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Vertex stress related parameters for certain Kneser graphs

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Abstract. This paper presents results for some vertex stress related parameters in respect of specific subfamilies of Kneser graphs. Kneser graphs for which $diam(KG(n,k)) = 2$ and $k \geq 2$ are considered. The note establishes the foundation for researching similar results for Kneser graphs for which $diam(KG(n,k)) \geq 3$. In addition some important vertex stress related properties are stated. Finally some results for specific bipartite Kneser graphs i.e. $BK(n,1)$, $n \geq 3$ will be presented. In the conclusion some worthy research avenues are proposed.

1 Introduction

It is assumed that the reader has good working knowledge of set theory. For the general notation, notions and important introductory results in set theory, see [3]. For the general notation, notions and important introductory results in graph theory, see [2, 4].

Only non-trivial, finite, undirected and connected simple graphs are considered. Let X_i, $i = 1, 2, 3, \ldots, \binom{n}{k}$, $k \geq 1$ be the k-element subsets of the set,

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$\{1, 2, 3, \ldots, n\}$. A Kneser graph denoted by $KG(n, k)$, $n, k \in \mathbb{N}$ is a graph with vertex set,

$$V(KG(n, k)) = \{v_i : v_i \mapsto X_i\}$$

and the edge set,

$$E(KG(n, k)) = \{v_i v_j : X_i \cap X_j = \emptyset\}.$$

Without any relation between n and k the following subfamilies of Kneser graphs follow easily.
1. For $k > n$ the Kneser graph has an empty vertex set implying that the edge set is empty. Hence, for $n \in \mathbb{N}$ the empty graph is obtained.
2. For $k = 1$ the Kneser graph $KG(n, k) \cong K_n$ $\forall$ n.
3. For $k = n$ the Kneser graph is always a trivial graph (i.e. K_1).
4. For $\frac{n}{2} < k < n$ the vertex set is non-empty. However, the edge set is empty so for the permissible values of k and n the corresponding null graphs $\mathfrak{N}_t$, $t = \binom{n}{k}$ are obtained.
5. For n even and $k = \frac{n}{2}$ a corresponding matching graph is obtained.
6. For n even and $2 \leq k \leq \frac{n-2}{2}$ and; for n odd and $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$ the subfamily of non-trivial, connected and non-complete Kneser graphs is obtained.

Note that the author draws a distinction between an empty graph G, (both $V(G) = \emptyset$ and $E(G) = \emptyset$) and a null graph H, ($V(H) \neq \emptyset$ and $E(H) = \emptyset$). This distinction is not common in the literature. Conventionally, the 0-element subset is not considered. However, from set theory it is known that the empty-set is indeed a subset of any set. Therefore, it seems proper to state that $KG(n, 0)$ is a trivial graph say, $v_1 \mapsto \emptyset$.

It follows directly from the structure of a Kneser graph that the order of a Kneser graph is given by $\nu(KG(n, k)) = \binom{n}{k}$. Furthermore, because vertex adjacency as it relates to a k-element subset is defined without loss of generality, a Kneser graph is inherently a degree regular graph. The number of neighbors of any vertex v_i is given by, $deg(v_i) = \binom{n-k}{k}$. From the aforesaid it follows that the number of edges is given by, $\varepsilon(KG(n, k)) = \frac{1}{2} \times \frac{n!}{k!k!(n-2k)!}$.

Example: $KG(5, 2)$**:** Let $V(KG(5, 2))$ be define as: $v_1 \mapsto \{1, 2\}$, $v_2 \mapsto \{1, 3\}$, $v_3 \mapsto \{1, 4\}$, $v_4 \mapsto \{1, 5\}$, $v_5 \mapsto \{2, 3\}$, $v_6 \mapsto \{2, 4\}$, $v_7 \mapsto \{2, 5\}$, $v_8 \mapsto \{3, 4\}$, $v_9 \mapsto \{3, 5\}$, $v_{10} \mapsto \{4, 5\}$. See figure 1 as illustration. It is known that, $KG(5, 2) \cong$ *Petersen graph.*

