Acta Univ. Sapientiae, Mathematica, **9**, 2 (2017) 377–390

DOI: 10.1515/ausm-2017-0029

Uniqueness of polynomial and differential monomial

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Abstract. In this paper, we discuss the problem of meromorphic functions sharing small function and present one theorem which extend a result of K. S. Charak and Banarasi Lal [16].

1 Introduction and main results

In this paper, a meromorphic function always mean a function which is meromorphic in the whole complex plane.

Definition 1 *Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions, $a \in \mathbb{C} \cup \{\infty\}$. We say that f and g share the value a CM if $f-a$ and $g-a$ have the same zeros with the same multiplicities.*

Definition 2 *We denote by $N_{k)}\left(r, \frac{1}{(f-a)}\right)$ the counting function for zeros of $f-a$ with multiplicity $\leq k$, and by $\overline{N}_{k)}\left(r, \frac{1}{(f-a)}\right)$ the corresponding one for*

2010 Mathematics Subject Classification: 30D35
Key words and phrases: meromorphic function, sharing values, differential monomial, polynomial

which multiplicity is not counted. Let $N_{(k}\left(r, \frac{1}{(f-a)}\right)$ *be the counting function for zeros of* $f-a$ *with multiplicity at least* k *and* $\overline{N}_{(k}\left(r, \frac{1}{(f-a)}\right)$ *the corresponding one for which multiplicity is not counted. Set*

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \ldots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

Definition 3 *For two positive integers* n, p *we define* $\mu_p = \min\{n, p\}$ *and* $\mu_p^* = p + 1 - \mu_p$. *Then it is clear that*

$$N_p(r, 0; f^n) \leq \mu_p N_{\mu_p^*}(r, 0; f).$$

Definition 4 [17] *Let* z_0 *be a zero of* $f - a$ *of multiplicity* p *and* a *zero of* $g-a$ *of multiplicity* q. *We denote by* $\overline{N}_L(r, a; f)$ *the counting function of those* a*-points of* f *and* g *where* $p > q \geq 1$, *by* $N_E^{1)}(r, a; f)$ *the counting function of those* a*-points of* f *and* g *where* $p = q = 1$ *and by* $\overline{N}_E^{(2}(r, a; f)$ *the counting function of those* a*-points of* f *and* g *where* $p = q \geq 2$, *each point in these counting functions is counted only once. In the same way we can define* $\overline{N}_L(r, a; g), N_E^{1)}(r, a; g), \overline{N}_E^{(2}(r, a; g)$.

Definition 5 [18] *Let* k *be a non-negative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $E_k(a; f)$ *the set of all* a*-points of* f, *where an* a*-point of multiplicity* m *is counted* m *times if* $m \leq k$ *and* $k+1$ *times if* $m > k$. *If* $E_k(a; f) = E_k(a; g)$, *we say that* f, g *share the value* a *with weight* k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

With the notion of weighted sharing of values Lahiri-Sarkar [13] improved the result of Zhang [14]. In [15] Zhang extended the result of Lahiri-Sarkar [13] and replaced the concept of value sharing by small function sharing.

In 2008, Zhang and Lü [12] obtained the following result.

Theorem A *Let* k, n *be the positive integers,* f *be a non-constant meromorphic function, and* $a(\not\equiv 0, \infty)$ *be a meromorphic function satisfying* $T(r, a) = o(T(r, f))$ *as* $r \to \infty$*. If* f^n *and* $f^{(k)}$ *share* a *IM and*

$$(2k+6)\Theta(\infty, f) + 4\Theta(0, f) + 2\delta_{2+k}(0, f) > 2k + 12 - n,$$

or f^n *and* $f^{(k)}$ *share* a *CM and*

$$(k+3)\Theta(\infty, f) + 2\Theta(0, f) + \delta_{2+k}(0, f) > k + 6 - n,$$

then $f^n = f^{(k)}$.

In the same paper, T. Zhang and W. Lü asked the following question:

Question 1 What will happen if f^n and $(f^{(k)})^m$ share a meromorphic function $a(\not\equiv 0, \infty)$ satisfying $T(r, a) = o(T(r, f))$ as $r \to \infty$?

