

Tripartite graphs with given degree set

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Abstract. If $k \geq 1$, then the *global degree set* of a k -partite graph $G = (V_1, V_2, \dots, V_k, E)$ is the set of the distinct degrees of the vertices of G , while if $k \geq 2$, then the *distributed degree set* of G is the family of the k degree sets of the vertices of the parts of G . We propose algorithms to construct bipartite and tripartite graphs with prescribed global and distributed degree sets consisting from *arbitrary* nonnegative integers. We also present a review of the similar known results on digraphs.

1 Introduction

In this paper we follow the terminology used in the monography of Chartrand, Lesniak and Zhang [5] and the handbook of Gross, Yellen and Zhang [11].

Let $m \geq 0$ and $n \geq 1$ be integers, $G = (V, E)$ be a finite, simple graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G) = \{e_1, \dots, e_m\}$. The

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degree d_i of a vertex v_i is the number of edges of G which are incident on v_i . The *degree sequence* $\mathbf{d} = [d_1, d_2, \dots, d_n]$ of G is the sequence of degrees of G , usually put in nonincreasing or nondecreasing order. The number of vertices is called the *order* of G , while the number of edges is called the *size* of G . A degree sequence $\mathbf{d}$ is said to be *graphic* if it is the degree sequence of some finite graph. The set of the disjoint degrees $\gamma = \{g_1, g_2, \dots, g_p\}$ of the vertices of G is called its *degree set*. If $k \geq 2$ is an integer and $G = (V_1, V_2, \dots, V_k, E)$ is a k -partite graph, then the degree set of the vertex set V_i ($1 \leq i \leq k$) is called the *global degree set of the i -th part* of G and is denoted by $\gamma_i(V_i)$; the union $\coprod_{i=1}^k \gamma_i$ is called the *global degree set of G* and is denoted by $\gamma(G)$. In the literature instead of the global degree set usually the shorter *degree set* expression is used, but we wish to underline the difference between the simple and multipartite graphs. The family of the degree sets γ_i is called the *distributed degree set of G* and is denoted by $\delta(G)$.

The papers of Tyshkevich and Chernyak [55, 56, 57, 58] contain a review on the different generalizations of score sequences.

Some early results on degree sets of simple graphs and trees (acyclic connected graphs) were published in 1977 by Kapoor, Polimeni and Wall. They introduced the concept of degree set and proved the following theorem

Theorem 1 (Kapoor, Polimeni, Wall [25]) *If p is a positive integer, then any set $\gamma = \{g_1, g_2, \dots, g_p\}$ of positive integers with $g_1 < g_2 < \dots < g_p$ is the degree set of a connected graph G and the minimum order of such a graph is $g_p + 1$.*

Proof. See [25, 37, 54]. □

In 1979 Koukichi and Katsuhiko [28] reproved Theorem 1. They defined (n, k) -sets as sets of integers $\{h_1, \dots, h_k\}$ with $n-1 \geq h_1 > h_2 > \dots > h_k \geq 0$. Further they defined $DGn(k)$ for any positive n and k with $1 \leq k \leq n-1$ as the set of all degree sets D of graphs G of order n with $|D| = k$, and $F_n(k)$ as the set of all (n, k) -sets $D = \{d_1, \dots, d_k\}$ satisfying: (i) if $d_1 = n-1$, then $d_k > 0$ and (ii) if $n \equiv 1 \pmod{k}$ then D contains at least one even number.. Among others they expressed $DGn(2)$ in terms of $F_n(2)$, and proved $DGn(3) = F_n(3)$ and $DGn(n-2) = F_n(n-2)$ for $n > 2$.

A short proof of this result is due to Tripathi and Vijay [54].

A simple consequence of Theorem 1 is the following assertion allowing $0 \in \gamma$ and not containing the condition of sorted γ .

Corollary 2 *If $p \geq 1$ is an integer, then any set $\gamma = \{g_1, g_2, \dots, g_p\}$ of non-negative integers is the degree set of a graph G and if $0 \notin \gamma$, then the minimum*

order of such graphs is $\max(\gamma) + 1$, otherwise $\max(\gamma) + 2$.

Proof. If $0 \notin \gamma$, then we can use the proof of Theorem 1, otherwise we can add an isolated vertex to the minimal size graph corresponding to $\gamma \setminus 0$. $\square$

In 2006 Ahuja and Tripathi investigated the possible sizes of graphs having prescribed degree set and extended Theorem 1 giving *all* possible size of graphs having a prescribed degree set. We say, that a graph G is a (p, γ) -graph, if its size is p and its degree set is γ .

Theorem 3 (Ahuja, Tripathi [1]) *Let $\gamma = \{g_1, g_2, \dots, g_p\}$ be any finite, non-empty set of positive integers and let $m = \max(\gamma)$. If all members of γ are odd, then there exists a (p, γ) -graph if and only if $p > m$ and p is even; otherwise there exists a (p, γ) -graph if and only if $p > m$, provided also that $p \neq m + 2$ in the special case, where $\gamma = \{1, m\}$ for any even integer $m \geq 4$.*

Proof. See [1]. $\square$

One can ask, what is the answer, if we allow $0 \in \gamma$? Using Theorem 3 it is easy to show, that in this case all sizes greater than p are possible.

In 1980 Sipka investigated the problem or the possible orders of graphs having a prescribed global degree set $\gamma = \{g_1, g_2, \dots, g_n\}$ of integers with $1 \leq g_1 < g_2 < \dots < g_n$. His results are summarized in the following theorem.

