



# Uniqueness and weighted value sharing of differential polynomials of meromorphic functions

Abhijit Banerjee

Department of Mathematics  
West Bengal State University  
Barasat, North 24 Prgs.-700126  
West Bengal, India.  
email: [abanerjee\\_kal@yahoo.co.in](mailto:abanerjee_kal@yahoo.co.in)

Pulak Sahoo

Department of Mathematics  
Netaji Subhas Open University  
1-Woodburn Park, Kolkata-700020  
West Bengal, India.  
email: [sahoopulak1@gmail.com](mailto:sahoopulak1@gmail.com)

**Abstract.** In the paper, with the aid of weighted sharing we investigate the uniqueness problems of meromorphic functions concerning differential polynomials that share one value and prove three uniqueness results which rectify, improve and supplement some recent results of [3].

## 1 Introduction

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [7, 18, 21]. Let  $E$  denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function  $f$ , we denote by  $T(r, f)$  the Nevanlinna characteristic of  $f$  and by  $S(r, f)$  any quantity satisfying  $S(r, f) = o\{T(r, f)\}(r \rightarrow \infty, r \notin E)$ .

Let  $f$  and  $g$  be two non-constant meromorphic functions. We say that  $f$  and  $g$  share the value  $a$  CM (counting multiplicities), if  $f - a$  and  $g - a$  have the same zeros with the same multiplicities. Similarly, we say that  $f$  and  $g$  share

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the value  $a$  IM, provided that  $f - a$  and  $g - a$  have the same zeros ignoring multiplicities. Throughout this paper, we need the following definition.

$$\Theta(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{\overline{N}(r, a; f)}{T(r, f)},$$

where  $a$  is a value in the extended complex plane.

In the recent past a number of authors worked on the uniqueness problem of meromorphic functions when differential polynomials generated by them share certain values (cf. [1, 2, 4, 5, 8, 11]). In [8] following question was asked: What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 CM ?

Since then the progress to investigate the uniqueness of meromorphic functions which are the generating functions of different types of nonlinear differential polynomials is remarkable and continuous efforts are being put in to relax the hypothesis of the results. (see [1], [4], [5], [14], [15]). In 1997, Yang and Hua [17] proved the following result.

**Theorem 1** *Let  $f$  and  $g$  be two non-constant meromorphic functions,  $n(\geq 11)$  an integer and  $a \in \mathbb{C} - \{0\}$ . If  $f^n f'$  and  $g^n g'$  share the value  $a$  CM, then either  $f = tg$  for some  $(n+1)$ th root of unity 1 or  $f(z) = c_1 e^{cz}$ ,  $g(z) = c_2 e^{-cz}$ , where  $c, c_1, c_2$  are constants satisfying  $(c_1 c_2)^{n+1} c^2 = -a^2$ .*

In 2004 Lin-Yi [15] proved the following results.

**Theorem 2** *Let  $f$  and  $g$  be two non-constant meromorphic functions satisfying  $\Theta(\infty, f) > 2/(n+1)$ ,  $n \geq 12$  an integer. If  $f^n(f-1)f'$  and  $g^n(g-1)g'$  share the value 1 CM, then  $f \equiv g$ .*

**Theorem 3** *Let  $f$  and  $g$  be two non-constant meromorphic functions,  $n \geq 13$  an integer. If  $f^n(f-1)^2 f'$  and  $g^n(g-1)^2 g'$  share the value 1 CM, then  $f \equiv g$ .*

Also in [4] Fang-Fang proved the following theorem.

**Theorem 4** *Let  $f$  and  $g$  be two non-constant meromorphic functions and  $n(\geq 28)$  be an integer. If  $f^n(f-1)^2 f'$  and  $g^n(g-1)^2 g'$  share the value 1 IM, then  $f \equiv g$ .*

Recently, in [3] Dyavanal proved the following results, which to the knowledge of the authors probably are the first approach in which in order to consider the value sharing of two differential polynomials the multiplicities of zeros and poles of  $f$  and  $g$  are taken into account.

