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Unification of extensions of zip rings

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Abstract. In this note, we investigate that a ring R is a right zip ring if and only if skew monoid ring $R*G$ (induced by a monoid homomorphism $\lambda : G \to \mathrm{Aut}(R)$) is a right zip ring when R is a right (G, λ)-McCoy ring, where G be a u.p.-monoid. Moreover, we study the relationship between right zip property of a ring R and skew generalized power series ring $R[[G, \omega]]$ (induced by a monoid homomorphism $\omega : G \to \mathrm{End}(R)$) over R when R is (G, ω)-Armendariz and G-compatible, where G is a strictly ordered monoid, which provides a unified solution to the questions raised by Faith [9].

1 Introduction

Throughout this article, R and G denote an associative ring with identity and monoid, respectively. For any subset X of a ring R, $r_R(X)$ denotes the right annihilator of X in R. Faith [8] called a ring R right zip provided that if the right annihilator $r_R(X)$ of a subset X of R is zero, then there exists a finite subset $Y \subseteq X$ such that $r_R(Y) = 0$; equivalently, for left ideal L of R with $r_R(L) = 0$, there exists a finitely generated left ideal $L_1 \subseteq L$ such that $r_R(L_1) = 0$. R is zip if it is both right and left zip. The concept of zip rings was initiated by Zelmanowitz [31] and appeared in various papers [3, 4, 7, 8, 9] and

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references therein. Zelmanowitz stated that any ring satisfying the descending chain conditions on right annihilators is a right zip ring, but the converse does not hold. Beachy and Blair [3] studied rings that satisfy the condition that every faithful right ideal I of a ring R (a right ideal I of a ring R is faithful if $r_R(I) = 0$) is cofaithful (a right ideal I of a ring R is cofaithful if there exists a finite subset $I_1 \subseteq I$ such that $r_R(I_1) = 0$). Right zip rings have this property and conversely for commutative ring R.

Extensions of zip rings were studied by several authors. Beachy and Blair [3] showed that if R is a commutative zip ring, then polynomial ring $R[x]$ over R is a zip ring. Afterwards, Cedo [4] proved that if R is a commutative zip ring, then the $n \times n$ full matrix ring $Mat_n(R)$ over R is zip; moreover, he settled negatively the following questions which were posed by Faith [8]: Does R being a right zip ring imply $R[x]$ being right zip?; Does R being a right zip imply $\mathrm{Mat}_n(R)$ being right zip?; Does R being a right zip ring imply $R[G]$ being right zip when G is a finite group? Based on the preceding results, Faith [9] again raised the following questions: When does R being a right zip ring imply $R[x]$ being right zip?; Characterize a ring R such that $Mat_n(R)$ is right zip; When does R being a right zip ring imply $R[G]$ being right zip when G is a finite group? Also he proved that if R is a commutative ring and G is a finite Abelian group, then the group ring $R[G]$ of G over R is zip.

In [14], Hong et al. studied above questions and proved that R is a right zip ring if and only if $R[x]$ is a right zip ring when R is an Armendariz ring. They also showed that if R is a commutative ring and G a u.p.-monoid that contains an infinite cyclic submonoid, then R is a zip ring if and only if $R[G]$ is a zip ring. Further, Cortes [7] studied the relationship between right (left) zip property of R and skew polynomial extensions over R by using skew versions of Armendariz rings and generalized the results of Hong et al. [14]. Later, Hashemi [10] showed that R is a right zip ring if and only if $R[G]$ is a right zip ring when R be a reversible ring and G a strictly totally ordered monoid. In this paper, we prove the above mentioned results to skew monoid ring $R * G$ (induced by a monoid homomorphism $\lambda : G \to \mathrm{Aut}(R)$) and skew generalized power series ring $R[[G, \omega]]$ (induced by a monoid homomorphism $\omega : G \to \mathrm{End}(R)$).

This paper is organized as follows. In Section 2, we introduce the concept of right (G, λ)-McCoy ring and extend the above mentioned results proved by Hong et al. [14], Cortes [7] and Hashemi [10] to skew monoid ring $R * G$ (induced by a monoid homomorphism $\lambda : G \to \mathrm{Aut}(R)$). In Section 3, we discuss a unification of the above extensions and prove that if R is (G, ω)-Armendariz ring and G-compatible, then skew generalized power series ring $R[[G, \omega]]$ (induced by a monoid homomorphism $\omega : G \to \mathrm{End}(R)$) is right zip if and only if R is right zip. This provides a unified generalization of the results due to Hong et al. [14] and Cortes [7].