Theorem 4 (Sipka [52]) *Let $\gamma = \{g_1, g_2, \dots, g_p\}$ be a nonempty set of positive integers with $g_1 < g_2 < \dots < g_p$.*

(i) *If p is even, g_i is odd for all $1 \leq i \leq p$, $p > g_p$, then there exists a graph G of order p with $\gamma(G) = \gamma$.*

(ii) *If g_i is even for some $1 \leq i \leq p$, $t \geq 2$ and $\gamma \neq \{1, 2, \dots, 2t\}$, then there exists a graph G of order p such that $\gamma(G) = \gamma$ for all positive $p > g_n$, then there exists a graph G of order p such that $\gamma(G) = \gamma$.*

(iii) *If $t \geq 2$ is an integer, $\gamma = \{1, 2, \dots, 2t\}$, then there exists a graph G of order p for all positive p exceeding g_p , with the exception of $g_n + 2$.*

Proof. See [52]. $\square$

Let $p \geq 1$ be an integer, further let $\gamma = \{g_1, g_2, \dots, g_p\}$ be a set of integers with $0 \leq g_1 < g_2 < \dots < g_p$. Then $\mu_{dc}(\gamma) = \mu_{dc}\{g_1, g_2, \dots, g_p\}$ denotes the minimum order of disconnected graphs G for which $\gamma(G) = \gamma$.

In 2004 Manoussakis et al. investigated disconnected graphs. Let $\gamma = \{g_1, g_2, \dots, g_p\}$ be a nonempty set of nonnegative integers with $0 \leq g_1 < g_2 < \dots < g_p$ and let $\mu_{dc}(\gamma) = \mu_{dc}\{g_1, g_2, \dots, g_p\}$ denote the minimum order of a disconnected graph G for which $\gamma(G) = \gamma$.

Manoussakis, Patil and Sankar assert [32], that if g is a nonnegative integer, then $\mu(g) = 2(g+1)$. This assertion is not correct. We give the correct formula.

Theorem 5 *If g is a nonnegative integer, then $\mu(g) = g + 2$.*

Proof. If G is disconnected and has a vertex with degree g , then it has at least $(g+1)+1 = g+2$ vertices. But a star with $g+1$ vertices plus an isolated vertex form a corresponding graph. $\square$

For the case $p \geq 2$ Manoussakis et al. proved the following assertion.

Theorem 6 (Manoussakis, Patil, Sankar [30, 31, 32]). *Let $p \geq 2$ be an integer and $\gamma = \{g_1, g_2, \dots, g_p\}$ be a set of nonnegative integers with $g_1 < g_2 < \dots < g_p$. Then there exists a disconnected graph G such that $\gamma(G) = \gamma$. Further, $\mu_{dc} = g_1 + g_p + 2$.*

Proof. See [31, 32]. $\square$

Manoussakis, Patil and Sankar [30, 32] also investigated degree sets for k -connected graphs, k -edge connected graphs, unicyclic graphs and maximal k -degenerate graphs.

Let m be a positive integer. An (m, γ) -graph has m edges and degree set γ . We denote by $e(\gamma)$ the least m , for which there exists an (m, γ) -graph, and by $l_e(\gamma)$ this least $e(\gamma)$.

In 2006 Tripathi and Vijay determined $l_e(\gamma)$ in the following cases:

- a) $|\gamma| \leq 3$;
- b) $t \geq 1$ is an integer and $\gamma = \{1, 2, \dots, t\}$;
- c) $\min(\gamma) \geq |\gamma|$.

Further, they gave lower and upper bounds for $l_e(\gamma)$ in all cases and exhibited the cases, when the bounds are tight.

In their paper Tripathi and Vijay use the following notations. If $s = [d_1, d_2, \dots, d_n]$ is an increasing degree sequence, then its short form is $s = [d_1^{m_1}, d_2^{m_2}, \dots, d_p^{m_p}]$, where $d_i^{m_i}$ denotes m_i copies of d_i .

Theorem 7 (Tripathi, Vijay [53]). *Let g be a nonnegative integer. If $\gamma = \{g\}$, then there exists a (q, γ) -graph if and only if*

$$e \in \begin{cases} \left\{ mg : m \geq \frac{g+1}{2} \right\} & \text{if } g \text{ is odd,} \\ \left\{ m \frac{g}{2} : m \geq g+1 \right\} & \text{if } g \text{ is even.} \end{cases}$$

In particular, $l_e(\{g\}) = \frac{1}{2}g(g+1)$.

Theorem 8 (Tripathi, Vijay [53]). *Let a and b be positive integers with $a > b$ and let $\gamma = \{a, b\}$. Then there exists a (q, γ) -graph if and only if q has the form $\frac{1}{2}(ma + mb)$, where m and n are positive integers, $m + n \geq a + b$, and*

$$1 \leq m \leq b \text{ or } m \geq a + 1 \text{ or } m(a + 1 - m) \leq mb \text{ with } b + 1 \leq m \leq a.$$

In particular,

$$l_q(\{a, b\}) = \begin{cases} \frac{a(b+1)}{2} & \text{if } a \text{ is even or } b \text{ is odd,} \\ \frac{a(b+1) + (a-b)}{2} & \text{if } a \text{ is odd or } b \text{ is even.} \end{cases}$$

Theorem 9 (Tripathi, Vijay [53]). *Let t be a positive integer and let $\gamma = \{1, 2, \dots, t\}$. Then there exists an (e, α) -graph if and only if*

$$e \geq \left\lceil \frac{t}{2} \right\rceil + 1.$$

In particular, $l_e(\gamma) = \lceil \frac{t}{2} \rceil (\lfloor \frac{t}{2} \rfloor + 1)$.