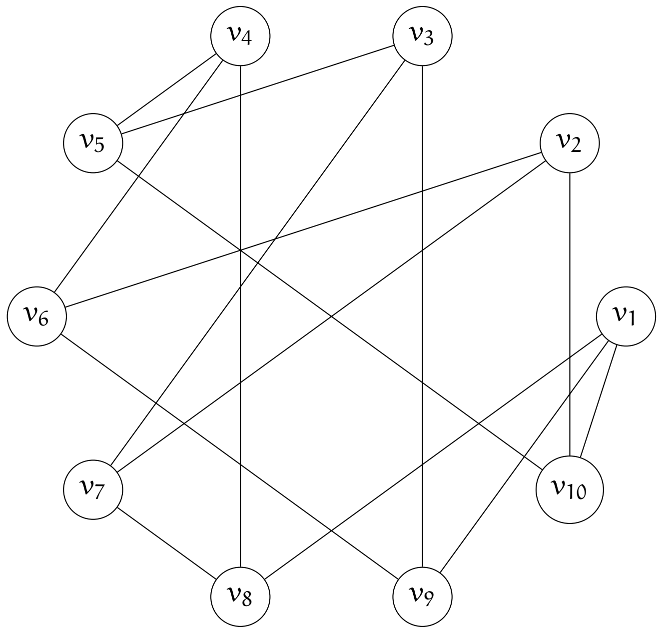

Figure 1: $KG(5,2)$ of order 10.

2 Total induced vertex stress, total vertex stress and vertex stress

The vertex stress of vertex $v \in V(G)$ is the number of times v is contained as an internal vertex in all shortest paths between all pairs of distinct vertices in $V(G)\backslash\{v\}$. Formally stated, $\mathcal{S}_G(v) = \sum\limits_{u \neq w \neq v \neq u} \sigma(v)$ with $\sigma(v)$ the number of shortest paths between vertices u, w which contain v as an internal vertex. Such a shortest uw-path is also called a uw-distance path. See [8, 9]. The *total vertex stress* of G is given by $\mathcal{S}(G) = \sum\limits_{v \in V(G)} \mathcal{S}_G(v)$, [5]. From [10, 11] we recall the definition of total induced vertex stress denoted by, $\mathfrak{s}_G(v_i)$, $v_i \in V(G)$.

Definition 1 [11] *Let* $V(G) = \{v_i : 1 \leq i \leq n\}$. *For the ordered vertex pair* (v_i, v_j) *let there be* $k_G(i,j)$ *distinct shortest paths of length* $l_G(i,j)$ *from* v_i *to* v_j. *Then,* $\mathfrak{s}_G(v_i) = \sum\limits_{j=1, j \neq i}^{n} k_G(i,j)(\ell_G(i,j) - 1)$.

The notion of vertex stress finds application in research related to *centrality* in graphs. In dynamical graph theory the parameter assists to identify vertices which are more prone to system failure. The nodes within road networks are a good example. A more subtle example is identifying the possibility of a high number of step-through certain steps in an algorithm. Such high number step-through may lead to excessive memory requirements. Highly congested airports can be pre-empted by determining the vertex stress of airports serving as the nodes of a flight route network.

The families no. 1 to 5 have the vertex stress related parameters equal to 0. The only interesting family of Kneser graphs to consider with regards to vertex stress and related parameters is family no. 6. Thus only Kneser graphs within the range $2 \leq k < \frac{n}{2}$ will be studied. From [12] we have that $diam(KG(n,k)) = 1 + \lceil \frac{k-1}{n-2k} \rceil$. Hence, for $2 \leq k \leq \frac{n+1}{3}$ a corresponding Kneser graph has $diam(KG(n,k)) = 2$. This section seeks to find results for $k \geq 2$ subject to, $n \geq 3k-1$.