S. S. Bhoosnurmath and Kabbur [3] proved:

Theorem B *Let* f *be a non-constant meromorphic function and* $a(\not\equiv 0, \infty)$ *be a meromorphic function satisfying* $T(r, a) = o(T(r, f))$ *as* $r \to \infty$*. Let* $P[f]$ *be a non-constant differential polynomial in* f*. If* f *and* $P[f]$ *share* a *IM and*

$$(2Q+6)\Theta(\infty, f) + (2 + 3\underline{d}(P))\delta(0, f) > 2Q + 2\underline{d}(P) + \overline{d}(P) + 7,$$

or if f *and* $P[f]$ *share* a *CM and*

$$3\Theta(\infty, f) + (\underline{d}(P) + 1)\delta(0, f) > 4,$$

then $f \equiv P[f]$.

Banerjee and Majumder [2] considered the weighted sharing of f^n and $(f^m)^{(k)}$ and proved the following result:

Theorem C *Let* f *be a non-constant meromorphic function,* $k, n, m \in N$ *and* l *be a non negative integer. Suppose* $a(\not\equiv 0, \infty)$ *be a meromorphic function satisfying* $T(r, a) = o(T(r, f))$ *as* $r \to \infty$ *such that* f^n *and* $(f^m)^{(k)}$ *share* (a, l)*. If* $l \geq 2$ *and*

$$(k+3)\Theta(\infty, f) + (k+4)\Theta(0, f) > 2k + 7 - n,$$

or $l = 1$ *and*

$$\left(k+\frac{7}{2}\right)\Theta(\infty, f) + \left(k+\frac{9}{2}\right)\Theta(0, f) > 2k + 8 - n,$$

or $l = 0$ *and*

$$(2k+6)\Theta(\infty, f) + (2k+7)\Theta(0, f) > 4k + 13 - n,$$

then $f \equiv (f^m)^{(k)}$.

In 2015, Kuldeep S. Charak and Banarasi Lal [16] proved the following result:

Theorem D *Let* f *be a non-constant meromorphic function,* n *be a positive integer and* $a(\not\equiv 0, \infty)$ *be a meromorphic function satisfying* $T(r, a) = o(T(r, f))$ *as* $r \to \infty$*. Let* $P[f]$ *be a non-constant differential polynomial in* f*. Suppose* f^n *and* $P[f]$ *share* (a, l) *such that any one of the following holds*:
(i) *when* $l \geq 2$ *and*

$$(Q+3)\Theta(\infty, f) + 2\Theta(0, f) + \overline{d}(P)\delta(0, f) > Q + 5 + 2\overline{d}(P) - \underline{d}(P) - n,$$

(ii) *when* $l = 1$ *and*

$$\left(Q+\frac{7}{2}\right)\Theta(\infty, f) + \frac{5}{2}\Theta(0, f) + \overline{d}(P)\delta(0, f) > Q + 6 + 2\overline{d}(P) - \underline{d}(P) - n,$$

(iii) *when* $l = 0$ *and*

$$(2Q+6)\Theta(\infty, f) + 4\Theta(0, f) + 2\overline{d}(P)\delta(0, f) > 2Q + 10 + 4\overline{d}(P) - 2\underline{d}(P) - n.$$

Then $f^n \equiv P[f]$.

Through the paper we shall assume the following notations. Let

$$\mathcal{P}(\omega) = a_{m+n}\omega^{m+n} + \ldots + a_n\omega^n + \ldots + a_0 = a_{n+m}\prod_{i=1}^{s}(\omega - \omega_{p_i})^{p_i}$$

where $a_j (j = 0, 1, 2, \ldots, n+m-1), a_{n+m} \neq 0$ and $\omega_{p_i} (i = 1, 2, \ldots, s)$ are distinct finite complex numbers and $2 \leq s \leq n+m$ and $p_1, p_2, \ldots, p_s, s \geq 2, n, m$ and k are all positive integers with $\sum_{i=1}^{s} p_i = n+m$. Also let $p > max_{p \neq p_i, i=1,\ldots,r}\{p_i\}, r = s-1$, where s and r are two positive integers.

