, in [11], the AIM group observed that $\mathbb{Z}(K_{2,n}) = 2 + n - 2 = n$. Therefore the result follows. □

In the next Proposition we consider complete graphs of order n. In [11] the AIM group observed that for the complete graph K_n, $\mathbb{Z}(K_n) = n - 1$. Using this fact and considering that the degree splitting graph of K_n is K_{n+1}, we have the following:

Proposition 14 *For a complete graph of order* n, $\mathbb{Z}[\mathcal{DS}(K_n)] = n$.

We recall the following result from [11].

Proposition 15 [11] *For the complete graph* K_n *of order* $n \geq 2$ *and for the path* P_k *of order* $k \geq 2$, $\mathbb{Z}(K_n \Box P_k) = n$.

Now we consider the degree splitting graph of the ladder graph and find its zero forcing number. The cartesian product graph $P_n \Box K_2$ is known as the ladder graph.

Proposition 16 *Let* G *be the degree splitting graph of the ladder graph* $P_n \Box K_2$ *with* $n \geq 4$ *vertices. Then* $\mathbb{Z}(G) = 4$.

Proof. We have from Proposition 15, $\mathbb{Z}(K_2 \Box P_k) = 2$. Assume that G be the degree splitting graph of $K_2 \Box P_k$. The degree splitting graph of $K_2 \Box P_k$ contains two newly introduced vertices and hence $t = 2$. Therefore, from Theorem 4

$$\mathbb{Z}(G) \leq \mathbb{Z}(K_2 \Box P_k) + 2 = 4. \tag{3}$$

Consider the n-ladder graph as $L_n = P_n \Box K_2$. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n in L_n and $v_1', v_2', \ldots, v_n'$ be the corresponding vertices of $v_1, v_2, \ldots, v_n$ in L_n. Let $B_1 = \{v_1, v_1', v_n, v_n'\}$ be the set of vertices of degree 2 in L_n and let $B_2 = \{v_2, v_3, \ldots, v_{n-1}, v_2', v_3', \ldots, v_{n-1}'\}$ be the set of vertices of degree 3 in L_n. Consider the graph $G \equiv \mathcal{DS}(L_n)$. Let $A_1 = \{B_1 \cup \{a_1\}\}$ be the set of vertices in G with $\deg(a_1) = 4$ and $A_2 = \{B_2 \cup \{a_2\}\}$ with $\deg(a_2) = 2(n-2)$.

To prove the reverse part assume that there exist a zero forcing set consisting of three vertices u, v and w. Degree of each vertex in G is at least three, therefore, to force at least one vertex it is necessary that uv and vw should form edges in G.

Case 1 Assume that the vertices u, v and w are in A_2. In A_2 each vertices have degree at least four, therefore u, v and w does not form a zero forcing set, a contradiction.

Case 2 Assume that the vertices u and v are in A_2 and the vertex w is in A_1. In this case u and v have degree at lest four and w has degree three therefore, u, v and w does not form a zero forcing set, a contradiction.

Case 3 Assume that the vertices u and v are in A_1 and the vertex w is in

A_2. $u = v_1$, $v = v_2$ and $w = v_1'$. Now v_1 forces the vertex a_1 and v_1' forces the vertex v_2' after this forcing, no more color changing is possible, a contradiction.

Case 4 Assume that the vertices u, v and w are in A_1. We have the following two sub cases.

Subcase 4.1 $u = v_1, v = v_1'$ and $w = a_1$. Now v_1 forces v_2 and v_1' forces v_2' after this forcing, no more color changing is possible, a contradiction.

Subcase 4.2 $u = v_1, v = a_1$ and $w = v_n$. In this case $\deg(u) = 3$, $\deg(v) = 4$ and $\deg(w) = 3$ and each of these vertices have two white neighbors, color changing is not possible, a contradiction.
Hence

$$4 \leq \mathbb{Z}(G). \tag{4}$$

Therefore, from (3) and (4) the result follows. □

2 Classes of graphs with $\mathbb{Z}[\mathcal{DS}(G)] < k + t$

In this section, we study simple graphs with $\mathbb{Z}[\mathcal{DS}(G)] < k+t$, where $\mathbb{Z}(G) = k$ and t be the newly introduced vertices in $\mathcal{DS}(G)$. Let G be the path P_4 and $\mathcal{DS}(P_4)$ be the degree splitting graph of P_4 as shown in Figure 2. Then the black vertices depicted in Figure 2 will act as a zero forcing set for $\mathcal{DS}(P_4)$ and hence, $\mathbb{Z}\mathcal{DS}(P_4) = 2 < 1 + 2$.