2 Right zip skew monoid rings

In this section, we study the fundamental concept of a skew monoid ring and give the definition of right (G, λ)-McCoy ring which is a generalization of right G-McCoy ring. Moreover, we investigate a relationship between right zip property of a ring R and skew monoid ring $R * G$ over R, and we also extend some results of [7, 10, 14].

Definition 2.1 *A monoid* G *is called a unique product monoid (or a u.p.-monoid) if for any two nonempty finite subsets* $A, B \subseteq G$ *there exist* $a \in A$ *and* $b \in B$ *such that* $ab \neq a'b'$ *for every* $(a', b') \in A \times B \setminus \{(a, b)\}$*; the element* ab *is called a u.p.-element of* $AB = \{cd : c \in A, d \in B\}$.

The class of u.p.-monoids includes the right and left totally ordered monoids, submonoids of a free group, and torsion-free nilpotent groups (for details see [23, 24]).

From [15, 22], let R be a ring and G a u.p.-monoid. Assume that there exists a monoid homomorphism $\lambda : G \to \mathrm{Aut}(R)$. We denote by $\lambda^g(r)$ the image of $r \in R$ under $g \in G$. Then we can form a skew monoid ring $R * G$ (induced by the monoid homomorphism $\lambda : G \to \mathrm{Aut}(R)$) by taking its elements to be finite formal combinations $\sum_{i=1}^{n} a_i g_i$, where $a_i \in R$, $g_i, \in G$ for all i, with multiplication rule defined by $gr = \lambda^g(r)g$.

It is well known that if R is a commutative ring and $f(x)$ is a zero divisor in $R[x]$, there is a nonzero element $r \in R$ with $f(x)r = 0$, as proved by McCoy [26, Theorem 2]. Based on this result, Nielsen [28] called a ring R right McCoy if for each pair of nonzero polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$ there exists a nonzero element $r \in R$ with $f(x)r = 0$. Left McCoy ring can be defined similarly. A ring R is McCoy if it is both right and left McCoy. Thus every commutative ring is McCoy. Further, Hashemi [10] generalized the concept of McCoy ring to monoid ring and called a ring R right G-McCoy ring (right McCoy ring relative to monoid) if whenever $0 \neq \alpha = a_1 g_1 + a_2 g_2 + \cdots + a_n g_n, 0 \neq \beta = b_1 h_1 + b_2 h_2 + \cdots + b_m h_m \in R[G]$, with $a_i, b_j \in R$, $g_i, h_j \in G$ satisfy $\alpha\beta = 0$ implies $\alpha r = 0$ for some nonzero $r \in R$. The left G-McCoy ring can be defined similarly. If R is both right and left G-McCoy, then R is

G-McCoy. Here, we extend the concept of McCoy ring to skew monoid ring $R * G$ (induced by a monoid homomorphism $\lambda : G \to \mathrm{Aut}(R)$) with the help of a construction of skew monoid rings $R * G$.

Definition 2.2 *Let R be a ring, G a u.p.-monoid and $\lambda : G \to \mathrm{Aut}(R)$ a monoid homomorphism. A ring R is called right (G,λ)-McCoy if whenever $\alpha = \sum_{i=1}^{n} a_i g_i$, $\beta = \sum_{j=1}^{m} b_j h_j \in R * G$, with $a_i, b_j \in R$, $g_i, h_j \in G$ satisfy $\alpha\beta = 0$, then $\alpha r = 0$ or $\lambda^{-g_i}(a_i) r = 0$ for some nonzero $r \in R$. The left (G,λ)-McCoy ring is defined similarly. If R is both right and left (G,λ)-McCoy, then R is (G,λ)-McCoy.*

Example 2.3 *We give some special cases of right (G,λ)-McCoy rings.*

(1) *Suppose G be trivial order monoid and $\lambda = 1 : G \to \mathrm{Aut}(R)$ a monoid homomorphism. Then R is right (G,λ)-McCoy if and only if R is right G-McCoy. Thus right G-McCoy [10] is special case of right (G,λ)-McCoy.*