Let

$$P(\omega_1) = a_{n+m}\prod_{i=1}^{s-1}(\omega_1+\omega_p-\omega_{p_i})^{p_i} = b_q\omega_1^q + b_{q-1}\omega_1^{q-1} + \dots + b_0,$$

where $a_{n+m} = b_q, \omega_1 = \omega - \omega_p, q = n+m-p$. Therefore, $\mathcal{P}(\omega) = \omega_1^p P(\omega_1)$. Next we assume

$$P(\omega_1) = b_q \prod_{i=1}^{r}(\omega_1 - \alpha_i)^{p_i},$$

where $\alpha_i = \omega_{p_i} - \omega_p, (i = 1, 2, \dots, r)$, be distinct zeros of $P(\omega_1)$.

In this paper we will prove one theorem which will improve and generalize Theorem D.

Theorem 1 *Let* $k(\geq 1)$,$n(\geq 1)$,$p(\geq 1)$ *and* $m(\geq 0)$ *be integers and* f *and* $f_1 = f - \omega_p$ *be two nonconstant meromorphic functions and* $M[f]$ *be a differential monomial of degree* d_M *and weight* Γ_M *and* k *is the highest derivative in* $M[f]$. *Let* $\mathcal{P}(z) = a_{m+n}z^{m+n} + \dots + a_n z^n + \dots + a_0$, $a_{m+n} \neq 0$, *be a polynomial in z of degree m+n such that* $\mathcal{P}(f) = f_1^p P(f_1)$. *Also let* $a(z)(\not\equiv 0, \infty)$ *be a small function with respect to* f. *Suppose* $\mathcal{P}(f) - a$ *and* $M[f] - a$ *share* $(0, l)$. *If* $l \geq 2$ *and*

$$(3+2\lambda)\Theta(\infty, f) + \mu_2\delta_{\mu_2^*}(\omega_p, f) + 2d_M\delta_{1+k}(0, f) > 2\Gamma_M + 2\mu_2 + 3 - p \quad (1)$$

or $l = 1$ *and*

$$\left(\frac{7}{2}+2\lambda\right)\Theta(\infty, f) + \frac{1}{2}\Theta(\omega_p, f) + \mu_2\delta_{\mu_2^*}(\omega_p, f) + 2d_M\delta_{1+k}(0, f) > 2\Gamma_M + \mu_2 + 4 + \frac{(m+n)-3p}{2} \quad (2)$$

or $l = 0$ *and*

$$(6+3\lambda)\Theta(\infty, f) + 2\Theta(\omega_p, f) + \mu_2\delta_{\mu_2^*}(\omega_p, f) + 3d_M\delta_{1+k}(0, f) > 3\Gamma_M + \mu_2 + 8 + 2(m+n) - 3p \quad (3)$$

then $\mathcal{P}(f) \equiv M[f]$.

Following example shows that in Theorem 1 $a(z) \not\equiv 0$ is essential.

Example 1 *Let us take* $f(z) = e^{Lz}$ *where* $L \neq 0, \pm 1$ *and* $\mathcal{P}(f) = f^3, M[f] = f^{(2)}$. *Then* $\mathcal{P}(f)$ *and* $M[f]$ *share* $a = 0(\text{or}, \infty)$. *Here* $m = 0, p = n = 1, \omega_p = 0, d_M = 1, \mu_2 = 1, \Gamma_M = 3$ *and* $\lambda = 2$. *Also* $\Theta(\infty; f) = 1 = \Theta(0; f)$ *and* $\delta_p(0; f) = 1, \forall q \in \mathbb{N}$. *Thus we see that the deficiency conditions stated in Theorem 1 are satisfied but* $\mathcal{P}(f) \not\equiv M[f]$.

The next example shows that the deficiency conditions stated in Theorem 1 are not necessary.

Example 2 *Let* $f(z) = C\cos z + D\sin z$, $CD \neq 0$. *Then* $\overline{N}(r, f) = S(r, f)$ *and*

$$\overline{N}(r, 0; f) = \overline{N}\left(r, \frac{C + iD}{C - iD}; e^{2iz}\right) \sim T(r, f).$$

Here $m = 0, p = n = 1, \omega_p = 0, d_M = 1, \mu_2 = 1, \Gamma_M = 4k + 1$ *and* $\lambda = 4k$. *Again* $\Theta(\infty, f) = 1$ *and* $\Theta(0, f) = \delta_p(0, f) = 0$. *Let* $m = 0$, *hence* $\mathcal{P}(f) = f$.

Therefore it is clear that $M[f] = f^{(4k)}$, *for* $k \in \mathbb{N}$ *and* $\mathcal{P}(f)$ *share* $a(z)$ *and the deficiency conditions in Theorem 1 are not satisfied, but* $\mathcal{P}(f) \equiv M$.