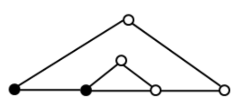

Figure 2:

Example 2 *Let* $G \equiv \mathcal{DS}(C_5 \circ K_1)$ *be the graph depicted in Figure 3. One can easily verify that the set* $\{v_7, v_4, v_8, v_9\}$ *forms a zero forcing set since there is no smaller zero forcing set exist for the graph* G*, therefore,* $\mathbb{Z}(G) = 4$*. Here* v_1 *and* v_{10} *are the newly introduced vertices in* $C_5 \circ K_1$ *to form* $\mathcal{DS}(C_5 \circ K_1)$ *, therefore* $t = 2$*. We have from [11],* $\mathbb{Z}(C_5 \circ K_1) = k = 3$*. Therefore,* $\mathbb{Z}(G) = 4 < k+t = 5$*.*

Proposition 17 *If* G *is the complete bipartite graph* $K_{m,n}$*, where* $m, n \geq 2$ *and* $m \neq n$*, then* $\mathbb{Z}(\mathcal{DS}(G)) = m + n - 1$*.*

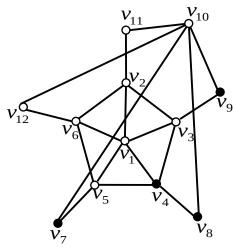

Figure 3:

Proof. Without loss of generality assume G is the complete bipartite graph $K_{m,n}$ and $H = \mathcal{DS}(G)$. Assume that we have a zero forcing set $\mathbb{Z}$ of H consisting of $m + n - 2$ vertices. Then the number of white vertices in H is $m+n+2-(m+n-2) = 4$. Now we divide the vertex set of H into four sets A, B, C and D as depicted in Figure 4. Where $A = \{u\}$, $B = \{u_1, u_2, \ldots, u_m\}$, $C = \{v_1, v_2, \ldots, v_n\}$ and $D = \{v\}$. Assume that the four white vertices are distributed among the sets A, B, C and D.

Claim 1. If H has a zero forcing set, then the total number of white vertices in the set B will never exceed one. On the contrary assume that there exist two white vertices u_i and u_j in the set B. Then for all vertices $v_i, 1 \leq i \leq n$ in the set C, the open neighborhood of $N(v_i)$ contains two white neighbors in the set B also the vertex u will never force the vertices u_i and u_j. Therefore, color changing rule is not applicable in this case, a contradiction to our assumption that there exist two white vertices u_i and u_j in the set B.

Claim 2. If H has a zero forcing set, then the total number of white vertices in the set C will never exceed one. On the contrary assume that there exist two white vertices v_i and v_j in the set C. Then for all vertices $u_i, 1 \leq i \leq n$ in the set B, the open neighborhood of $N(u_i)$ contains two white neighbors in the set C also the vertex v will never force the vertices v_i and v_j. Therefore, color changing rule is not applicable in this case, a contradiction to our assumption that there exist two white vertices v_i and v_j in the set C.

Now assume that we have distributed the white vertices one each in all sets A, B, C and D. Immediately, we can see that any black vertices in the set B and the set C have two white neighbors also the vertices u and v are white, color changing rule is not applicable, a contradiction to our assumption that there exist a zero forcing set in H consisting of $m+n-2$ vertices. Therefore,

$$\mathbb{Z}(\mathcal{DS}(G)) \geq m+n-1. \quad (5)$$

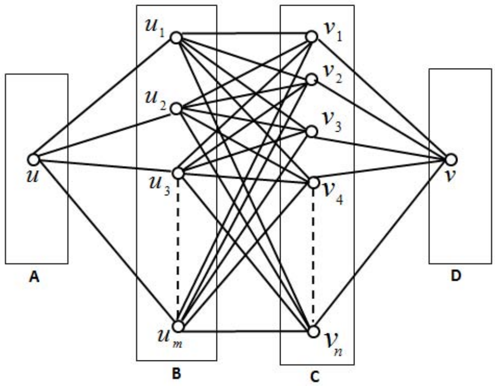

Figure 4:

To prove the reverse part consider the set $\mathbb{E} = \{u_2, u_3, \ldots, u_m, v_2, v_3, \ldots, v_n, v\}$ of vertices from the Figure 4. Color the vertices in the set $\mathbb{E}$ as black. Clearly the vertex $v \rightarrowtail v_1$, $v_1 \rightarrowtail u_1$, and $u_1 \rightarrowtail u$. Now the set $\mathbb{E}$ forms a zero forcing set and $|\mathbb{E}| = m-1+n-1+1 = m+n-1$. Therefore,

$$\mathbb{Z}(\mathcal{DS}(G)) \leq m+n-1. \quad (6)$$

Hence from (5) and (6) the result follows. □

The following Lemma can be found in [7].