(2) *Suppose $G = (\mathbb{N} \cup \{0\}, +)$ and $\lambda = 1 : G \to \mathrm{Aut}(R)$. Then R is right (G,λ)-McCoy if and only if R is right McCoy. Thus right McCoy [28] is special case of right (G,λ)-McCoy.*

In the following theorem, we extend the results of Hong et al. [14, Theorem 11, Corollary 13, Proposition 14, Theorem 16 and Corollary 17], Cortes [7, Theorem 2.8(i)] and Hashemi [10, Theorem 1.25 and Corollary 1.26] to skew monoid ring $R * G$ (induced by a monoid homomorphism $\lambda : G \to \mathrm{Aut}(R)$) using right (G,λ)-McCoy ring, and also provides a generalized solution to the questions posed by Faith [9] for noncommutative zip rings.

Theorem 2.4 *Let G be a u.p.-monoid and $\lambda : G \to \mathrm{Aut}(R)$ a monoid homomorphism. If R is a right (G,λ)-McCoy ring, then R is right zip if and only if $R * G$ is right zip.*

Proof. Suppose R is right zip and Y a nonempty subset of $R * G$ such that $r_{R*G}(Y) = 0$. Let V be the set of all coefficients of elements of Y and defined by $V = C_Y = \bigcup_{\alpha \in Y} C_\alpha$ such that $C_\alpha = \{\lambda^{-g_i}(a_i) : 1 \leq i \leq n\}$, where $\alpha = \sum_{i=1}^{n} a_i g_i \in R * G$. Take any $a \in r_R(V)$ then $a \in r_R(\bigcup_{\alpha \in Y} C_\alpha)$ which implies $a \in r_R(C_\alpha)$ for all $\alpha \in Y$. Thus $a \in r_{R*G}(\alpha) = 0$ for all $\alpha \in Y$. Therefore $r_R(V) = 0$. Since R is right zip, there exists a nonempty subset

$V_1 = \{\lambda^{-g_{i_1}}(a_{i_1}), \lambda^{-g_{i_2}}(a_{i_2}), \ldots, \lambda^{-g_{i_n}}(a_{i_n})\}$ such that $r_R(V_1) = 0$. For each $\lambda^{-g_{i_j}}(a_{i_j}) \in V_1$, there exists $\alpha_{i_j} \in Y$ such that some of the coefficients of α_{i_j} are a_{i_j} for each $1 \leq j \leq n$. So we have $Y_0 = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}\}$ be a nonempty subset of Y whose some of the coefficients are a_{i_j} for each $1 \leq j \leq n$. Suppose V_0 be a set of all the coefficients of Y_0. Then $V_1 \subseteq V_0$ which implies $r_R(V_0) \subseteq r_R(V_1) = 0$. Now we will show that $r_{R*G}(Y_0) = 0$. Consider $r_{R*G}(Y_0) \neq 0$, so $0 \neq \beta \in r_{R*G}(Y_0)$ which gives $\alpha_{i_j}\beta = 0$ for all $\alpha_{i_j} \in Y_0$. Since R is right (G, λ)-McCoy, there exists a nonzero element $r_1 \in R$ with $\alpha_{i_j} r_1 = 0$ for all $\alpha_{i_j} \in Y_0$. Thus $\lambda^{-g_{i_j}}(a_{i_j})r_1 = 0$ for each $1 \leq j \leq n$, it follows that $r_1 \in r_R(V_1) = 0$. Therefore $r_1 = 0$, which is a contradiction and so $r_{R*G}(Y_0) = 0$. Hence $R * G$ is a right zip ring.

Conversely, suppose $R*G$ is right zip and V a nonempty subset of R such that $r_R(V) = 0$. Then $r_{R*G}(V) = 0$. Since $R * G$ is right zip, there exists a nonempty subset V_1 of V such that $r_{R*G}(V_1) = 0$. Thus $r_R(V_1) = r_{R*G}(V_1) \cap R = 0$. Hence R is right zip. □

In 1974, Armendariz [2] proved that $a_i b_j = 0$ for every i and j whenever polynomials $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$ over a reduced ring R (a ring R with no nonzero nilpotent elements) satisfy $f(x)g(x) = 0$, where x is an indeterminate over R, following which, Rege and Chhawchharia [29] called such a ring (not necessarily reduced) an Armendariz ring.