2 Lemmas

Lemma 1 [17] *For the differential monomial* $M[f]$,

$$N_p(r, 0; M[f]) \leq d_M N_{p+k}(r, 0; g) + \lambda \overline{N}(r, \infty, f) + S(r, f).$$

Lemma 2 [17] *Let* F *and* G *share* $(1, l)$. *Then*

$$\overline{N}_L(r, 1; F) \leq \frac{1}{l+1}\overline{N}(r, \infty; F) + \frac{1}{l+1}\overline{N}(r, 0; F) + S(r, F) \text{ if } l \geq 1,$$

and

$$\overline{N}_L(r, 1; F) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + S(r, F) \text{ if } l = 0.$$

Lemma 3 *Let* f *be a non-constant meromorphic function and* $a(z)$ *be a small function of* f. *Let us define* $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ *and* $G = \frac{M[f]}{a}$. *Then* $FG \not\equiv 1$.

Proof. *On contrary suppose* $FG \equiv 1$ *i.e*

$$f_1^p P(f_1) M[f] = a^2.$$

From above it is clear that the function f can't have any zero and poles. Therefore $\overline{N}(r, 0; f) = S(r, f) = \overline{N}(r, \infty; f)$. So by the First Fundamental Theorem and Lemma 1, we have

$$\begin{aligned}(m + n + d_M)T(r, f) &= T\left(r, \frac{a^2}{f_1^p P(f_1) f^{d_M}}\right) + S(r, f) \leq T\left(r, \frac{M[f]}{f^{d_M}}\right) + S(r, f)\\ &\leq m\left(r, \frac{M[f]}{f^{d_M}}\right) + N\left(r, \frac{M[f]}{f^{d_M}}\right) + S(r, f)\\ &\leq N\left(r, \frac{M[f]}{f^{d_M}}\right) + S(r, f)\end{aligned}$$

Then using Lemma 2 and from above inequality, we get

$$(m+n+d_M)T(r,f) \leq d_M N(r,0;f) + \lambda\overline{N}(r,f) + S(r,f) \leq S(r,f),$$

which is not possible.

Lemma 4 [17] *Let* f *be a non-constant meromorphic function and* $a(z)$ *be a small function of* f*. Let* $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ *and* $G = \frac{M[f]}{a}$ *such that* F *and* G *shares* $(1,\infty)$*. Then one of the following cases holds:*
1. $T(r) \leq N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_L(r,\infty;F) + \overline{N}_L(r,\infty;G) + S(r)$,
2. $F \equiv G$,
3. $FG \equiv 1$.
where $T(r) = max\{T(r,F), T(r,G)\}$ *and* $S(r) = o(T(r))$, $r \in I$, I *is a set of infinite linear measure of* $r \in \{0,\infty\}$.

3 Proof of the Theorem

Proof.

Let $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ and $G = \frac{M[f]}{a}$. Then $F-1 = \frac{f_1^p P(f_1)-a}{a}$ and $G-1 = \frac{M[f]-a}{a}$. Since $\mathcal{P}(f)$ and $M[f]$ share (a,l), it follows that F and G share $(1,l)$, except the zeros and poles of $a(z)$.

Define

$$\psi = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{4}$$

We consider the following cases:

Case 1. When $\psi \not\equiv 0$. Then from (4), we have $m(r,\psi) = S(r,f)$.
By the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned} T(r,F) + T(r,G) \leq 2\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G}\right) \\ + \overline{N}\left(r,\frac{1}{G-1}\right) - N_0\left(r,\frac{1}{F'}\right) - N_0\left(r,\frac{1}{G'}\right) + S(r,f), \end{aligned} \tag{5}$$

where $N_0\left(r,\frac{1}{F'}\right)$ denotes the counting function of the zeros of F' which are not the zeros of $F(F-1)$ and $N_0\left(r,\frac{1}{G'}\right)$ denotes the counting function of the zeros of G' which are not the zeros of $G(G-1)$.