In their paper Tripathi and Vijay constructed a special $\gamma = \{g_1, g_2, \dots, g_t\}$ and determined its $l(\gamma)$ as follows:

$$l(\gamma) = \begin{cases} \frac{1}{2}p_0(\gamma) & \text{if } p_0(\gamma) \text{ is even} \\ (\frac{1}{2}p_0(\gamma) + g_r - g_p) & \text{if } p_0(\gamma) \text{ is odd and } g_p \text{ is even} \\ \min(\frac{1}{2}p_0(\alpha) + a_r - a_t, \frac{1}{2}(p_0(\alpha) + a_p)) & \text{if both } p_0(\gamma) \text{ and } g_p \text{ are odd.} \end{cases}$$

Their following result shows that the above bounds are achieved for infinite number of sets.

Theorem 10 (Tripathi, Vijay [53]). *Let γ be a finite set of positive integers such that $\min(\gamma) \geq |\gamma|$. Then $l_e(\gamma) = l(\gamma)$.*

The proofs of these theorems due to Tripathi and Vijay can be found in [53].

In 2011 Volkmann extended Theorem 1 to multigraphs, proving the following assertion.

Theorem 11 (Volkmann [59]) *Let $p \geq 1$ and integer and $\gamma = \{g_1, g_2, \dots, g_p\}$ be a set of integers such that*

$$g_1 > g_2 > \dots > g_p \geq 1.$$

- (i) If $k = 1$, then γ is the degree set of a multigraph of order two.
 Assume now that $k \geq 2$ and $g_1 \leq \sum_{i=2}^p g_i$.
 (ii) If $\sum_{i=2}^p g_i$ is even, then γ is the degree set of a multigraph of order k .
 (iii) If $\sum_{i=2}^p g_i$ is odd, then γ is the degree set of a multigraph of order $k + 1$.
 Next assume that $k \geq 2$ and $g_1 > \sum_{i=2}^p g_i$.
 (iv) If $g_1 + \sum_{i=2}^k g_i$ even, then γ is the degree set of a multigraph of order $k + 1$.
 In addition, assume in the following that $g_1 + \sum_{i=1}^p g_i$ is odd.
 (v) Let $\sum_{i=1}^k g_i$ be even. If there exists an index $2 \leq k$ such that g_j is even and $g_1 \leq g_j + \sum_{i=2}^k g_i$, then δ is the degree set of a multigraph of order $k + 1$. If there is no such index, then γ is the degree set of a multigraph of order $k + 2$.
 (vi) Let $\sum_{i=1}^k g_i$ be odd. If there exists an index $2 \leq j \leq k$ such that g_j is odd and $g_1 + \sum_{i=2}^k g_i$, then γ is the degree set of a multigraph of order $k + 2$.
 In all cases of the multigraph is the least possible one.

Proof. See [59]. □

The *girth* of a graph is defined as the length of a shortest cycle in the graph. For integers $r \geq 2$ and $g \geq 3$ $f(r, g)$ is defined as the smallest order of an r -regular graph, having girth g . Such graphs are called *cages* [5, 37]. In 1963 Erdős and Sachs [5, 10] not only proved the existence of all cages but gave an upper bound for their order.

Theorem 12 (Erdős, Sachs [10]). *If $r \geq 2$ and $g \geq 3$, then*

$$1 + r \sum_{t=0}^{\lceil (g-3)/2 \rceil} (r-1)^t \leq f(r, g) \leq 4 \sum_{t=1}^g (r-1)^t.$$

Proof. See [10]. □

Erdős and Sachs remarked that Theorem 12 can be improved using the method proposed by Ferenc Kárteszi [26].

For $k \geq 1$, $n \geq 3$ and a set of integers $\gamma = \{g_1, g_2, \dots, g_k\}$ with $2 \leq g_1 < \dots < g_k$ we define

$$f(\gamma, g) = f(g_1, g_2, \dots, g_k; g)$$

to be the smallest order of graph having girth g and degree set D .

In 1981 Chartrand, Gould and Kapoor proved the following four theorems on the values of $f(D, g)$.

Theorem 13 (Chartrand, Gould, Kapoor [3]) *If $n \geq 1$ and $\gamma = \{g_1, g_2, \dots, g_n\}$ is a set of integers with $2 \leq g_1 < g_2 < \dots < g_n$, then $f(\gamma; 3) = 1 + a_n$.*

Theorem 14 (Chartrand, Gould, Kapoor [3]) *For $m \geq 3$ and $n \geq 3$*

$$f(2, m; n) = \begin{cases} \frac{m(n-2)+4}{2} & \text{if } n \text{ is even,} \\ \frac{m(n-1)+2}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 15 (Chartrand, Gould, Kapoor [3]) *If $2 \leq s$, then*

$$f(r, s; 4) = s.$$

Theorem 16 (Chartrand, Gould, Kapoor [3]) $f(3, 4; 5) = 13$, $f(3, 4; 6) = 18$.

In 1982 Wang published a survey [61] on the results connected with cages.

In 1988 Chernyak [6] continued the investigation of $f(r, g)$.