Recall for a ring R and a ring automorphism $\sigma : R \to R$, the skew polynomial ring $R[x; \sigma]$ (skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$) consists of polynomials in the form $f(x) = \sum_{i=0}^{n} a_i x^i$ ($f(x) = \sum_{j=q}^{m} b_j x^j$), where the addition is defined as usual and multiplication defined by the rule $xa = \sigma(a)x$ ($x^{-1}a = \sigma^{-1}(a)x$) for any $a \in R$. In [13], Hong et al. extended Armendariz property to skew polynomial ring $R[x; \sigma]$ and defined that a ring R with an endomorphism σ is σ-skew Armendariz if whenever polynomials $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x; \sigma]$ satisfy $f(x)g(x) = 0$ which implies $a_i \sigma^i(b_j) = 0$ for every i and j. Further, Liu [21] introduced the concept of Armendariz ring relative to monoid as is a generalization of Armendariz ring and called a ring R G-Armendariz ring (Armendariz ring relative to monoid), if whenever elements $\alpha = a_1 g_1 + a_2 g_2 + \ldots + a_n g_n, \beta = b_1 h_1 + b_2 h_2 + \ldots + b_m h_m \in R[G]$ satisfy $\alpha\beta = 0$ implies $a_i b_j = 0$ for each i and j, where $a_i, b_j \in R$, $g_i, h_j \in G$ and G is a monoid. Now we define Armendariz ring to skew monoid ring $R * G$.

Definition 2.5 *Let* R *be a ring,* G *a u.p.-monoid and* $\lambda : G \to \mathrm{Aut}(R)$ *a* *monoid homomorphism. A ring* R *is called* (G, λ)*-Armendariz if whenever* $\alpha = \sum_{i=1}^{n} a_i g_i$, $\beta = \sum_{j=1}^{m} b_j h_j \in R * G$, *with* $a_i, b_j \in R$, $g_i, h_j \in G$ *satisfy* $\alpha\beta = 0$, *then* $a_i \lambda^{g_i}(b_j) = 0$ *for all* i, j.

Notice that all the above mentioned classes of Armendariz rings are special cases of (G, λ)-Armendariz, whereas (G, λ)-Armendariz ring is a special case of (G, ω)-Armendariz ring which was investigated by Marks et al. [24] (for details see section 3). It is also clear from the definition 2.1 and definition 2.4 that every (G, λ)-Armendariz ring is a right (G, λ)-McCoy ring. In the following example, we show that converse need not be true.

Given a ring R and a bimodule ${}_R\mathcal{M}_R$, the trivial extension of R by $\mathcal{M}$ is the ring $T(R, \mathcal{M}) = R \bigoplus \mathcal{M}$ with usual addition and the multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$, where $r_1 r_2 \in R$ and $m_1 m_2 \in M$.

Example 2.6 *There exists a right* (G, λ)*-McCoy ring which is not* (G, λ)*-Armendariz.*

Proof. Let $\mathbb{Z}_8$ be a ring of integers of modulo 8 then its trivial extension is $T(\mathbb{Z}_8, \mathbb{Z}_8)$. Suppose $\lambda : G \to \mathrm{Aut}(T)$ defined by $\lambda^g(a, b) = (b, a)$, for any $g \in G$ and $(a, b) \in T(\mathbb{Z}_8, \mathbb{Z}_8)$, where G be a u.p.-monoid. It is easy to check that λ is a monoid homomorphism. Assume $e, g \in G$ with $e \neq g$ and $\alpha = (4, 0)e + (4, 1)g$, $\beta = (0, 4)e + (1, 4)g \in R * G$. Then $\alpha\beta = 0$, while $(4, 0)\lambda^g(1, 4) \neq 0$. Thus T is not (G, λ)-Armendariz. Consider a nonzero element $t = (4, 0) \in T$, so $\alpha t = ((4, 0)e + (4, 1)g)(4, 0) = 0$. Therefore R is a right (G, λ)-McCoy ring. □

Now, we get the following corollary.

Corollary 2.7 *Let* G *be a u.p.-monoid and* $\lambda : G \to \mathrm{Aut}(R)$ *a monoid homomorphism. If* R *is* (G, λ)*-Armendariz then* R *is right zip if and only if* $R * G$ *is right zip.*

Proof. Since R is a (G, λ)-Armendariz ring so R is a right (G, λ)-McCoy ring. Thus by Theorem 2.4, R is right zip if and only if $R * G$ is right zip. □

We also deduce some important results as corollaries of above theorem.