Case 1: Let $k = 2$, then $n \geq 5$. For ease of reasoning the following convention for 2-subsets of the set $\{1, 2, 3, \dots, n\}$ will be used. The vertices are defined as: $v_1 \mapsto \{1,2\}$, $v_2 \mapsto \{1,3\}, \dots, v_{n-1} \mapsto \{1,n\}$, $v_n \mapsto \{2,3\}$, $v_{n+1} \mapsto \{2,4\}, \dots, v_{2n-3} \mapsto \{2,n\}, \dots, v_{\binom{n}{2}} \mapsto \{n-1,n\}$.

Remark. As stated before, since vertex adjacency is defined without loss of generality (for brevity, the wlg-principle) all results in respect of vertex v_1 are (immediately) valid for all $v_i \in V(KG(n,k))$. Hence, for ease of reasoning the results for v_1 will be determined and then generalized. Such generalization is axiomatically valid and requires no further proof.

Recall that the set of vertices adjacent to vertex v_i is called the open neighborhood of v_i and it is denoted by $N(v_i)$ (or $N_G(v_i)$ if reference to G is important). The closed neighborhood of vertex v_i is defined as, $N[v_i] = N(v_i) \cup \{v_i\}$ (or $N_G[v_i]$ if reference to G is important).

Proposition 2 *For a Kneser graph* $KG(n,2)$*,* $n \geq 5$ *the total induced vertex stress of* v_1 *is given by*

$$\mathfrak{s}_{KG(n,2)}(v_1) = \left[\binom{n}{2} - \left(\binom{n-2}{2} + 1\right)\right] \times \binom{n-3}{2}.$$

Proof. Clearly, $N(v_1) = \{v_i : 1, 2 \notin v_i\}$. It is known that $|N(v_1)| = \binom{n-2}{2}$. Because $diam(KG(n,k)) = 2$ there does not exist any shortest 3-path in $KG(n,2)$. Hence, there only exist shortest 2-paths from v_1 to the remaining $\binom{n}{2} - \left(\binom{n-2}{2} + 1\right)$ vertices of $KG(n,2)$. Without loss of generality consider vertex $v_2 \mapsto \{1,3\}$. Since $N(v_2) = \{v_j : 1, 3 \notin v_j\}$ it follows easily that $|N(v_1) \cap N(v_2)| = \binom{n-3}{2}$. The aforesaid is true because the elements $1, 2, 3$ are (must be) excluded as elements of the 2-element subsets in $N(v_1) \cap N(v_2)$. Thus, the *partial total vertex stress* induced by vertex v_1 along all shortest v_1v_2-paths is settled. By the wlg-principle the principle of immediate induction is valid. Therefore,

$$\mathfrak{s}_{KG(n,2)}(v_1) = \left[\binom{n}{2} - \left(\binom{n-2}{2} + 1\right)\right] \times \binom{n-3}{2}. \qquad \square$$

The generalized corollaries follow immediately.

Corollary 3 *For a Kneser graph* $KG(n,2)$*,* $n \geq 5$ *the total induced vertex stress of* $v_i \in V(KG(n,2))$ *is given by,*

$$\mathfrak{s}_{KG(n,2)}(v_i) = \left[\binom{n}{2} - \left(\binom{n-2}{2} + 1\right)\right] \times \binom{n-3}{2}.$$

Corollary 4 *For a Kneser graph* $KG(n,2)$*,* $n \geq 5$ *the total vertex stress is given by*

$$\mathcal{S}(KG(n,2)) = \frac{1}{2}\binom{n}{2}\left[\binom{n}{2} - \left(\binom{n-2}{2} + 1\right)\right] \times \binom{n-3}{2}.$$

Proof. The result follows from Definition 1 read together with the proof of Proposition 2 and Corollary 3. $\square$

Corollary 5 *For a Kneser graph* $KG(n,2)$*,* $n \geq 5$ *the vertex stress is given by*

$$\mathcal{S}_{KG(n,2)}(v_i) = \frac{1}{2}\left[\binom{n}{2} - \left(\binom{n-2}{2} + 1\right)\right] \times \binom{n-3}{2}.$$