A graph having the minimum number of vertices in the class of graphs with degree set $\gamma = \{g_1, g_2, \dots, g_p\}$ and girth m is called a $(\gamma; p)$ -cage; the order of a $(\gamma; p)$ -cage is denoted by $f(\gamma; p)$. In this paper some new values of the function f are determined constructively: $f(3, 4, r; 5) = 3k + 1$ for $k = 5$ and $r = 4$, as well as for $k \geq 6$ and $4 \leq r \leq 3 + 2\lceil(k-5/3)/2\rceil$; $f(3, 4, k; 6) = 4k + 1$ for $k \geq 5$; $f(3, k; 6) = 4k + 2$ for $k \geq 4$; $f(3, 4, k; 7) = 7k + 1$ for $k \geq 4$, and $f(3, 4; 8) = 39$.

In 1985 Mynhardt [34] determined the condition of the existence of a degree uniform graph having prescribed global degree set.

A *signed graph* G is a graph in which to each edge is assigned a positive or negative sign. The set of distinct signed degrees D of a signed graph G is called its *global signed degree set*.

The concept of signed graphs was introduced and firstly characterized by Harary in 1953 [15]. In the first paper he proved the following assertions. According to his paper a signed graph, $G = (V, L^+, L^-)$ consists of a vertex set $V = \{V_1, V_2, \dots, V_n\}$, and two disjoint sets of edges L^+ and L^- .

Theorem 17 (Harary [15]) *A complete signed graph is balanced if and only if its vertex set V can be partitioned into two disjoint subsets V_1 and V_2 , one of which may be empty, such that all edges between the vertices of the same subset are positive, and all edges between vertices of the two different subsets are negative.*

Proof. See [15]. □

Theorem 18 (Harary [15]) *A signed graph is balanced if and only if for each pair of distinct vertices A and B all paths joining A and B are positive.*

Proof. See [15]. □

Theorem 19 (Harary [15]) *A signed graph is balanced if and only if its vertex set can be partitioned into two disjoint subsets V_1 and V_2 in such a way that each positive edge of G joins two vertices of different subsets.*

Proof. See [15]. □

In 1955 Harary [16] presented enumeration results on the different types of graphs including also signed graphs.

A sequence $\sigma = (d_1, d_2, \dots, d_n)$ of integers is *standard*, if it is nonincreasing, the sum of its element is even, $d_1 > 0$, each $|d_i| < n$, and $|d_1| \geq |d_n|$.

In 1968 Chartrand et al. published the following assertion, similar to the well-known theorem of Hakimi [13] for the degree sets of graphs.

Theorem 20 (Chartrand, Gavlas, Harary, Schulz [2]) *If $\gamma = (g_1, g_2, \dots, g_p)$ is a standard sequence, then there exists a signed graph G with global signed degree set γ if and only if there exists a signed graph G' with signed global degree set*

$$\sigma' = (g_2 - 1, g_3 - 1, \dots, g_{g_1+s+1} - 1, g_{g_1+s+2}, \dots, g_{p-s}, \dots, g_{p-s+1} + 1, \dots, g_{p+1})$$

for some s , $0 \leq s \leq \frac{p-1-g_1}{2}$.

Theorem 21 (Yan, Lih, Kuo, Chang [62]) *let γ be a standard sequence. There exists a signed graph with global signed degree sequence γ if and only if there exist integers r and s with $g_1 = r - s$, $0 \leq s \leq \frac{p-1-g_1}{2}$ such that there exist a a isigned graph G' with global signed score set*

$$\gamma' = \{g_2 - 1, g_3 - 1, \dots, g_{g_1+m+1} - 1, g_{g_1+m+2}, \dots, g_{p-m}, g_{p-m+1} + 1, g_p + 1\}.$$

In 2007 Pirzada et al. improved Theorem 1 proved by Kapoor, Polimeni and Wall in 1977.

Theorem 22 (Pirzada, Naikoo, Dar [47]) *Let p be a positive integer and $\gamma = \{g_1, g_2, \dots, g_p\}$ be a set of integers with $g_1 < g_2 < \dots < g_p$. Then γ is the signed global score set of some connected signed graph G and the minimal order of such signed graphs is $g_p + 1$.*

Proof. See [47] □

In 2013 Kumar, Sarma, and Sawlami [29] studied the number of vertices and multiplicity of degrees as parameters of directed and undirected tree realizations of prescribed degree sets.

2 Bipartite graphs with prescribed global degree sets

A graph $B(V, E)$ is said to be bipartite (or bigraph or 2-partite graph) if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 with $V = V_1 \cup V_2$ so that if $uv \in E$, then u and v belong to different vertex sets. We will use the notation $B(V_1, V_2, E)$. A bipartite graph is *complete* if $uv \in E$ for every $u \in V_1$ and every $v \in V_2$. If $|V_1| = n_1$ and $|V_2| = n_2$, then the complete bipartite graph is denoted by K_{n_1, n_2} . Examples of bipartite graphs are trees, cycle graphs with even number of vertices, planar graphs whose faces all have even length (special cases of this are grid graphs and square graphs), hypercube graphs, partial cubes and median graphs. Bipartite graphs can be characterized in several different ways such as (i) A graph is bipartite if and only if it does not contain an odd cycle, (ii) A graph is bipartite if and only if it is 2-colorable.

The set of distinct degrees $\{g_1, g_2, \dots, g_p\}$ of $B = (V_1, V_2, E)$ is called the *global degree set* of B and is denoted by $\gamma(B)$ (or simply by γ). For any non-empty subset U of $V_1 \cup V_2$ $\gamma(U)$ denotes the set of degrees of vertices in U . Then, the *global degree set* of a bipartite graph B with a bipartition (V_1, V_2) is the set $\gamma(B)$ which is the union of the sets of degrees in V_1 and in V_2 , i.e. $\gamma(B) = \gamma(V_1) \cup \gamma(V_2)$.