Recall that a ring R is called reversible if $ab = 0$ implies $ba = 0$ for all $a, b \in R$. Let $(G, \leq)$ be an ordered monoid. If for any $g, g', h \in G, g < g'$ implies that $gh < g'h$ and $hg < hg'$, then $(G, \leq)$ is called strictly ordered monoid.

Corollary 2.8 (Hashemi [10, Theorem 1.25]) *Let* R *be a reversible ring and* G *a strictly totally ordered monoid. Then* R *is right zip if and only if* $R[G]$ *is right zip.*

Proof. Since R is reversible ring and G a strictly totally ordered monoid, by [10, Corollary 1.5], R is a G-McCoy ring. Suppose $\lambda = 1 : G \to \mathrm{Aut}(R)$ a monoid homomorphism then R is right (G, λ)-McCoy ring if and only if R is a right G-McCoy ring. Thus by Theorem 2.4, R is right zip if and only if $R[G]$ is right zip. □

Definition 2.9 *A ring* R *is called right duo if all right ideals are two sided ideals. Left duo rings are defined similarly, and a ring is called duo if it is both right and left duo.*

Corollary 2.10 *Let* R *be a right duo ring and* G *a strictly totally ordered monoid. Then* R *is right zip if and only if* $R[G]$ *is right zip.*

Proof. Since R be a right duo ring and G a strictly totally ordered monoid, by [10, Theorem 1.8], R is a right G-McCoy ring. Suppose $\lambda = 1 : G \to \mathrm{Aut}(R)$ a monoid homomorphism then R is a right (G, λ)-McCoy ring if and only if R is a right G-McCoy ring. Thus by Theorem 2.4, R is right zip if and only if $R[G]$ is right zip. □

The following definition is taken from [7].

Definition 2.11 (1) *A ring* R *satisfies* $SA1'$ *if for* $f(x) = \sum_{i=0}^{n} a_i x^i$ *and* $g(x) = \sum_{j=0}^{m} b_j x^j$ *in* $R[x; \sigma]$, $f(x)g(x) = 0$ *implies that* $a_i \sigma^i(b_j) = 0$ *for all* i *and* j, *where* σ *be an endomorphism of* R.

(2) *A ring* R *satisfies* $SA3'$ *if for* $f(x) = \sum_{i=p}^{n} a_i x^i$ *and* $g(x) = \sum_{j=q}^{m} b_j x^j$ *in* $R[x, x^{-1}; \sigma]$, $f(x)g(x) = 0$ *implies that* $a_i \sigma^i(b_j) = 0$ *for all* i *and* j, *where* σ *be an automorphism of* R.

Corollary 2.12 (Cortes [7, Theorem 2.8(i)]) *Let* σ *be an automorphism of* R *and satisfies* $SA1'$. *The following conditions are equivalent:*

(1) R *is right zip;*

(2) $R[x; \sigma]$ *is right zip;*

(3) $R[x, x^{-1}; \sigma]$ *is right zip.*

Proof. (1) $\Leftrightarrow$ (2) Given σ be an automorphism of R and satisfies $SA1'$(R is a σ-skew Armendariz ring). Suppose $G = (\mathbb{N} \cup \{0\}, +)$ and $\lambda(1) = \sigma$. Then R is a σ-skew Armendariz ring if and only if R is a (G, λ)-Armendariz ring. Thus by Corollary 2.7, R is right zip if and only if $R[x, \sigma]$ is right zip.
(1) $\Leftrightarrow$ (3) Given σ be an automorphism of R and satisfies $SA3'$ then by [7, Lemma 2.3] R satisfies $SA1'$(R is a σ-skew Armendariz ring). Suppose $G = (\mathbb{Z} \cup \{0\}, +)$ and $\lambda(1) = \sigma$. Then R is a σ-skew Armendariz ring if and only if R is a (G, λ)-Armendariz ring. Thus by Corollary 2.7, R is right zip if and only if $R[x, x^{-1}; \sigma]$ is right zip. □

Corollary 2.13 (Hong et al. [14, Theorem 11]) *Let* R *be an Armendariz ring. Then* R *is a right zip ring if and only if* $R[x]$ *is a right zip ring.*