Proof. It is known that the Kneser graphs $KG(n,2)$ are distance regular, see [1]. By Theorem 3.6 in [9] the Kneser graphs $KG(n,2)$ are stress regular as well. Thus, the result of Corollary 4 must simply be divided by $\binom{n}{2}$. $\square$

Note that since $2 \leq k \leq \frac{n+1}{3}$ it follows that $n \geq 3k_1 - 1$ for a $k_1 \in \mathbb{N}\backslash\{1\}$ to ensure that $diam(KG(n,k_1)) = 2$. This observation enables immediate generalizations. The vertices which may be used for reasoning of proof are:

$$v_1 \mapsto \{1,2,3,\dots,k_1-1,k_1\},\ v_2 \mapsto \{1,2,3,\dots,k_1-1,k_1+1\}$$
$$\text{and } v_i \in N(v_1) \cap N(v_2).$$

Because the reasoning of proof is similar to that found in Proposition 2 and Corollaries 3 to 5 and the fact that the wlg-principle applies throughout in all Kneser graph embodiments, no further proofs are presented.

Theorem 6 *For a Kneser graph* $KG(n,k_1)$*,* $k_1 \in \mathbb{N}\backslash\{1,2\}$*,* $n \geq 3k_1 - 1$ *the total induced vertex stress of* $v_i \in V(KG(n,k_1))$ *is given by*

$$\mathfrak{s}_{KG(n,k_1)}(v_i) = \left[\binom{n}{k_1} - (\binom{n-k_1}{k_1} + 1)\right] \times \binom{n-(k_1+1)}{k_1}.$$

Corollary 7 *For a Kneser graph* $\mathsf{KG}(n, k_1)$*,* $k_1 \in \mathbb{N}\backslash\{1, 2\}$*,* $n \geq 3k_1 - 1$ *the total vertex stress is given by*

$$\mathcal{S}(\mathsf{KG}(n, k_1)) = \frac{1}{2}\binom{n}{k_1}\Big[\binom{n}{k_1} - \Big(\binom{n-k_1}{k_1} + 1\Big)\Big] \times \binom{n-(k_1+1)}{k_1}.$$

Corollary 8 *For a Kneser graph* $\mathsf{KG}(n, k_1)$*,* $k_1 \in \mathbb{N}\backslash\{1, 2\}$*,* $n \geq 3k_1 - 1$ *the vertex stress is given by,*

$$\mathcal{S}_{\mathsf{KG}(n,k_1)}(v_i) = \frac{1}{2}\Big[\binom{n}{k_1} - \Big(\binom{n-k_1}{k_1} + 1\Big)\Big] \times \binom{n-(k_1+1)}{k_1}.$$

2.1 Vertex stress related properties of $\mathsf{KG}(n, 2)$

Recall some results from [9]. A graph G for which $\mathcal{S}_G(v_i) = \mathcal{S}_G(v_j)$ for all distinct pairs $v_i, v_j \in V(G)$ is said to be stress regular.

Theorem 9 [9] *Every distance regular graph is stress regular.*

Corollary 10 [9] *Every strongly regular graph is stress regular.*

Corollary 11 [9] *Every distance transitive graph is stress regular.*

Since it is known that the family of Kneser graphs $\mathsf{KG}(n, 2)$ are distance regular graphs it follows from Theorem 9 that the Kneser graphs $\mathsf{KG}(n, 2)$ are stress regular. Furthermore, it is known from [1] that every distance regular graph G with $\mathrm{diam}(G) = 2$, is strongly regular. The aforesaid read together with Corollary 10 permit the next corollary without further proof.

Corollary 12 *Kneser graphs* $\mathsf{KG}(n, k_1)$*,* $k_1 \in \mathbb{N}\backslash\{1, 2\}$*,* $n \geq 3k_1 - 1$ *are stress regular.*

In fact, a general result (without further proof) is permitted from the knowledge that all Kneser graphs $\mathsf{KG}(n, k)$, $n \geq k$ are vertex transitive.