In 1977 Kapoor et al. proved the following assertion on the existence of trees (ie., connected, bipartite acyclic graphs) having prescribed global degree set.

Theorem 23 (Kapoor, Polimeni, Wall [25]) *Let $\gamma = \{g_1, g_2, \dots, g_p\}$ be a non-empty set of positive integers. Then there exists a nontrivial tree T (i.e. a connected, acyclic bipartite graph) with global degree set $\gamma(T) = \gamma$ if and only if $1 \in \gamma$. Further, if $1 \in \gamma$, then the minimum order of a nontrivial tree T with $\gamma(T) = \gamma$ is $\sum_{i=1}^n (g_i - 1) + 2$.*

Proof. See [25]. □

If $q \geq 2$, then every even cycle C_{2q} is bipartite with $\gamma(C_{2q}) = \{2\}$ and moreover, $\mu(C_{2q}) = 4$.

In 1979 Kapoor and Lesniak [24] studied the minimal order of bipartite graphs, having a prescribed global degree set. They received partial results: in

some special cases determined the minimal order of triangle-free graphs having prescribed degree set.

In 1994 Ellis [9] published a paper on layered graphs called by him $(k, 2)$ -partite graphs in which he proposed effective sequential and parallel algorithms to decide whether a given graph is $(k, 2)$ -layered, and also for the effective solution of several connected problems.

In 2007 Pirzada, Naikoo and Dar proved the following assertion.

Theorem 24 (Pirzada, Naikoo, Dar [48]) *Every set of positive integers is the global degree set of some connected bipartite graph.*

Proof. See [48]. □

Recently Manoussakis and Patil determined the families of connected unicyclic bipartite graphs having prescribed global degree set.

Theorem 25 (Manoussakis, Patil [33]). *Let $p \geq 2$ be an integer and $\gamma = \{g_1, g_2, \dots, g_p\}$ be a set of positive integers with $g_1 < g_2 < \dots < g_p$. Then there exists a connected unicyclic bipartite graph B with $\gamma(B) = \gamma$ if and only if either (a) or (b) below holds:*

- a) $p = 2$, $g_1 = 1$ and $g_2 \geq 3$. In this case $\mu(\gamma) = 4(g_2 - 1)$.
- b) $p \geq 3$ and $g_1 = 1$. In this case

$$\mu(\gamma) = \begin{cases} 3g_2 + g_3 - 4, & \text{if } p = 3, \\ 2g_2 + g_3 + g_4 - 4, & \text{if } p = 4, \\ \sum_{i=2}^n (g_i - 1), & \text{if } p \geq 5. \end{cases}$$

Proof. See [33]. □

The paper of Manoussakis and Patil contains the following lemma too.

Lemma 26 (Manoussakis, Patil [33]). *For any given positive integer n , there exists a complete bipartite graph B with bipartition (X, Y) such that $\gamma(B) = \{n\}$ if and only if $n = |X| = |Y|$.*

Proof. See [33]. □

Here we prove that every finite set of positive integers is the global degree set of some connected bipartite graph. Our approach is constructive and is different from that used in [33, 48, 39].

At first we prove a useful lemma.

Lemma 27 *If $g_1, g_2, \dots, g_p$ is a nonempty set of nonnegative integers with $0 \leq g_1 < g_2 < \dots < g_p$, then there exists a bipartite graph $B = (V_1, V_2, E)$ with global degree set $\gamma(B) = \left\{ g_1, \sum_{i=1}^2 g_i, \dots, \sum_{i=1}^p g_i \right\}$.*

Proof. We consider two cases, (a) when $g_1 = 0$ and (b) $g_1 > 0$.

Case (a). Let $g_1 = 0$. We have three subcases to consider.

(i) Suppose $p = 1$. We choose the null bipartite graph $G(V_1, V_2, E)$ with $|V_1| = |V_2| = 1$ and $E = \emptyset$. In this case the degrees of the vertices $v_1 \in V_1$ and $v_2 \in V_2$ are $g_{v_1} = g_{v_2} = 0 = g_1$. So degree set of $G(V_1, V_2, E)$ is $\gamma = g_1$.

(ii) Now let $n = 2$. We construct a bipartite graph $G(V_1, V_2, E)$ as follows.

Let $V_1 = X_1 \cup X_2, V_2 = Y_1$ with $X_1 \cap X_2 = \emptyset, |X_1| = 1, |X_2| = |Y_1| = g_2$. Take an edge from each vertex of X_2 to every vertex of Y_1 . The degrees of the vertices of $G(V_1, V_2, E)$ are as follows.

For $x_1 \in X_1, g_{x_1} = 0 = g_1$ and for all $x_2 \in X_2, g_{x_2} = |Y_1| = g_2 = g_1 + g_2$; and for all $y_1 \in Y_1, g_{y_1} = |X_2| = g_2 = g_1 + g_2$.

Thus the degree set of $G(V_1, V_2, E)$ is $\gamma = \{g_1, g_1 + g_2\}$.