Subcase 1.1. When $l \geq 1$. Then from (4), we have,

$$\begin{aligned} N_E^{1)}\left(r,\frac{1}{F-1}\right) &\leq N\left(r,\frac{1}{\psi}\right)+S(r,f)\leq T(r,\psi)+S(r,f)=N(r,\psi)+S(r,f)\\ &\leq \overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\ &\quad+\overline{N}_L\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f), \end{aligned}$$

and so

$$\begin{aligned} &\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)=N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)\\ &\quad+\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,f)\\ &\leq \overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)\\ &\quad+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f). \end{aligned} \tag{6}$$

For $l \geq 2$, we have

$$\begin{aligned} &2\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\ &\leq N\left(r,\frac{1}{G-1}\right)+S(r,f). \end{aligned}$$

Thus from (6), we obtain

$$\begin{aligned} &\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\leq \overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)\\ &+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+N\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f). \end{aligned} \tag{7}$$

Now from Lemma 1, (5) and (7) we obtain

$$\begin{aligned} T(r,F) &\le 3\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + S(r,f) \\ &\le 3\overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{G}\right) + S(r,f) \\ &\le 3\overline{N}(r,f) + \mu_2 N_{\mu_2^*}(r,\omega_p;f) + 2d_M N_{1+k}\left(r,\frac{1}{f}\right) + 2\lambda\overline{N}(r,f) + S(r,f) \end{aligned}$$

$$\begin{aligned} (n+m)T(r,f) \le (3+2\lambda)\overline{N}(r,f) + \mu_2 N_{\mu_2^*}(r,\omega_p;f) + (m+n-p)T(r,f) \\ +2d_M N_{1+k}\left(r,\frac{1}{f}\right) + S(r,f) \end{aligned}$$

$$\begin{aligned} \{(3+2\lambda)\Theta(\infty,f) + \mu_2\delta_{\mu_2^*}(r,\omega_p;f) + 2d_M\delta_{1+k}(0,f)\}T(r,f) \\ \le (3+2\lambda+2\mu_1+m+n-2p+2d_M)T(r,f) + S(r,f). \end{aligned}$$

$$\begin{aligned} \{(3+2\lambda)\Theta(\infty,f) + \mu_2\delta_{\mu_2^*}(0,f) + 2d_M\delta_{1+k}(0,f) - \epsilon\}T(r,f) \\ \le (2\Gamma_M + 3 + 2\lambda + 2\mu_2 - p)T(r,f) + S(r,f). \end{aligned}$$

which violates (1).
Next, consider the case when $l = 1$.
First note that

$$\overline{N}_L(r,\frac{1}{F-1}) \le \frac{1}{2}N(r,\frac{1}{F'}|F\neq 0) \le \frac{1}{2}\overline{N}(r,F) + \frac{1}{2}\overline{N}(r,\frac{1}{F}), \tag{8}$$

when $N\left(r,\frac{1}{F'}|F\neq 0\right)$ denotes the zeros of F', that are not the zeros of F.
From (4) and (8), we have

$$\begin{aligned} 2\overline{N}_L\left(r,\frac{1}{F-1}\right) + 2\overline{N}_L\left(r,\frac{1}{F-1}\right) + \overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) \\ \le N\left(r,\frac{1}{G-1}\right) + \overline{N}_L\left(r,\frac{1}{F-1}\right) + S(r,f) \\ \le N\left(r,\frac{1}{G-1}\right) + \frac{1}{2}\overline{N}(r,F) + \frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right) + S(r,f) \end{aligned} \tag{9}$$

Thus, from (5) and (9) , we have

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)&\le\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\&+\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+T(r,G)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f).\end{aligned}\tag{10}$$

From Lemma 1, (5) and (10) we obtain

$$\begin{aligned}T(r,F)&\le 3\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\&\quad+\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+S(r,f)\\&\le\frac{7}{2}\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{F}\right)+2\overline{N}\left(r,\frac{1}{G}\right)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+S(r,f)\\&\le\frac{7}{2}\overline{N}(r,f)+\mu_2N_{\mu_2^*}(r,\omega_p;f)+(m+n-p)T(r,f)+2d_MN_{1+k}\left(r,\frac{1}{f}\right)\\&\quad+2\lambda\overline{N}(r,f)+\frac{1}{2}\{\overline{N}(r,\omega_p;f)+(m+n-p)T(r,f)+S(r,f)\}\end{aligned}$$