Theorem 13 *All Kneser graphs* $\mathsf{KG}(n, k)$*,* $n \geq k$ *are stress regular.*

2.2 Stress balanced graphs

Definition 14 *A graph* G *is said to be stress-balanced if and only if*

$$\sum_{v_t \in N[v_i]} \mathcal{S}_G(v_t) = \sum_{v_l \in N[v_j]} \mathcal{S}_G(v_l)$$

for all pairs of distinct vertices $v_i, v_j \in V(G)$*.*

The value $\gamma(v_i) = \sum\limits_{v_t \in N[v_i]} \mathcal{S}_G(v_t)$ is called the *vertex stress index* of the vertex v_i. A star graph (for brevity, a star) is a tree which has a central vertex v_0 with $m \geq 0$ pendent vertices (or leafs) attached to v_0. The star is denoted by $S_{1,m}$ and the leafs are labeled, v_i, $i = 1, 2, 3, \ldots, m$. It is straightforward to verify that the respective vertex stress are, $\mathcal{S}_{S_{1,m}}(v_i) = 0$, $1 \leq i \leq m$ and $\mathcal{S}_{S_{1,m}}(v_0) = \frac{m(m-1)}{2}$. Since, $\sum\limits_{v_j \in N_{S_{1,m}}[v_0]} \mathcal{S}_{S_{1,m}}(v_j) = \frac{m(m-1)}{2} + m \times 0 = \frac{m(m-1)}{2}$ and $\sum\limits_{v_j \in N_{S_{1,m}}[v_i]} \mathcal{S}_{S_{1,m}}(v_j) = 0 + \frac{m(m-1)}{2} = \frac{m(m-1)}{2}$, $1 \leq i \leq m$ a star is stress-balanced. A star shows that, despite not being degree regular or stress regular, a star is stress-balanced. Figure 2 depicts another example. The graph $G = C_4 + v_1v_3$ is not degree regular and has $\mathcal{S}_G(v_1) = \mathcal{S}_G(v_3) = 1$ and $\mathcal{S}_G(v_2) = \mathcal{S}_G(v_4) = 0$. So G is not stress regular but it is stress-balanced.

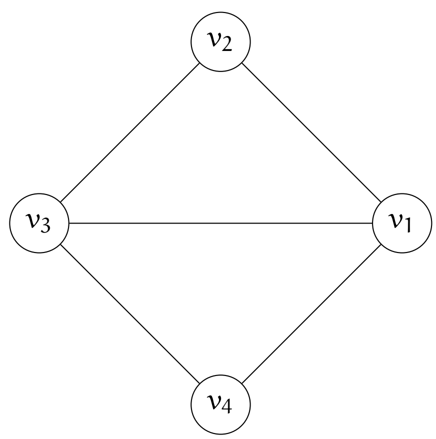

Figure 2: $G = C_4 + v_1v_3$.

Lemma 15 *A graph G which is degree regular (or regular for brevity) and stress regular is stress-balanced.*

Proof. The result is a direct consequence of Definition 14. □

We present the main result of this subsection.

Theorem 16 *All Kneser graphs $KG(n,k)$, $n \geq k$ are stress-balanced.*

Proof. Since all Kneser graphs $KG(n,k)$, $n \geq k$ are degree regular and stress regular (see Theorem 13), read together with Definition 14 and Lemma 15, settles the result. □