Lemma 1 [7] *Let $G = (V, E)$ be a connected graph with a cut-vertex $v \in V(G)$. Let $C_1, \ldots, C_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[C_i \cup \{v\}]$. Then $\mathbb{Z}(G) \geq \sum_{i=1}^{k} \mathbb{Z}(G_i) - k + 1$.*

Definition 1 *The Pineapple graph K_m^n is obtained by coalescing any vertex of the complete graph K_m with the star $K_{1,n}$ at the vertex of degree n. The number of vertices in K_m^n is $m+n$, number of edges in K_m^n is $\frac{m^2-m+2n}{2}$. These graphs were defined and studied in [12] and [10].*

The authors in [12] and [10] studied about the spectral properties of Pineapple Graphs.

We recall the following results from [13].

Proposition 18 [13] *If* G *is the Pineapple graph* K_m^n *with* $n \geq 2, m \geq 3$*, then* $\mathbb{Z}(G) = m + n - 3$.

Proposition 19 *If* G *is the Pineapple graph* K_m^1 *with* $m \geq 3$*, then* $\mathbb{Z}(G) = m - 1$.

Proposition 20 *If* G *is the Pineapple graph* K_m^n *with* $m \geq 3$ *and* $n \geq 1$*, then* $\mathbb{Z}(\mathcal{DS}(K_m^n)) = m + n - 2$.

Proof. Case 1 Without of loss of generality assume that $n = 1$. Let $\mathcal{DS}(K_m^1)$ be the degree splitting graph of K_m^1 and let v be the newly introduced vertex in $\mathcal{DS}(K_m^1)$ to construct it. Let u' be the coalesced vertex of the complete graph K_m with the star $K_{1,n}$ in K_m^1 and let u be the corresponding vertex of u' in $\mathcal{DS}(K_m^1)$. Let w be the pendant vertex in $\mathcal{DS}(K_m^1)$ and let x be an arbitrary vertex of $\mathcal{DS}(K_m^1)$ other than u, v and w. Color all vertices except u, x and w in $\mathcal{DS}(K_m^1)$ as black. Clearly the vertex $v \rightarrowtail x$, $x \rightarrowtail u$ and $u \rightarrowtail w$ and hence

$$\mathbb{Z}(\mathcal{DS}(K_m^1)) \leq m - 1. \tag{7}$$

To prove the reverse part we use the following

$$\mathbb{Z}(K_{m+1} - e) = m - 1 \tag{A}$$

$$\mathbb{Z}(K_2) = 1. \tag{B}$$

Now Lemma 1, (A) and (B) yields,

$$\mathbb{Z}(\mathcal{DS}(K_m^1)) \geq \sum_{i=1}^{2} \mathbb{Z}(G_i) - 2 + 1 = \mathbb{Z}(K_{m+1} - e) + \mathbb{Z}(K_2) - 1 = m - 1. \tag{8}$$

Thus the result follows from (7) and (8).

Case 2 Assume that $n = 2$. Let $\mathcal{DS}(K_m^2)$ be the degree splitting graph of K_m^2. Let u' be the coalesced vertex of the complete graph K_m with the star $K_{1,n}$ in K_m^1 and let u be the corresponding vertex of u' in $\mathcal{DS}(K_m^2)$. Let w_1, w_2 and w_3 be the vertices of degree two in $\mathcal{DS}(K_m^2)$. The subgraph induced by the vertices w_1, w_2, w_3 and u forms a cycles C_4 in $\mathcal{DS}(K_m^2)$. Let x be an arbitrary vertex of $\mathcal{DS}(K_m^2)$ other than w_1, w_2, w_3 and u. Color all vertices except w_2, w_3, x and

u in $\mathcal{DS}(K_m^2)$ black. Let y be an arbitrary black colored vertex other than w_1 in $\mathcal{DS}(K_m^2)$. Clearly $y \rightarrowtail x$, $x \rightarrowtail u$, $u \rightarrowtail w_3$ and $w_3 \rightarrowtail w_2$, hence

$$\mathbb{Z}(\mathcal{DS}(K_m^2)) \leq m. \tag{9}$$

To prove the reverse inequality use the the following

$$\mathbb{Z}(K_{2,2}) = 2. \tag{C}$$

Now Lemma 1, (A) and (C) yields the following,

$$\mathbb{Z}(\mathcal{DS}(K_m^2)) \geq \sum_{i=1}^{2} \mathbb{Z}(G_i) - 2 + 1 = \mathbb{Z}(K_{m+1} - e) + \mathbb{Z}(K_{2,2}) - 1 = m - 1 + 2 - 1 = m. \tag{10}$$

Therefore, from (9) and (10) the result follows.