$$\begin{aligned}(m+n)T(r,f)\le\left(\frac{7}{2}+2\lambda\right)(1-\Theta(\infty,f))+\mu_2(1-\delta_{\mu_2^*}(\omega_p,f))+\frac{3}{2}(m+n-p)\\+2d_M(1-\delta_{1+k}(0,f))+\frac{1}{2}(1-\Theta(\omega_p,f))+S(r,f).\end{aligned}$$

$$\begin{aligned}&\left\{\left(\frac{7}{2}+2\lambda\right)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(\omega_p,f))+2d_M\delta_{1+k}(0,f)+\frac{1}{2}\Theta(\omega_p;f)\right\}\\&\le\left(\frac{7}{2}+2\lambda+\mu_2+\frac{3}{2}(m+n-p)+2d_M+\frac{1}{2}-m-n+\epsilon\right)T(r,f)+S(r,f)\\&\le(2\Gamma_M+4+\mu_2+\frac{1}{2}m+\frac{1}{2}n-\frac{3}{2}p)T(r,f)+S(r,f)\end{aligned}$$

which violates (2).

Subcase 1.2. When $l=0$. Then, we have

$$N_E^{1)}\left(r,\frac{1}{F-1}\right)=N_E^{1)}\left(r,\frac{1}{G-1}\right)+S(r,f),$$
$$\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)=\overline{N}_E^{(2}\left(r,\frac{1}{G-1}\right)+S(r,f),$$

and also from (4), we have

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\le N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)\\
&\quad+\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,f)\\
&\le N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+N\left(r,\frac{1}{G-1}\right)+S(r,f) \qquad (11)\\
&\le \overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)\\
&+N\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f)
\end{aligned}$$

From Lemma 2, (5) and (9), we obtain

$$\begin{aligned}
T(r,F)&\le 3\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\
&\quad+2\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+S(r,f)\\
&\le 6\overline{N}(r,f)+\mu_2 N_{\mu_2^*}(r,0;f)+(m+n-p)T(r,f)+3(d_M N_{1+k}(r,0;f)\\
&\quad+\lambda\overline{N}(r,f))+2\{\overline{N}(r,\omega_p;f)+(m+n-p)T(r,f)\}+S(r,f)
\end{aligned}$$

$$\begin{aligned}
(m+n)T(r,f)&\le(6+3\lambda)(1-\Theta(\infty,f))+\mu_2(1-\delta_{\mu_2^*}(r,f))+3(m+n-p)\\
&\quad+3d_M(1-\delta_{1+k}(0,f))+2(1-\Theta(\omega_p,f))+S(r,f).
\end{aligned}$$

$$\begin{aligned}
&\{(6+3\lambda)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(r,f)+3d_M\delta_{1+k}(0,f)+2\Theta(\omega_p,f)-\epsilon\}T(r,f)\\
&\quad\le(6+3\lambda+\mu_2+3m+3n-3p+3d_M+2-m-n)T(r,f)+S(r,f)\\
&\quad\le(3\Gamma_M+\mu_2+2m+2n-3p+8-\epsilon)T(r,f)+S(r,f)
\end{aligned}$$

which violates (3).

Case 2. Let $H\equiv 0$.
On Integration we get

$$\frac{1}{G-1}\equiv\frac{A}{F-1}+B,$$

where $A(\neq 0), B$ are complex constants.
It is clear that F and G share $(1,\infty)$. Also by construction of F and G we see that F and G share $(\infty,0)$ also.

So using Lemma 1 and condition (2), we obtain

$$\begin{aligned}
&N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_L(r,\infty;F) \\
&\quad + \overline{N}_L(r,\infty;G) + S(r) \leq 2\overline{N}(r,0;F) + 2\overline{N}(r,0;G) + 3\overline{N}(r,\infty;f) + S(r) \\
&\leq 2(\overline{N}(r,\omega_p;f) + (m+n-p)T(r,f)) + 2(d_M N_{1+k}\left(r,\frac{1}{f}\right) + \lambda\overline{N}(r,f)) \\
&\quad + 3\overline{N}(r,f) + S(r) \leq 2(1-\Theta(\omega_p,f)) + 2d_M(1-\delta_{1+k}(0,f)) \\
&\quad + (3+2\lambda)(1-\Theta(\infty,f)) + S(r) + (m+n-p)T(r,f) \\
&\leq (3+2\lambda+2d_M+2+m+n-p) - (3+2\lambda+2d_M+2-p)T(r,f) + S(r) \\
&\leq (m+n)T(r,f) + S(r) < T(r,F) + S(r).
\end{aligned} \tag{12}$$

Hence inequality (1) of Lemma 4 does not hold. Again in view of Lemma 3, we get $FG \not\equiv 1$. Therefore $F \equiv G$ i.e., $\mathcal{P}(f) \equiv M[f]$. □

Acknowledgement

The author (VH) is greatful to the University Grants Commission(UGC), New Delhi, India for supporting her research work by providing her with a Maulana Azad National Fellowship(MANF).