(iii) For $n \geq 3$, we construct a bipartite graph $G(V_1, V_2)$ whose

$$V_1 = \left(\bigcup_{i=1}^p X_i \right) \bigcup \left(\bigcup_{i=3}^n X'_i \right) \text{ and } V_2 = \left(\bigcup_{i=1}^{n-1} Y_i \right) \bigcup \left(\bigcup_{i=2}^{n-1} Y'_i \right),$$

where for all $i \neq j, X_i \cap X_j = \emptyset, X_i \cap X'_j = \emptyset, Y_i \cap Y'_j = \emptyset, Y'_i \cap X'_j = \emptyset$;

for all $i, 2 \leq i \leq p, |X_1| = 1, |X_i| = |Y_{i-1}| = d_i$;

for all $i, 3 \leq i \leq p, |X'_i| = |Y'_{i-1}| = \sum_{r=2}^{i-1} g_r$.

We choose an edge from each vertex of X_i to every vertex of Y_j whenever $i \geq j$; an edge from each vertex of X'_i to every vertex of Y_{i-1} for all $i, 3 \leq i \leq p$; and an edge from each vertex of X'_i to every vertex of Y'_{i-1} for all $i, 3 \leq i \leq p$.

The degrees of the vertices of the bipartite graph $G(V_1, V_2, E)$ constructed above are as follows.

For $x_1 \in X_1, g_{x_1} = 0 = g_1$ and for $2 \leq i \leq p$, for all $x_i \in X_i$,

$$\begin{aligned} g_{x_i} &= \sum_{j=1}^{i-1} |Y_j| = \sum_{j=1}^{i-1} |g_{j+1}| \\ &= g_2 + g_3 + \cdots + g_i = g_1 + g_2 + \cdots + g_i; \end{aligned}$$

for $3 \leq i \leq n$, for all $x'_i \in X'_i$,

$$\begin{aligned} d(x'_i) &= |Y_{i-1}| + |Y'_{i-1}| \\ &= g_i + g_2 + g_3 + \cdots + g_{i-1} \\ &= g_1 + g_2 + g_3 + \cdots + g_i; \end{aligned}$$

for all $y_1 \in Y_1$,

$$\begin{aligned} d(y_1) &= \sum_{i=2}^n |X_i| = \sum_{i=2}^p g_i \\ &= g_2 + g_3 + \cdots + g_n \\ &= g_1 + g_2 + g_3 + \cdots + g_n; \end{aligned}$$

for $2 \leq i \leq n-1$, for all $y_i \in Y_i$

$$\begin{aligned} d(y_i) &= \sum_{j=i+1}^p |X_j| + |X'_{i+1}| = \sum_{j=i+1}^p g_j + (g_2 + \cdots + g_i) \\ &= g_{i+1} + g_{i+1} + \cdots + g_p + g_2 + g_3 + \cdots + g_i \\ &= g_1 + g_2 + \cdots + g_p; \end{aligned}$$

for $2 \leq i \leq p-1$, for all $y'_i \in Y'_i$

$$\begin{aligned} d(y'_i) &= |X'_{i+1}| = g_2 + \cdots + g_i \\ &= g_1 + g_2 + g_3 + \cdots + g_i. \end{aligned}$$

Therefore we see that the degree set of $G(V_1, V_2)$ is

$$\gamma = \{g_1, \sum_{i=1}^2 g_i, \dots, \sum_{i=1}^p g_i\}.$$

Case (b) Now let $g_1 > 0$. We have two subcases.

(i) Let $p = 1$. We consider the bipartite graph $G(V_1, V_2)$ with $|V_1| = |V_2| = g_1$ in which there is an edge from each vertex of V_1 to every vertex of V_2 . Here the degrees of the vertices of $G(V_1, V_2)$ are given as $d(v_1) = |V_2| = g_1$ and $d(v_2) = |V_1| = g_1$, for all $v_1 \in V_1, v_2 \in V_2$. Thus the degree set of $G(V_1, V_2)$ is $D = g_1$.

(ii) Let $p \geq 2$. Consider the bipartite graph $G(V_1, V_2)$ whose

$$\begin{aligned} V_1 &= \left(\bigcup_{i=1}^n X_i \right) \bigcup \left(\bigcup_{i=2}^n X'_i \right), \\ V_2 &= \left(\bigcup_{i=1}^p Y_i \right) \bigcup \left(\bigcup_{i=2}^p Y'_i \right), \end{aligned}$$

where (for $i \neq j$) each $X_i \cap X_j$, $X_i \cap X'_j$, $X'_i \cap X'_j$, $Y_i \cap Y_j$, $Y_i \cap Y'_j$ and $Y'_i \cap Y'_j$ are empty, for all i , $1 \leq i \leq p$, $|X_i| = |Y_i| = g_i$. Also for all i , $2 \leq i \leq p$, we have $|X'_i| = |Y'_i| = g_1 + g_2 + \cdots + g_{i-1}$.

Take an edge from each vertex of X_i to every vertex of Y_j whenever $i \geq j$; an edge from each vertex of X'_i to every vertex of Y_i for all i , $2 \leq i \leq n$ and an edge from each vertex of X'_i to every vertex of Y'_i for all i , $2 \leq i \leq p$. The following are the degrees of the vertices of the bipartite graph $G(U, V)$ constructed above.