Case 3 Assume $n \geq 3$. Let $\mathcal{DS}(K_m^n)$ be the degree splitting graph of K_m^n. Let u' be the coalesced vertex of the complete graph K_m with the star $K_{1,n}$ in K_m^n and let u be the corresponding vertex of u' in $\mathcal{DS}(K_m^n)$. Similarly let t be the newly introduced vertex in $\mathcal{DS}(K_m^n)$ obtained by joining the pendant vertices in K_m^n. Let $w_1, w_2, \ldots, w_n$ be the vertices of degree two in $\mathcal{DS}(K_m^n)$. The subgraph induced by the vertices $\{w_1, w_2, \ldots, w_n\} \cup \{t, u\}$ forms the complete bipartite graph $K_{2,n}$ in $\mathcal{DS}(K_m^n)$.

Let x be the newly introduced vertex in $\mathcal{DS}(K_m^n)$ other than the vertex t in $\mathcal{DS}(K_m^n)$. Let y be a vertex in $\mathcal{DS}(K_m^n)$ other than $w_1, w_2, \ldots, w_n, u, x$ and t. Color all vertices except the vertices w_n, t, y and u in $\mathcal{DS}(K_m^n)$ as black. Clearly $x \rightarrowtail y$, $y \rightarrowtail u$, $u \rightarrowtail w_n$, $w_n \rightarrowtail t$ hence

$$\mathbb{Z}(\mathcal{DS}(K_m^n)) \leq m + n - 2. \tag{11}$$

To prove the reverse inequality use the the following result from [13]

$$\mathbb{Z}(K_{m,n}) = m + n - 2. \tag{D}$$

Now Lemma 1, (A) and (D) yields the following,

$$\begin{aligned} \mathbb{Z}(\mathcal{DS}(K_m^n)) &\geq \sum_{i=1}^{2} \mathbb{Z}(G_i) - 2 + 1 = \mathbb{Z}(K_{m+1} - e) + \mathbb{Z}(K_{m,n}) - 1 \\ &= (m - 1) + (2 + n - 2) - 1 = m + n - 1. \end{aligned} \tag{12}$$

Therefore, from (11) and (12) the result follows. □

3 Conclusion and open problems

This paper deals with the problem of determination of the zero forcing number of graphs and their degree splitting graphs. Characterization of classes graphs for which $\mathbb{Z}[\mathcal{DS}(\mathsf{G})] = \mathsf{k} + \mathsf{t}$ is an open problem.

Acknowledgments

The author is thankful to the referee for going through the manuscript very carefully and suggesting many changes.

References

[1] D. Amos, Y. Caro, R. Davila, R. Pepper, Upper bounds on the k-forcing number of a graph, *Discrete Appl. Math.*, **181** (2015), 1–10.

[2] D. Burgarth, V. Giovannetti, L. Hogben, S. Severini, M. Young, Logic circuits from zeroforcing, *Natural Computing*, **14** (2015), 485–490.

[3] F. Harary, *Graph Theory*, Addison-Wesely, Massachusethes, (1969).

[4] R. Ponraj, S. Somasundaram, On the degree splitting graph of a graph, *NATL ACAD SCI LETT*, **27** (7 & 8) (2004), 275–278.

[5] Premodkumar, Charles Dominic, Baby Chacko, On the zeroforcing number of graphs and their splitting graphs, *Algebra Discrete Math.*, Accepted, 28 (2019).

[6] D. D. Row, *Zero forcing number: Results for computation and comparison with other graph parameters*, Iowa State University, Ames, Iowa (2011).

[7] D. D. Row, A technique for computing the zero forcing number of a graph with a cut vertex, *Linear Algebra Appl.*, **436** (2012), 4423–4432.

[8] E. Sampathkumar, H. B. Walikaer, On the splitting graph of a graph, *Journal of Karnatak University Science*, **25** (1981), 13–16.

[9] F. A. Taklimi, Zeroforcing sets for graphs, arXiv:1311.7672, (2013).

[10] Hatice Topcu, Sezer Sorgun, Willem H. Haemers, *On the spectral characterization of pineapple graphs*, arXiv:1511.08674v3 [math.CO] 10 June 2016.

[11] Hein van der Holst et al., Zero forcing sets and the minimum rank of graphs, *Linear Algebra Appl.*, 428 (2008), 1628–1648,.

[12] Xiaolin Zhang, Heping Zhang, Some graphs determined by their spectra, *Linear Algebra Appl*, 431 (2009), 1443–1454.

[13] https://aimath.org/pastworkshops/catalog2.html

Received: October 11, 2018