**Theorem 5 (Theorem 1.1, [3])** *Let  $f$  and  $g$  be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least  $s$ , where  $s$  is a positive integer. Let  $n \geq 2$  be an integer satisfying  $(n+1)s \geq 12$ . If  $f^n f'$  and  $g^n g'$  share the value 1 CM, then either  $f = dg$  for some  $(n+1)$ -th root of unity  $d$  or  $f(z) = c_1 e^{cz}$ ,  $g(z) = c_2 e^{-cz}$ , where  $c, c_1, c_2$  are constants satisfying  $(c_1 c_2)^{n+1} c^2 = -1$ .*

**Theorem 6 (Theorem 1.2, [3])** *Let  $f$  and  $g$  be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least  $s$ , where  $s$  is a positive integer and  $\Theta(\infty, f) > 2/(n+1)$ . Let  $n \geq 4$  be an integer satisfying  $(n+1)s \geq 12$ . If  $f^n(f-1)f'$  and  $g^n(g-1)g'$  share the value 1 CM, then  $f \equiv g$ .*

**Theorem 7 (Theorem 1.3, [3])** *Let  $f$  and  $g$  be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least  $s$ , where  $s$  is a positive integer. Let  $n \geq 3$  be an integer satisfying  $(n+1)s \geq 12$ . If  $f^n(f-1)^2 f'$  and  $g^n(g-1)^2 g'$  share the value 1 CM, then  $f \equiv g$ .*

The results are new and seems fine. However in page 7, in the proof of Theorem 1.2 [3] there is a serious lacuna when a counting function is being elaborated and then restricted in terms of Nevanlinna's characteristic function.

Actually in Page 7, line 8 onwards from bottom should be

$$\begin{aligned} \overline{N}\left(r, \frac{1}{f}\right) &= \overline{N}\left[r, \frac{1}{f^{n+1}\left(f - \frac{n+2}{n+1}\right)}\right] \leq \frac{1}{s(n+1)} N\left(r, \frac{1}{f^{n+1}}\right) \\ &+ \overline{N}\left(r, \frac{1}{f - \frac{n+2}{n+1}}\right) \not\leq \frac{1}{s(n+1)} N\left(r, \frac{1}{f}\right), \end{aligned}$$

since nowhere in the paper it has been assumed that the zeros of  $f - \frac{n+2}{n+1}$  are of multiplicities  $s(n+1)$ . Since the counting function just mentioned above plays a vital role in the proofs of *Theorems 1.2, 1.3 and 1.5* in [3], the validity of the three theorems namely *Theorems 1.2, 1.3 and 1.5* in [3] cease to hold.

So it would be quite natural to investigate the accurate forms of the above theorems and at the same time to combine all the theorems in [3] to a single one. In this paper we are taking up these problems. In fact, we will not only rectify the above three theorems but also improve and supplement all the theorems of [3] by relaxing the nature of sharing the values with the aid of the notion of weighted sharing of values defined as follows.

**Definition 1** Let  $k$  be a nonnegative integer or infinity. For  $a \in \mathbb{C} \cup \{\infty\}$  we denote by  $E_k(a; f)$  the set of all  $a$ -points of  $f$  where an  $a$ -point of multiplicity  $m$  is counted  $m$  times if  $m \leq k$  and  $k+1$  times if  $m > k$ . If  $E_k(a; f) = E_k(a; g)$ , we say that  $f, g$  share the value  $a$  with weight  $k$ .

The definition implies that if  $f, g$  share a value  $a$  with weight  $k$ , then  $z_0$  is an  $a$ -point of  $f$  with multiplicity  $m(\leq k)$  if and only if it is an  $a$ -point of  $g$  with multiplicity  $m(\leq k)$  and  $z_0$  is an  $a$ -point of  $f$  with multiplicity  $m(> k)$  if and only if it is an  $a$ -point of  $g$  with multiplicity  $n(> k)$ , where  $m$  is not necessarily equal to  $n$ .

We write  $f, g$  share  $(a, k)$  to mean that  $f, g$  share the value  $a$  with weight  $k$ . Clearly if  $f, g$  share  $(a, k)$  then  $f, g$  share  $(a, p)$  for any integer  $p$ ,  $0 \leq p < k$ . Also we note that  $f, g$  share a value  $a$  IM or CM if and only if  $f, g$  share  $(a, 0)$  or  $(a, \infty)$  respectively.

We now state the main results of the paper.