Proof. Since Armendariz ring is a special case of (G, λ)-Armendariz when $G = (\mathbb{N} \cup \{0\}, +)$ and $\lambda = 1 : G \to \mathrm{Aut}(R)$. Thus by Corollary 2.7, R is a right zip ring if and only if $R[x]$ is a right zip ring. □

Corollary 2.14 (Hong et al. [14, Propsition 2]) *Let* R *be a reduced ring and* G *a u.p.-monoid. Then* R *is right zip if and only if* $R[G]$ *is right zip.*

Proof. Since R be a reduced ring and G a u.p.-monoid, by [21, Proposition 1.1] R is G-Armendariz. Suppose $\lambda = 1 : G \to \mathrm{Aut}(R)$ a monoid homomorphism then R is a (G, λ)-Armendariz ring if and only if R is a G-Armendariz ring. Thus by Corollary 2.7, R is right zip if and only if $R[G]$ is right zip. □

Corollary 2.15 (Hong et al. [14, Theorem 16]) *Suppose that* R *is a commutative ring and* G *a u.p.-monoid that contains an infinite cyclic submonoid. Then* R *is a zip ring if and only if* $R[G]$ *is a zip ring.*

3 Skew generalized power series rings

In this section, we study the concept of skew generalized power series rings, (G, ω)-Armendariz rings and G-compatible rings which were introduced by Mazurek et al. [25]. Moreover, we investigate a relationship between right zip property of a ring R and skew generalized power series ring $R[[G, \omega]]$ over R. This relationship generalizes some of the results of [7, 14].

Recall from [24, 25] that for a construction of skew generalized power series ring, we need some definitions. Let $(G, \leq)$ be a partially ordered set. Then $(G, \leq)$ is called artinian if every strictly decreasing sequence of elements of G is finite, and $(G, \leq)$ is called narrow if every subset of pairwise order-incomparable elements of G is finite. Thus, $(G, \leq)$ is artinian and narrow if and only if every nonempty subset of G has at least one but only a finite number of minimal elements.

An ordered monoid is a pair $(G, \leq)$ consisting of a monoid G and an order $\leq$ on G such that for all $a, b, c \in G$, $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$. An ordered monoid $(G, \leq)$ is said to be strictly ordered if for all $a, b, c \in G$, $a < b$ implies $ca < cb$ and $ac < bc$.

Let R be a ring, $(G, \leq)$ a strictly ordered monoid and $\omega : G \to \mathrm{End}(R)$ a monoid homomorphism. For $s \in G$, let ω_s denote the image of s under ω, that is, $\omega_s = \omega(s)$. Let $\mathcal{A}$ be the set of all functions $\alpha : G \to R$ such that the $\mathrm{supp}(\alpha) = \{s \in G : \alpha(s) \neq 0\}$ is artinian and narrow. Then for any $s \in G$ and $\alpha, \beta \in \mathcal{A}$ the set

$$X_s(\alpha, \beta) = \{(x, y) \in \mathrm{supp}(\alpha) \times \mathrm{supp}(\beta) : s = xy\}$$

is finite. Thus one can define the product $\alpha\beta : G \to R$ of $\alpha, \beta \in \mathcal{A}$ as follows:

$$(\alpha\beta)(s) = \sum_{(x,y) \in X_s(\alpha,\beta)} \alpha(x) \cdot \omega_x(\beta(y))\,.$$

With pointwise addition and multiplication as defined above, $\mathcal{A}$ becomes a ring, called the ring of skew generalized power series with coefficients in R and exponents in G (see [25]), denoted by $R[[G, \omega, \leq]]$ (or by $[[G, \omega]]$). The skew generalized power series ring $R[[G, \omega]]$ is a compact generalization of (skew) polynomial rings, (skew) Laurent polynomial rings, (skew) power series rings, (skew) group rings, (skew) monoid rings, Mal'cev Neumann Laurent series rings and generalized power series rings.

The symbol 1 denote the identity elements of the multiplicative monoid G, the ring R, and the ring $R[[S, \omega]]$, as well as the trivial monoid homomorphism $\omega = 1 : G \to \mathrm{End}(R)$ that sends every element of G to the identity endomorphism.