References

[1] Banerjee, Abhijit, Meromorphic functions sharing one value. Int. *J. Math. Math. Sci.*, **22** (2005), 3587–3598.

[2] A. Banerjee, S. Majumder, Some uniqueness results related to meromorphic function that share a small function with its derivative. *Math. Rep.* (Bucur.), **16** (66) (2014), no. 1, 95–111.

[3] Subhas S. Bhoosnurmath, Smita R. Kabbur, On entire and meromorphic functions that share one small function with their differential polynomial, *Int. J. Anal.*, (2013), Art. ID 926340, 8 pp.

[4] S. S. Bhoosnurmath, Anupama J. Patil, On the growth and value distribution of meromorphic functions and their differential polynomials, *J. Indian Math. Soc.* (N.S.), **74** (2007), no. 3-4, 167–184 (2008).

[5] W. K. Hayman, Meromorphic functions, Oxford Mathematical Monographs Clarendon Press, Oxford 1964 xiv+191 pp.

[6] H. -X. Yi, Uniqueness of meromorphic functions and a question of C. C. Yang. *Complex Var. Theory and Appl.*, **14** (1990), no. 1–4, 169–176.

[7] H. -X. Yi, Uniqueness theorems for meromorphic functions whose nth derivatives share the same 1-points, C*omplex Var. Theory and Appl.*, **34** (1997), no. 4, 421–436.

[8] H. Huang, B. Huang, Uniqueness of meromorphic functions concerning differential monomials, *Appl. Math.* (Irvine), **2** (2011), no. 2, 230–235.

[9] N. Li, L.-Z. Yang, , Meromorphic function that shares one small function with its differential polynomial, *Kyungpook Math. J.*, **50** (2010), no. 3, 447–454.

[10] E. Mues, N. Steinmetz, Meromorphe Funktionen, die mit ihrer Ableitung Werte teilen. (German) *Manuscripta Math.*, **29** (1979), no. 2-4, 195–206.

[11] C.-C. Yang, H.-X. Yi, *Uniqueness theory of meromorphic functions. Mathematics and its Applications*, 557. Kluwer Academic Publishers Group, Dordrecht, 2003. viii+569 pp. ISBN: 1-4020-1448-1

[12] T. Zhang, W. Lü, Notes on a meromorphic function sharing one small function with its derivative, *Complex Var. Elliptic Equ.*, **53** (2008), no. 9, 857–867.

[13] I. Lahiri, A. Sarkar, Uniqueness of a meromorphic function and its derivative, *JIPAM. J. Inequal. Pure Appl. Math.*, **5** (2004), no. 1, Article 20, 9 pp. (electronic).

[14] Zhang, Qing Cai, The uniqueness of meromorphic functions with their derivatives, *Kodai Math. J.*, **21** (1998), no. 2, 179–184.

[15] Q. C. Zhang, Meromorphic function that shares one small function with its derivative, *J. Inequal. Pure. Appl. Math.*, **6** (2015),1–13.

[16] K. S. Charak, Banarsi Lal, Uniqueness of f^n and $P[f]$, arXiv:1501.05092v1.[math.CV]21 Jan 2015.

[17] A. Banerjee, B. Chakraborty, Further investigations on a question of Zhang and Lü, *Ann. Univ. Paedagog. Crac. Stud. Math.*, **14** (2015), 105–119.

[18] A. Banerjee, S. Majumder, Sujoy, On the uniqueness of a power of a meromorphic function sharing a small function with the power of its derivative, *Comment. Math. Univ. Carolin.*, **51** (2010), no. 4, 565–576.

[19] I. Lahiri, Weighted value sharing and uniqueness of meromorphic functions, *Complex Var. Theory and Appl.*, **46** (2001), no. 3, 241–253.