**Theorem 8** Let  $f$  and  $g$  be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least  $s$ , where  $s$  is a positive integer. Let  $f^n(f-1)^mf'$  and  $g^n(g-1)^mg'$  share  $(b, 2)$  where  $m \geq 0$ ,  $n > \max\{m+1+2m/s, m+1+9/s\}$  are integers and  $b(\neq 0)$  is a constant. Then each of the following holds:

(i) If  $m = 0$ , then either  $f = tg$  for some  $(n+1)$ -th root of unity  $t$  or  $f(z) = c_1 e^{cz}$ ,  $g(z) = c_2 e^{-cz}$ , where  $c, c_1, c_2$  are constants satisfying  $(c_1 c_2)^{n+1} c^2 = -b^2$ .

(ii) If  $m = 1$  and  $\Theta(\infty, f) + \Theta(\infty, g) > 4/(n+1)$  or  $m = 2$ , then  $f \equiv g$ .

(iii) If  $m \geq 3$ , then

$$f^{n+1} \sum_{i=0}^m {}^m C_i \frac{(-1)^i}{n+m-i+1} f^{m-i} \equiv g^{n+1} \sum_{i=0}^m {}^m C_i \frac{(-1)^i}{n+m-i+1} g^{m-i}.$$

**Remark 1** Putting  $s = 1$  in the above theorem we get the rectified, improved as well as generalised form of all the theorems in [3].

**Theorem 9** Let  $f$  and  $g$  be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least  $s$ , where  $s$  is a positive integer. Let  $f^n(f-1)^mf'$  and  $g^n(g-1)^mg'$  share  $(b, 1)$  where  $m \geq 0$ ,  $n > \max\{m+1+2m/s, m+2+21/2s\}$  are integers and  $b(\neq 0)$  is a constant. Then the conclusions (i)-(iii) of Theorem 8 hold.

**Theorem 10** Let  $f$  and  $g$  be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least  $s$ , where  $s$  is a positive integer. Let

$f^n(f-1)^mf'$  and  $g^n(g-1)^mg'$  share  $(b, 0)$  where  $m \geq 0$ ,  $n > \max\{m+1+2m/s, m+7+18/s\}$  are integers and  $b(\neq 0)$  is a constant. Then the conclusions (i)-(iii) of Theorem 8 hold.

Though we use the standard notations and definitions of the value distribution theory available in [7], we explain the following definition and notation which is used in the paper.

**Definition 2** [13] Let  $p$  be a positive integer or infinity. We denote by  $N_p(r, \alpha; f)$  the counting function of  $\alpha$ -points of  $f$ , where an  $\alpha$ -point of multiplicity  $m$  is counted  $m$  times if  $m \leq p$  and  $p$  times if  $m > p$ . Then

$$N_p(r, \alpha; f) = \overline{N}(r, \alpha; f) + \overline{N}(r, \alpha; f | \geq 2) + \dots + \overline{N}(r, \alpha; f | \geq p).$$

## 2 Lemmas

In this section we present some lemmas which will be needed to prove the theorem.

**Lemma 1** [16] Let  $f$  be a non-constant meromorphic function and  $P(f) = a_0 + a_1f + a_2f^2 + \dots + a_nf^n$ , where  $a_0, a_1, a_2, \dots, a_n(\neq 0)$  are constants. Then

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

**Lemma 2** [19] Let  $f$  be a non-constant meromorphic function. Then

$$N\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N(r, 0; f) + S(r, f).$$

**Lemma 3** [22] Let  $f$  be a non-constant meromorphic function and  $p, k$  be a positive integers. Then

$$N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f).$$

**Lemma 4** [9] Let  $f$  and  $g$  be two non-constant meromorphic functions sharing  $(1, 2)$ . Then one of the following cases holds:

- (i)  $T(r) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r)$ ,
- (ii)  $f \equiv g$ ,
- (iii)  $fg \equiv 1$ .

**Lemma 5** [1] *Let  $f$  and  $g$  be two non-constant meromorphic functions sharing  $(1, m)$  and*

$$\frac{f''}{f'} - \frac{2f'}{f-1} \neq \frac{g''}{g'} - \frac{2g'}{g-1}.$$

*Now the following hold:*

- (i) *if  $m = 1$  then  $T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + \frac{1}{2}\overline{N}(r, 0; f) + \frac{1}{2}\overline{N}(r, \infty; f) + S(r, f) + S(r, g)$ ;*  
(ii) *if  $m = 0$  then  $T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + 2\overline{N}(r, 0; f) + \overline{N}(r, 0; g) + 2\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, f) + S(r, g)$ .*