For $1 \leq i \leq p$, for all $x_i \in X_i$,

$$d(x_i) = \sum_{j=1}^{i-1} |Y_j| = \sum_{j=1}^i g_j = g_1 + g_2 + \cdots + g_i.$$

For $2 \leq i \leq p$, for all $x'_i \in X'_i$,

$$d(x'_i) = |Y_i| + |Y'_i| = g_i + (g_1 + g_2 + \cdots + g_{i-1}) = g_1 + g_2 + \cdots + g_i.$$

For $1 \leq i \leq p$, for all $y_i \in Y_i$,

$$\begin{aligned} d(y_i) &= \sum_{j=1}^n |X_j| + |X'_i| \\ &= \left(\sum_{j=1}^n g_j \right) + (g_1 + g_2 + \cdots + g_{i-1}) \\ &= (g_i + g_{i+1} + \cdots + g_n) + (g_1 + g_2 + \cdots + g_{i-1}) \\ &= g_1 + g_2 + \cdots + g_p. \end{aligned}$$

For $2 \leq i \leq p$, for all $y'_i \in Y'_i$,

$$d(y'_i) = |X'_i| = g_1 + g_2 + \cdots + g_{i-1}.$$

Therefore the degree set of the bipartite graph $G(U, V)$ constructed above is $D = \{g_1, \sum_{i=1}^2 g_i, \dots, \sum_{i=1}^p g_i\}$. $\square$

Using Lemma 27, we can prove the following assertion.

Theorem 28 *Every set of nonnegative integers is the global degree set of some bipartite graph.*

Proof. Let $\gamma = \{a_1, a_2, \dots, a_n\}$ be any set of distinct nonnegative integers. We choose

$$b_1 = a_2 - a_1, b_2 = a_3 - a_2, \dots, b_{p-1} = a_p - a_{p-1}.$$

Now γ can be rewritten as

$$\begin{aligned} \gamma &= \{a_1, a_2 - b_1 + b_1, a_3 - b_2 + b_2, \dots, a_p - b_{p-1} + b_{p-1}\} \\ &= \{a_1, a_1 + b_1, a_2 + b_2, \dots, a_{p-1} + b_{p-1}\} \\ &= \{a_1, (a_1 + b_1), (a_1 + b_1 + b_2), \dots, (a_1 + b_1 + b_2 + \dots + b_{p-1})\}. \end{aligned}$$

As seen in Theorem 1, the set $\gamma = \{a_1, (a_1 + b_1), (a_1 + b_1 + b_2), \dots, (a_1 + b_1 + b_2 + \dots + b_{p-1})\}$ is the global degree set of some bipartite graph which is equivalent to say that the set $\gamma = \{a_1, a_2, \dots, a_p\}$ is the global degree set of some bipartite graph. $\square$

Example 29 Consider $\gamma = \{0, 5, 7\}$. We can rewrite γ as $\gamma = \{0, 5+0, 5+0+2\}$ and apply Corollary 2, then we get the bipartite graph with degree set γ . In case $\gamma = \{3, 5, 10, 12\}$, we write γ as $\gamma = \{3, 3+2, 3+2+5, 3+2+5+2\}$.

From case (b) of Lemma 27, we have the following assertion.

Theorem 30 Every set of positive integers is the global degree set of some connected bipartite graph.

The following algorithm GLOBAL-BIPARTITE is based on Theorem 41. Therefore the algorithm is a slightly modified version of DISTRIBUTED-BIPARTITE. GLOBAL-BIPARTITE constructs a bipartite graph having prescribed global degree set.

Input. p : the number of elements in the prescribed degree set γ ;
 $\gamma = \{g_1, g_2, \dots, g_p\}$: the prescribed degree set for $B(V_1, V_2, E)$.

Output. $M(B)$: the incidence matrix of the constructed bipartite graph (V_1, V_2, B) .

n_1 : the number of lines of M , that is the size of the vertex set V_1 ;

n_2 : the number of columns of M , that is the size of the vertex set V_2 .

Work variables. i, j : cycle variables.

The pseudocodes of this paper are written using the conventions described in the textbook written by Cormen, Leiserson, Rivest, and Stein [7].

```

GLOBAL-BIPARTITE( $p, \gamma$ )
01  $\alpha = 0$  // lines 01–03: computation of  $\alpha$ 
02 for  $i = 1$  to  $p$ 
03    $\alpha = \alpha + g_i$ 
04  $v = \alpha^2$ 
08  $n_1 = v/p$ 
09  $n_2 = v/p$ 
10 for  $i = 1$  to  $v$  // lines 10–12: initialization of  $M$ 
11   for  $j = 1$  to  $v$ 
12      $M_{ij} = 0$ 
13  $i = 1$ 
14  $j = 1$ 
15  $x = n_1$ 
16  $y = n_2$ 
17 while  $j < v$ 
18   while  $x > 0$  and  $y > 0$ 
19      $M_{ij} = 1$ 
20      $x = x - 1$ 
21      $y = y - 1$ 
22      $j = j + 1$ 
23     if  $x == 0$ 
24        $x = n_1$ 
25        $i = i + 1$ 
26     if  $y == 0$ 
27        $y = n_2$ 
28 return  $\mu, M$ 

```

Theorem 31 *Let $\sum_{i=1}^p g_i = s$. Then the running time of GLOBAL-BIPARTITE is $\Theta(s^2/p^2)$ in all cases.*

Proof. See the proof of Theorem 43. □

Let m and n be positive integers. A *signed bipartite graph* $B = (U, V, E)$ with $U = \{u_1, u_2, \dots, u_{n_1}\}$ and $V = \{v_1, v_2, \dots, v_{n_2}\}$ is a bipartite graph in which to each edge is assigned a positive or negative sign. The *signed degree* of a u_i is defined as $g_{u_i} = g_i = g_i^+ - g_i^-$, where $1 \leq i \leq n_1$ and g_i^+ (g_i^-) is the number of positive (negative) edges incident to u_i , and the *signed degree* of a v_j is defined as $g_{v_j} = g_j = g_j^+ - g_j^-$, where $1 \leq j \leq n_2$ and g_j^+ (g_j^-) is the number of positive (negative) edges incident to v_j . The *global degree set* γ of a signed bipartite graph is the set of its distinct signed degrees.