To each $r \in R$ and $s \in G$, we associate elements $c_r, e_s \in R[[G, \omega]]$ defined by

$$c_r(x) = \begin{cases} r & \text{if } x = 1 \\ 0 & \text{if } x \in G \setminus \{1\} \end{cases}, \quad e_s(x) = \begin{cases} 1 & \text{if } x = s \\ 0 & \text{if } x \in G \setminus \{s\}. \end{cases}$$

It is clear that $r \mapsto c_r$ is a ring embedding of R into $R[[G,\omega]]$ and $s \mapsto e_s$ is a monoid embedding of G into a multiplicative monoid of ring $R[[S,\omega]]$, and $e_s c_r = c_{\omega_s(r)} e_s$. Moreover, for each nonempty subset X of R we put $X[[G,\omega]] = \{\alpha \in R[[G,\omega]] : \alpha(s) \in X \cup \{0\}$ for every $s \in G\}$ denotes a subset of $R[[G,\omega]]$, and for each nonempty subset Y of $R[[G,\omega]]$ we put $C_Y = \{\beta(t) : \beta \in Y, t \in G\}$ denotes a subset of R.

In [18], Kim et al. studied a stronger condition than Armendariz and defined a ring R is called powerserieswise Armendariz if whenever power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ in $R[[x]]$ satisfy $f(x)g(x) = 0$ then $a_i b_j = 0$ for all i and j. Further, Liu [20] generalized the definition of the powerserieswise Armendariz to (untwisted) generalized power series ring $[[R^{G\leq}]]$ (particular case of Skew generalized power series ring $R[[G,\omega]]$) and defined as follows: if R is a ring and $(G,\leq)$ is a commutative strictly ordered monoid, then R is called G-Armendariz if whenever generalized power series $\alpha, \beta \in R[[G,1]]$ satisfy $\alpha\beta = 0$ then $\alpha(s)(\beta(t)) = 0$ for all $s,t \in S$. With the help of a construction of skew generalized power series ring $R[[G,\omega]]$, Marks et al. [24] introduced the concept of Armendariz property to skew generalized power series ring $R[[G,\omega]]$ and gave a unified approach to all classes of Armendariz property.

Definition 3.1 *Let* R *be a ring,* $(G,\leq)$ *a strictly ordered monoid and* $\omega : G \to \mathrm{End}(R)$ *a monoid homomorphism. A ring* R *is called* (G,ω)*-Armendariz if whenever* $\alpha\beta = 0$ *for* $\alpha, \beta \in R[[G,\omega]]$*, then* $\alpha(s).\omega_s(\beta(t)) = 0$ *for all* $s,t \in G$*. If* $G = \{1\}$ *then every ring is* (G,ω)*-Armendariz*

We recall the definition of compatible endomorphism from [24, Definition 2.3].

Definition 3.2 *An endomorphism* σ *of a ring* R *is compatible if for all* $a, b \in R$*,* $ab = 0 \Leftrightarrow a\sigma(b) = 0$*.*

Definition 3.3 *Let* R *be a ring,* $(G,\leq)$ *a strictly ordered monoid and* $\omega : G \to \mathrm{End}(R)$ *a monoid homomorphism. Then* R *is* G*-compatible if* ω_s *is compatible for every* $s \in G$*.*

To prove the main result of this section, we need Lemma 3.4 and Lemma 3.5 which were proved by Marks et al. [24]. Here we quote only the statements.

Lemma 3.4 *Let* R *be a ring,* $(G,\leq)$ *a strictly ordered monoid and* $\omega : G \to \mathrm{End}(R)$ *a monoid homomorphism. The following conditions are equivalent:*

(1) R *is* G*-compatible;*

(2) *for any* $a \in R$ *and any nonempty subset* $Y \subseteq R[[G, \omega]]$,

$$a \in ann_r^R(C_Y) \Leftrightarrow c_a \in ann_r^{R[[G,\omega]]}(Y).$$

Proof. See [24, Lemma 3.1]. □

Lemma 3.5 *Let* R *be a ring,* $(G, \leq)$ *a strictly ordered monoid and* $\omega : G \to \text{End}(R)$ *a monoid homomorphism. If* R *is* G*-compatible, then for any nonempty subset* $X \subseteq R$, $ann_r^R(X)[[G, \omega]] = ann_r^{R[[G,\omega]]}(X[[G, \omega]])$.