**Lemma 6** [20] *Let  $f$  and  $g$  be two non-constant meromorphic functions. If*

$$\frac{f''}{f'} - \frac{2f'}{f-1} \equiv \frac{g''}{g'} - \frac{2g'}{g-1}$$

*and*

$$\limsup_{r \rightarrow \infty, r \notin E} \frac{\overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)}{T(r)} < 1$$

*then either  $f \equiv g$  or  $fg \equiv 1$ , where  $T(r)$  is the maximum of  $T(r, f)$  and  $T(r, g)$ .*

**Lemma 7** *Let  $f$  and  $g$  be two non-constant meromorphic functions whose zeros and poles are of multiplicities at least  $s$ , where  $s$  is a positive integer. Let  $n$  and  $m$  are positive integers such that  $n > m + 1 + 2m/s$ . Then*

$$f^n(f-1)^mf'g^n(g-1)^mg' \neq b^2,$$

*where  $b$  is a nonzero constant.*

**Proof.** We suppose that

$$f^n(f-1)^mf'g^n(g-1)^mg' \equiv b^2. \quad (1)$$

Let  $z_0$  be a zero of  $f$  with multiplicity  $p_0 (\geq s)$ . Then  $z_0$  is a pole of  $g$  with multiplicity  $q_0 (\geq s)$ , say. From (1) we obtain

$$np_0 + p_0 - 1 = (n + m + 1)q_0 + 1$$

and so

$$(n + 1)(p_0 - q_0) = mq_0 + 2. \quad (2)$$

From (2) we get  $q_0 \geq \frac{n-1}{m}$  and so again from (2) we obtain

$$p_0 \geq q_0 + 1 \geq \frac{n+m-1}{m}.$$

Let  $z_1$  be a zero of  $f-1$  with multiplicity  $p_1$ . Then  $z_1$  is a pole of  $g$  with multiplicity  $q_1(\geq s)$ , say. So from (1) we get

$$(m+1)p_1 - 1 = (n+m+1)q_1 + 1$$

which gives

$$p_1 \geq \frac{(n+m+1)s+2}{m+1}.$$

Since a pole of  $f$  is either a zero of  $g^n(g-1)^m$  or a zero of  $g'$ , we have

$$\begin{aligned} \overline{N}(r, \infty; f) &\leq \overline{N}(r, 0; g) + \overline{N}(r, 1; g) + \overline{N}_0(r, 0; g') \\ &\leq \frac{m}{n+m-1} N(r, 0; g) + \frac{m+1}{(n+m+1)s+2} N(r, 1; g) \\ &\quad + \overline{N}_0(r, 0; g') \\ &\leq \left( \frac{m}{n+m-1} + \frac{m+1}{(n+m+1)s+2} \right) T(r, g) + \overline{N}_0(r, 0; g'), \end{aligned}$$

where  $\overline{N}_0(r, 0; g')$  denotes the reduced counting function of those zeros of  $g'$  which are not the zeros of  $g(g-1)$ .

Then by the second fundamental theorem of Nevanlinna we get

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + \overline{N}(r, 1; f) - \overline{N}_0(r, 0; f') + S(r, f) \\ &\leq \left( \frac{m}{n+m-1} + \frac{m+1}{(n+m+1)s+2} \right) \{T(r, f) + T(r, g)\} \\ &\quad + \overline{N}_0(r, 0; g') - \overline{N}_0(r, 0; f') + S(r, f). \end{aligned} \quad (3)$$

Similarly, we get

$$\begin{aligned} T(r, g) &\leq \left( \frac{m}{n+m-1} + \frac{m+1}{(n+m+1)s+2} \right) \{T(r, f) + T(r, g)\} \\ &\quad + \overline{N}_0(r, 0; f') - \overline{N}_0(r, 0; g') + S(r, g). \end{aligned} \quad (4)$$

Adding (3) and (4) we obtain

$$\left( 1 - \frac{2m}{n+m-1} - \frac{2(m+1)}{(n+m+1)s+2} \right) \{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

which leads to a contradiction as  $n > m+1+2m/s$ . This proves the lemma.  $\square$

**Lemma 8** Let  $f$  and  $g$  be two non-constant entire functions and  $n$  be a positive integer. If  $f^n f' g^n g' = b^2$ , where  $b$  is a nonzero constant, then  $f(z) = c_1 e^{cz}$ ,  $g(z) = c_2 e^{-cz}$ , where  $c, c_1, c_2$  are constants satisfying  $(c_1 c_2)^{n+1} c^2 = -b^2$ .