3 On bipartite Kneser graphs, $\mathsf{BK}(n,k)$

Without loss of generality let $n \geq 3$ and let $1 \leq k \leq \lceil \frac{n}{2} \rceil - 1$. Let X_i, $i = 1,2,3,\dots,\binom{n}{k}$ be the k-element subsets of the set, $\{1,2,3,\dots,n\}$. Let Y_i, $i = 1,2,3,\dots,\binom{n}{k}$ be the $(n-k)$-element subsets of the set, $\{1,2,3,\dots,n\}$. Let $V_1 = \{v_i : v_i \mapsto X_i\}$ and $V_2 = \{u_i : u_i \mapsto Y_i\}$. A connected bipartite Kneser graph denoted by $\mathsf{BK}(n,k)$ is a graph with vertex set,

$$V(BG(n,k)) = V_1 \cup V_2$$

and the edge set,

$$E(BG(n,k)) = \{v_i u_j : X_i \subset Y_j\}.$$

From the definition it is axiomatically true (or from applying the wlg-principle) that $\mathsf{BK}(n,k)$ is degree regular with $deg(v_i) = deg(u_j) = \binom{n-k}{k}$. Equally straightforward is that $\mathsf{BK}(n,k)$ is of order $2 \times \binom{n}{k}$. In fact, $|V_1| = |V_2| = \binom{n}{k}$.

Theorem 17 *A bipartite Kneser graph* $\mathsf{BK}(n,k)$ *has* $diam(\mathsf{BK}(n,k)) \geq 3$.

Proof. Since $\binom{n-k}{k} < \binom{n}{k}$ it follows that $N(v_i) \subset V_2$ and similarly, $N(u_j) \subset V_1$. Hence, there exists at least one shortest $v_i u_j$-path (or shortest $u_j v_i$-path) of distance greater or equal to 3. □

Similar to the notion of stress regularity it is said that a graph G is *induced vertex stress regular* (for brevity, IVS-regular) if and only if $\mathfrak{s}_G(v_i) = \mathfrak{s}_G(v_j)$ for all distinct pairs $v_i, v_j \in V(G)$.

Theorem 18 *An* IVS*-regular graph* G *is stress regular.*

Proof. The result follows from the fact that for any vertex v_i the vertex stress is given by, $\mathcal{S}_G(v_i) = \frac{\mathfrak{s}_G(v_i)}{2}$. □

Corollary 19 *An* IVS*-regular graph* G *with a singular adjacency regime for all vertices is stress-balanced.*

Proof. The result follows from Theorem 18 and the fact that $deg_G(v_i) = deg_G(v_j)$ for all distinct pairs $v_i, v_j \in V(G)$. □

Theorem 18 and Corollary 19 read together with the definition of $\mathsf{BK}(n,k)$ imply that bipartite Kneser graphs are stress-balanced.

3.1 Specific results for $BK(n, 1)$, $n \geq 3$

Theorem 20 *Bipartite Kneser graphs,* $BK(n, 1)$*,* $n \geq 3$ *are stress regular.*

Proof. Proposition 3.4 in [7] convinces that a graph $BK(n, 1)$ is distance-transitive. Also, *distance-transitive* $\Rightarrow$ *distance regular*. Therefore, the latter read together with Theorem 9 (Theorem 3.6 in [9]) settles the fact that bipartite Kneser graphs $BK(n, 1)$, $n \geq 3$ are stress regular. $\square$

Theorem 21 *A bipartite Kneser graph,* $BK(n, 1)$*,* $n \geq 3$ *has,*

$$diam(BK(n, 1)) = 3.$$

Proof. Let $V_1(BK(3, 1)) = \{v_i \mapsto \{i\} : i = 1, 2, 3$ and $V_2(BK(n, 1)) = \{u_1 \mapsto \{1, 2\}, u_2 \mapsto \{1, 3\}, u_3 \mapsto \{2, 3\}\}$. From the definition of $BK(n, 1)$ it follows immediately that $BK(3, 1) \cong C_6$. Hence, $diam(BK(3, 1)) = 3$. For $B(4, 1)$ each vertex $u_i \in V_2(B(3, 1))$ becomes $u_i \cup \{4\}$ and exactly two vertices are added. Therefore, $V_1(BK(4, 1)) = V_1(BK(3, 1)) \cup \{4\}$ and $V_2(BK(4, 1)) = \{u_i \cup \{4\} : u_i \in V_2(BK(3, 1))\} \cup \{1, 2, 3\}$. After adding the edges in accordance with the adjacency definition it follows easily that, $diam(BK(4, 1)) = 3$. Obviously the vertex changes and the addition of exactly two new vertices remain consistent as n progresses consecutively through $5, 6, 7 \ldots$.