Proof. See [24, Lemma 3.2]. □

Now, we are able to prove the main theorem of this section. The following theorem generalizes Corollary 2.7 (Section 2), Hong et al. [14, Theorem 11, Corollary 13 and Proposition 14], Cortes [7, Theorem 2.8] and propose a unified solution to the questions raised by Faith [9].

Theorem 3.6 *Let* R *be a ring,* $(G, \leq)$ *a strictly ordered monoid and* $\omega : G \to \text{End}(R)$ *a monoid homomorphism. If* R *is* (G, ω)*-Armendariz and* G*-compatible then* $R[[G, \omega]]$ *is right zip if and only if* R *is right zip.*

Proof. Suppose that $R[[G, \omega]]$ is a right zip ring. We show that R is a right zip ring. For this consider $Y \subseteq R$ with $r_R(Y) = 0$. Since $Y \subseteq R$, so we put $Y[[G, \omega]] = \{\alpha : \alpha(s) \in R \text{ and } s \in G\} \subseteq R[[G, \omega]]$. Let any $\beta \in r_{R[[G,\omega]]}(Y[[G, \omega]])$. Then $\alpha\beta = 0$ which implies $\alpha(s)\beta(t) = 0$ for all $s, t \in G$ since R is G-compatible and (G, ω)-Armendariz. Thus $\beta(t) \in r_R(\alpha(s)) = 0$ for all $\alpha(s) \in R$ which implies $\beta = 0$ for all $\beta \in R[[G, \omega]]$. It follows that $r_{R[[G,\omega]]}(Y[[G, \omega]]) = 0$. Since $R[[G, \omega]]$ is a right zip ring, there exists a subset $V \subseteq Y[[G, \omega]]$ such that $r_{R[[G,\omega]]}(V) = 0$. Then we put $C_V = \{\gamma(u) : u \in S \text{ and } \gamma \in V\}$ is a subset of Y. By Lemma 3.4, for any $a \in r_R(C_V) \Leftrightarrow c_a \in r_{R[[G,\omega]]}(V)$ since R is G-compatible. Thus we have $r_R(C_V) = 0$. Hence R is a right zip ring.

Conversely, suppose R is a right zip ring and a subset $U \subseteq R[[G, \omega]]$ with $r_{R[[G,\omega]]}(U) = 0$. We put $C_U = \{\beta(t) : \beta \in U \text{ and } t \in G\}$ which is nonempty subset of R. By Lemma 3.4, for any $p \in r_R(C_U) \Leftrightarrow c_p \in r_{R[[G,\omega]]}(U)$ since R is G-compatible. Thus $r_R(C_U) = 0$. Since R is a right zip ring, there exists a nonempty subset $X \subseteq C_U$ such that $r_R(X) = 0$. So we put $X[[G, \omega]] = \{\alpha \in R[[G, \omega]] : \alpha(s) \in X \cup \{0\} \text{ and } s \in G\}$. Thus by Lemma 3.5, $r_{R[[G,\omega]]}(X[[G, \omega]]) = r_R(X)[[G, \omega]] = 0$ since R is G-compatible. Therefore $R[[G, \omega]]$ is right zip. □

Now, we get following result as corollary which was proved by Cortes [7]. To get Corollary 3.8, we need the following definition.

Definition 3.7 ([7, Definition 2.2(ii)]) *A ring* R *satisfies* $SA2'$ *if for* $f(x) = \sum_{i=0}^{\infty} a_i x^i$ *and* $g(x) = \sum_{j=0}^{\infty} b_j x^j$ *in* $R[[x, \sigma]]$, $f(x)g(x) = 0$ *implies that* $a_i \sigma^i(b_j) = 0$ *for all* i *and* j, *where* σ *be an endomorphism of* R.

Corollary 3.8 (Cortes [7, Theorem 2.8(ii)]) *Let* σ *be an automorphism of* R *and* R *satisfies* $SA2'$. *The following conditions are equivalent:*

(1) R *is right zip;*

(2) $R[[x, \sigma]]$ *is right zip.*

Proof. Suppose $G = (\mathbb{N} \cup \{0\}, +)$ and $\omega(1) = \sigma$. Then R satisfies $SA2'$ if and only if R is (G, ω)-Armendariz. Thus by Theorem 3.6, R is right zip if and only if $R[x, \sigma]$ is right zip. □

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