**Proof.** We omit the proof since it can be proved in the line of the proof of Theorem 3 in [17].  $\square$

**Lemma 9** Let  $f$  and  $g$  be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least  $s$ , where  $s$  is a positive integer and

$$F = f^{n+1} \left( \sum_{i=0}^m m C_i \frac{(-1)^i}{n+m-i+1} f^{m-i} \right);$$

$$G = g^{n+1} \left( \sum_{i=0}^m m C_i \frac{(-1)^i}{n+m-i+1} g^{m-i} \right).$$

Further let  $F_0 = \frac{F'}{b}$  and  $G_0 = \frac{G'}{b}$ , where  $b (\neq 0)$  is a constant. Then  $S(r, F_0)$  and  $S(r, G_0)$  are replaceable by  $S(r, f)$  and  $S(r, g)$  respectively.

**Proof.** By Lemma 1 we get

$$\begin{aligned} T(r, F_0) &\leq T(r, F') + S(r, f) \\ &\leq 2T(r, F) + S(r, f) \\ &= 2(n+m+1)T(r, f) + S(r, f) \end{aligned}$$

and similarly

$$T(r, G_0) \leq 2(n+m+1)T(r, g) + S(r, g).$$

This proves the lemma.  $\square$

**Lemma 10** Let  $F, G, F_0$  and  $G_0$  be defined as in Lemma 9. We define  $F = f^{n+1} F_1$  and  $G = g^{n+1} G_1$  where

$$F_1 = \sum_{i=0}^m m C_i \frac{(-1)^i}{n+m-i+1} f^{m-i} \text{ and } G_1 = \sum_{i=0}^m m C_i \frac{(-1)^i}{n+m-i+1} g^{m-i}.$$

Then

- (i)  $T(r, F) \leq T(r, F_0) + N(r, 0; f) + N(r, 0; F_1) - mN(r, 1; f) - N(r, 0; f') + S(r, f),$
- (ii)  $T(r, G) \leq T(r, G_0) + N(r, 0; g) + N(r, 0; G_1) - mN(r, 1; g) - N(r, 0; g') + S(r, g).$



**Proof.** We prove (i) only as the proof of (ii) is similar. By Nevanlinna's first fundamental theorem and Lemma 1 we get

$$\begin{aligned}
 T(r, F) &= T\left(r, \frac{1}{F}\right) + O(1) \\
 &= N(r, 0; F) + m\left(r, \frac{1}{F}\right) + O(1) \\
 &\leq N(r, 0; F) + m\left(r, \frac{F_0}{F}\right) + m(r, 0; F_0) + O(1) \\
 &= N(r, 0; F) + T(r, F_0) - N(r, 0; F_0) + S(r, F) \\
 &= T(r, F_0) + N(r, 0; f) + N(r, 0; F_1) - mN(r, 1; f) \\
 &\quad - N(r, 0; f') + S(r, f).
 \end{aligned}$$

This proves the lemma.  $\square$

**Lemma 11** *Let  $F$  and  $G$  be defined as in Lemma 9, where  $m(\geq 0)$  and  $n(\geq m + 3/s)$  are positive integers. Then  $F' \equiv G'$  implies  $F \equiv G$ .*

**Proof.** Let  $F' \equiv G'$ . Then  $F \equiv G + C$ , where  $C$  is a constant. If possible, we suppose that  $C \neq 0$ . Then by the second fundamental theorem of Nevanlinna we get

$$\begin{aligned}
 T(r, F) &\leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, C; F) + S(r, F) \\
 &\leq \overline{N}(r, 0; f) + \overline{N}(r, 0; F_1) + \overline{N}(r, \infty; f) + \overline{N}(r, 0; g) \\
 &\quad + \overline{N}(r, 0; G_1) + S(r, f) \\
 &\leq \frac{1}{s}N(r, 0; f) + \overline{N}(r, 0; F_1) + \frac{1}{s}N(r, \infty; f) + \frac{1}{s}N(r, 0; g) \\
 &\quad + \overline{N}(r, 0; G_1) + S(r, f) \\
 &\leq (m + 2/s)T(r, f) + (m + 1/s)T(r, g) + S(r, f),
 \end{aligned}$$