Assume the result holds for $BK(n, 1)$, $5 \leq n \leq k$. By similar reasoning to show the result for the progression from $n = 3$ to $n = 4$, it follows by immediate induction that the results holds for the progression from $n = k$ to $n = k + 1$. Thus,

$$BK(n, 1),\ n \geq 3 \text{ has } diam(BK(n, 1)) = 3.$$ $\square$

Proposition 22 *A vertex* $v_i \in V_1(BK(n, 1))$ *(or* $u_j \in V_2(BK(n, 1))$*) has:*

$$\mathfrak{s}_{BK(n,1)}(v_i) = 3 \times deg_{BK(n,1)}(v_i)(deg_{BK(n,1)}(v_i) - 1).$$

Proof. It follows from Theorem 21 that a vertex $v_i \in V_1(BK(n, 1))$ (or $u_j \in V_2(BK(n, 1))$) has exactly,

$$deg_{BK(n,1)}(v_i)(deg_{BK(n,1)}(v_i) - 1) \text{ shortest 2-paths}$$

and exactly,

$$deg_{BK(n,1)}(v_i)(deg_{BK(n,1)}(v_i) - 1) \text{ shortest 3-paths.}$$

Obviously v_1 has $deg_{BK(n,1)}(v_i)$ shortests 1-paths (or edges). Applying Definition 1 settles the result. $\square$

Corollary 23 *(a) A vertex* $v_i \in V_1(BK(n,1))$ *(or* $u_j \in V_2(BK(n,1))$*) has:*

$$\mathcal{S}_{BK(n,1)}(v_i) = \frac{3(n-1)(n-2)}{2}.$$

(b) $BK(n,1)$*,* $n \geq 3$ *has,* $\mathcal{S}(BK(n,1)) = 3n(n-1)(n-2)$*.*
(c) $BK(n,1)$*,* $n \geq 3$ *has,* $\mathfrak{s}(BK(n,1)) = 6n(n-1)(n-2)$*.*

Proof. Trivial from the appropriate definitions. □

4 Conclusion

Author is of the view that an extension of this paper through a study of Kneser graphs with diameter greater than 2 is a worthy endeavor. To ensure that say, $diam(KG(n,k)) = 3$ it follows that for $k \geq 3$, $n \geq \frac{5k-1}{2}$. The number of vertices (k-subsets) of $\binom{\geq \frac{5k-1}{2}}{k}$ becomes large very rapidly. More so for $diam(KG(n,k)) = \ell$, $\ell \geq 4$. It is suggested that shortest path algorithms (existing or newly developed) or experimental mathematics through simulation programs be utilized in support of further research.

The new notion of stress-balanced graphs has only been briefly introduced. It is suggested to be an interesting concept with a wide scope for further research. A graph G which is not stress-balanced will have at least one vertex say, v_i such that $\gamma(v_i) = max\{\gamma(v_j) : \gamma(v_j) = \sum\limits_{v_t \in N[v_j]} \mathcal{S}_G(v_t)\}$. A closed neighborhood $N[v_i]$ which yields such maximum is called a *stress district* of G. Similarly a closed neighborhood $N[v_j]$ which yields $min\{\gamma(v_k) : \gamma(v_k) = \sum\limits_{v_t \in N[v_k]} \mathcal{S}_G(v_t)\}$ is called a *stress suburb* of G. Studying stress districts and stress suburbs remains open.

Various studies of other families of graphs which are constructed from the subsets of a set together with a well-defined adjacency regime have been published. We refer to this as the study of graphs from sets. A specific and perhaps less known family called set-graphs can be read in [6]. Hence, various research projects under the theme *Vertex stress related parameters for graphs from sets* remain open.

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