where  $F_1$  and  $G_1$  are defined as in Lemma 9. So by Lemma 1 we have

$$(n + 1 - 2/s)T(r, f) \leq (m + 1/s)T(r, g) + S(r, f). \quad (5)$$

Similarly, it can be shown that

$$(n + 1 - 2/s)T(r, g) \leq (m + 1/s)T(r, f) + S(r, g). \quad (6)$$

Adding (5) and (6) we obtain

$$(n - m + 1 - 3/s)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

which is a contradiction. Therefore  $C = 0$  and the lemma follows.  $\square$

**Lemma 12** *Let  $f$  and  $g$  be two non-constant meromorphic functions such that*

$$\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n+1},$$

*where  $n(\geq 2)$  is an integer. Then*

$$f^{n+1}(af+b) \equiv g^{n+1}(ag+b)$$

*implies  $f \equiv g$ , where  $a, b$  are two nonzero constants.*

**Proof.** We omit the proof since it can be carried out in the line of Lemma 6 [12].  $\square$

**Lemma 13** [6] *Let*

$$Q(w) = (n-1)^2(w^n-1)(w^{n-2}-1) - n(n-2)(w^{n-1}-1)^2,$$

*then*

$$Q(w) = (w-1)^4(w-\nu_1)(w-\nu_2)\dots(w-\nu_{2n-6}),$$

*where  $\nu_j \in \mathbb{C} \setminus \{0, 1\}$  ( $j = 1, 2, \dots, 2n-6$ ), which are distinct respectively.*

### 3 Proof of the Theorem

**Proof of Theorem 8.** Let  $F, G, F_0$  and  $G_0$  be defined as in Lemma 9. Since  $F_0$  and  $G_0$  share  $(1, 2)$ , one of the possibilities of Lemma 4 holds. We suppose that

$$\begin{aligned} T_0(r) \leq & N_2(r, 0; F_0) + N_2(r, 0; G_0) + N_2(r, \infty; F_0) + N_2(r, \infty; G_0) \\ & + S(r, F_0) + S(r, G_0), \end{aligned} \quad (7)$$

where  $T_0(r) = \max\{T(r, F_0), T(r, G_0)\}$ . Now by Lemma 2, Lemma 9 and Lemma 10 we get from (7)

$$\begin{aligned}
 T(r, F) &\leq T(r, F_0) + N(r, 0; f) + N(r, 0; F_1) - mN(r, 1; f) \\
 &\quad - N(r, 0; f') + S(r, f) \\
 &\leq N_2(r, 0; F_0) + N_2(r, 0; G_0) + N_2(r, \infty; F_0) + N_2(r, \infty; G_0) \\
 &\quad + N(r, 0; f) + N(r, 0; F_1) - mN(r, 1; f) - N(r, 0; f') \\
 &\quad + S(r, f) + S(r, g) \\
 &\leq 2\bar{N}(r, 0; f) + mN(r, 1; f) + N(r, 0; f') + 2\bar{N}(r, \infty; f) \\
 &\quad + 2\bar{N}(r, 0; g) + mN(r, 1; g) + N(r, 0; g') + 2\bar{N}(r, \infty; g) \\
 &\quad + N(r, 0; f) + N(r, 0; F_1) - mN(r, 1; f) - N(r, 0; f') \\
 &\quad + S(r, f) + S(r, g) \\
 &= \frac{2}{s}N(r, 0; f) + \frac{2}{s}N(r, \infty; f) + N(r, 0; f) + N(r, 0; F_1) \\
 &\quad + \frac{2}{s}N(r, 0; g) + mN(r, 1; g) + N(r, 0; g') + \frac{2}{s}N(r, \infty; g) \\
 &\quad + S(r, f) + S(r, g) \\
 &\leq \{m + 1 + 4/s\}T(r, f) + \{m + 1 + 5/s\}T(r, g) \\
 &\quad + S(r, f) + S(r, g) \\
 &\leq \{2m + 2 + 9/s\}T(r) + S(r),
 \end{aligned}$$

where  $T(r)$  is defined as in Lemma 6. So by Lemma 1 we obtain

$$(n + m + 1)T(r, f) \leq \{2m + 2 + 9/s\}T(r) + S(r). \quad (8)$$

Similarly we get

$$(n + m + 1)T(r, g) \leq \{2m + 2 + 9/s\}T(r) + S(r). \quad (9)$$

From (8) and (9) we see that

$$[n - m - 1 - 9/s]T(r) \leq S(r),$$

which is a contradiction. Hence (7) does not hold. So by Lemma 4 either  $F_0G_0 \equiv 1$  or  $F_0 \equiv G_0$ . Suppose that  $F_0G_0 \equiv 1$ . Then

$$f^n(f-1)^mf'g^n(g-1)^mg' \equiv b^2. \quad (10)$$

If  $m \geq 1$ , by Lemma 7 we arrive at a contradiction. If  $m = 0$ , by (10) we get

$$f^n f' g^n g' \equiv b^2. \quad (11)$$

Let  $z_0$  be a zero of  $f$  with multiplicity  $p(\geq s)$ . Then  $z_0$  is a pole of  $g$  with multiplicity  $q(\geq s)$ , say. From (11) we obtain

$$np + p - 1 = nq + q + 1$$

and so  $(n+1)(p-q) = 2$ , which is impossible as  $n \geq 2$  and  $p, q$  are positive integers. Therefore, we conclude that  $f$  and  $g$  are entire functions. Hence using Lemma 8, we get  $f(z) = c_1 e^{cz}$ ,  $g(z) = c_2 e^{-cz}$ , where  $c, c_1, c_2$  are constants satisfying  $(c_1 c_2)^{n+1} c^2 = -b^2$ .

Now we assume that  $F_0 \equiv G_0$ . And so by Lemma 11 we get  $F \equiv G$ , that is

$$\begin{aligned} & f^{n+1} \left( \sum_{i=0}^m {}^m C_i \frac{(-1)^i}{n+m-i+1} f^{m-i} \right) \\ & \equiv g^{n+1} \left( \sum_{i=0}^m {}^m C_i \frac{(-1)^i}{n+m-i+1} g^{m-i} \right). \end{aligned} \quad (12)$$

We now consider following three cases.

**Case 1** Let  $m = 0$ . Then from (12) we obtain  $f^{n+1} = g^{n+1}$ , which gives  $f = \tau g$  for some  $(n+1)$ -th root of unity 1.

**Case 2** Let  $m = 1$ . From (12) we obtain

$$f^{n+1} \left( \frac{1}{n+2} f - \frac{1}{n+1} \right) = g^{n+1} \left( \frac{1}{n+2} g - \frac{1}{n+1} \right),$$

which together with

$$\Theta(\infty, f) + \Theta(\infty, g) > 4/(n+1)$$

and Lemma 12 gives  $f \equiv g$ .

**Case 3** Let  $m = 2$ . Suppose that  $h = \frac{f}{g}$ . By (12) we get

$$\begin{aligned} & (n+2)(n+1)g^2(h^{n+3} - 1) - 2(n+3)(n+1)g(h^{n+2} - 1) \\ & + (n+2)(n+3)(h^{n+1} - 1) = 0. \end{aligned} \quad (13)$$

By (13) and by Lemma 13, we can conclude that

$$\begin{aligned} & \{(n+1)(n+2)(h^{n+3} - 1)g - (n+3)(n+1)(h^{n+2} - 1)\}^2 \\ & = -(n+3)(n+1)Q(h), \end{aligned}$$

where  $Q(h) = (h-1)^4(h-\nu_1)(h-\nu_2)\dots(h-\nu_{2n})$ , where  $\nu_j \in \mathbb{C} \setminus \{0, 1\}$  ( $j = 1, 2, \dots, 2n$ ), which are pairwise distinct.

If  $h$  is not a constant, this implies that every zero of  $h-\nu_j$  ( $j = 1, 2, \dots, 2n$ ), has a multiplicity of at least 2. By the second fundamental theorem of Nevanlinna we obtain that  $n \leq 2$ , which is again a contradiction. Therefore,  $h$  is a constant. We have from (13) that  $h^{n+1}-1=0$  and  $h^{n+2}-1=0$ , which imply  $h=1$ , and hence  $f \equiv g$ .

This completes the proof of theorem 8.  $\square$

**Proof of Theorem 9.** We put

$$H = \left( \frac{F_0''}{F_0'} - \frac{2F_0'}{F_0-1} \right) - \left( \frac{G_0''}{G_0'} - \frac{2G_0'}{G_0-1} \right).$$

We suppose that  $H \not\equiv 0$ . Since  $F_0$  and  $G_0$  share  $(1, 1)$ , by Lemma 2, Lemma 5(i), Lemma 9 and Lemma 10 we get

$$\begin{aligned} T(r, F) &\leq T(r, F_0) + N(r, 0; f) + N(r, 0; F_1) - mN(r, 1; f) \\ &\quad - N(r, 0; f') + S(r, f) \\ &\leq N_2(r, 0; F_0) + N_2(r, 0; G_0) + N_2(r, \infty; F_0) + N_2(r, \infty; G_0) \\ &\quad + \frac{1}{2}\overline{N}(r, 0; F_0) + \frac{1}{2}\overline{N}(r, \infty; F_0) + N(r, 0; f) + N(r, 0; F_1) \\ &\quad - mN(r, 1; f) - N(r, 0; f') + S(r, f) + S(r, g) \\ &\leq 2\overline{N}(r, 0; f) + mN(r, 1; f) + N(r, 0; f') + 2\overline{N}(r, \infty; f) \\ &\quad + 2\overline{N}(r, 0; g) + mN(r, 1; g) + N(r, 0; g') + 2\overline{N}(r, \infty; g) \\ &\quad + \frac{1}{2}\overline{N}(r, 0; f) + \frac{1}{2}\overline{N}(r, 1; f) + \frac{1}{2}\overline{N}(r, 0; f') + \frac{1}{2}\overline{N}(r, \infty; f) \\ &\quad + N(r, 0; f) + N(r, 0; F_1) - mN(r, 1; f) - N(r, 0; f') \\ &\quad + S(r, f) + S(r, g) \\ &\leq (m+2+11/2s)T(r, f) + (m+1+5/s)T(r, g) \\ &\quad + S(r, f) + S(r, g) \\ &\leq (2m+3+21/2s)T(r) + S(r). \end{aligned}$$

So by Lemma 1 we get

$$(n+m+1)T(r, f) \leq (2m+3+21/2s)T(r) + S(r).$$

Similarly we get

$$(n+m+1)T(r, g) \leq (2m+3+21/2s)T(r) + S(r).$$

Combining the above two inequalities we obtain

$$(n - m - 2 - 21/2s)T(r) \leq S(r),$$

which is a contradiction. Hence  $H \equiv 0$ .

Now by Lemma 1 we get

$$\begin{aligned} (n + m)T(r, f) &= T(r, f^n(f - 1)^m) + S(r, f) \\ &\leq T(r, F') + T(r, f') + S(r, f) \\ &\leq T(r, F_0) + 2T(r, f) + S(r, f) \end{aligned}$$

and so

$$T(r, F_0) \geq (n + m - 2)T(r, f) + S(r, f).$$

Similarly we get

$$T(r, G_0) \geq (n + m - 2)T(r, g) + S(r, g).$$

Also from Lemma 2 we have

$$\begin{aligned} &\bar{N}(r, 0; F_0) + \bar{N}(r, \infty; F_0) + \bar{N}(r, 0; G_0) + \bar{N}(r, \infty; G_0) \\ &\leq \bar{N}(r, 0; f) + \bar{N}(r, 1; f) + \bar{N}(r, 0; f') + \bar{N}(r, \infty; f) + \bar{N}(r, 0; g) \\ &\quad + \bar{N}(r, 1; g) + \bar{N}(r, 0; g') + \bar{N}(r, \infty; g) + S(r, f) + S(r, g) \\ &\leq (2 + 3/s)T(r, f) + (2 + 3/s)T(r, g) + S(r, f) + S(r, g) \\ &\leq \frac{4 + 6/s}{n + m - 2}T_0(r) + S(r), \end{aligned}$$

where  $S_0(r) = o\{T_0(r)\}$  as  $r \rightarrow \infty$  possibly outside a set of finite linear measure and  $\epsilon(> 0)$  is sufficiently small.

In view of the hypothesis we get from above

$$\limsup_{r \rightarrow \infty, r \notin E} \frac{\bar{N}(r, 0; F_0) + \bar{N}(r, \infty; F_0) + \bar{N}(r, 0; G_0) + \bar{N}(r, \infty; G_0)}{T_0(r)} < 1.$$

So by Lemma 6 we obtain either  $F_0 G_0 \equiv 1$  or  $F_0 \equiv G_0$ . Now by using Lemma 7, Lemma 8, Lemma 11 and proceeding in the same way as the proof of Theorem 8, we easily obtain the results of Theorem 9.  $\square$

**Proof of Theorem 10.** Using Lemma 5(ii) the theorem can be proved as the proof of Theorem 9.  $\square$

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