In 2006 Pirzada et al. proved the following properties of signed bipartite graphs.

Theorem 32 (Pirzada, Naikoo, Dar [46, 49]) *Let $\gamma = \{g_1, g_2, \dots, g_p\}$ be a nonempty set set of positive (or negative) integers. Then γ is the signed global degree set of some signed bipartite graph, and the minimal order of such graphs is $1 + \max_{1 \leq i \leq p} |g_i|$.*

Proof. See [49]. □

Theorem 33 (Pirzada, Naikoo, Dar [46, 49]) *Let p be a positive integer and $\gamma = \{g_1, g_2, \dots, g_p\}$ be a set of negative integers. Then γ is the signed global degree set of some signed bipartite graph, and the minimal order of such graphs is $|\min(\gamma)|$.*

Proof. See [46, 49]. □

As the following assertion shows, the requirement of the identical sign of the elements of the score set can be removed.

Theorem 34 (Pirzada, Naikoo, Dar [46, 49]) *Let γ be a set of integers. Then γ is the signed global degree set of some signed bipartite graph.*

Proof. See [46, 49]. □

In 2008 Pirzada et al. published the followig result.

Theorem 35 (Pirzada, Naikoo, Dar [46, 49]) *Let p be a positive integer and $\gamma = \{g_1, g_2, \dots, g_p\}$ be a set of positive integers. Then $\sum_{i=1}^1 g_i, \sum_{i=1}^2 g_i, \dots, \sum_{i=1}^p g_i$ is the signed global degree set of some signed bipartite graph, and the minimal order of such graphs is $|\min(\gamma)|$.*

Proof. See [49]. □

3 Bipartite graphs with prescribed distributed degree set

Let p be a positive integer and $B = (V_1, V_2, E)$ a bipartite graph, where $\delta(V_1) = \{a_1, a_2, \dots, a_p\}$, and $\delta(V_2) = \{b_1, b_2, \dots, b_p\}$. Then the pair (δ_1, δ_2) is called the *distributed degree set* of B .

Given a pair (δ_1, δ_2) of finite, nonempty sets of positive integers, let $\mu(\delta_1 \cup \delta_2) = \min\{|B| : B \in \mathcal{B}, \text{ where } \mathcal{B} \text{ is the family of all bipartite graphs } B \text{ with}$

$\delta(B) = (\delta_1 \cup \delta_2)$. Manoussakis and Patil [33] have shown for a given pair (δ_1, δ_2) of finite, nonempty sets of positive integers of same cardinality the existence of a bipartite graph $B(V_1, V_2)$ such that $\delta(V_1) = \delta_1$ and $\delta(V_2) = \delta_2$ and obtained the minimum orders of different types of such graphs.

Theorem 36 (Manoussakis, Patil [33]) *Let $\delta_1 = \{a_1, a_2, \dots, a_p\}$ and $\delta_2 = \{b_1, b_2, \dots, b_p\}$ be nonempty sets of positive integers with $a_1 < a_2 < \dots < a_p$, and $b_1 < b_2 < \dots < b_p$. Then there exists a bipartite graph $B = (V_1, V_2, E)$ with distributed score set (δ_1, δ_2) . Furthermore, B is connected if and only if the minimum order $\mu(\delta_1 \cup \delta_2) = a_p + b_p$, where $|V_1| = a_p$, and $|V_2| = b_p$.*

Proof. See [33]. The proof of 36 begins with the interesting remark that $B = \bigcup (\bar{K}_{a_i} + \bar{K}_{b_i})$ satisfies the required property. Then comes an inductive proof which is not correct. For example on page 387 in 6th and 5th lines from below if $n = 3$, then $3 \leq m < n$ is meaningless. On the next page in the third line a_1u is also an error. $\square$

Manoussakis and Patil published also the following corollaries of Theorem 36 (the proofs can be seen in the same paper [33]).

Corollary 37 *Let $\delta_1 = \{a_1, a_2, \dots, a_p\}$, and $\delta_2 = \{b_1, b_2, \dots, b_p\}$ be nonempty sets of different positive integers such that $a_1 < a_2 < \dots < a_p$, and $b_1 < b_2 < \dots < b_p$. Then there exists a connected, bipartite graph $B(V_1, V_2, E)$ of order $a_p + b_p$ such that $\delta(V_1) = \delta_1$ and $\delta(V_2) = \delta_2$, where $|V_1| = a_p$ and $|V_2| = b_p$.*

Corollary 38 *Let $\delta = \{a_1, a_2, \dots, a_p\}$ be nonempty set of positive integers with $a_1 < a_2 < \dots < a_p$. Then there exists a connected, bipartite graph B with bipartition (V_1, V_2) such that the global degree sets $\delta(V_1)$ and $\delta(V_2)$ are different, and the global degree set $\delta(B)$ is δ . Moreover, the minimum order of B with $\delta(B) = \delta$ is*

$$\mu(\delta) = \begin{cases} a_{p/2} + a_p & \text{if } p \text{ is even,} \\ a_{\lceil p/2 \rceil} + a_p & \text{if } p \text{ is odd.} \end{cases}$$

Corollary 39 *Let $\delta = \{a_1, a_2, \dots, a_p\}$ be a nonempty set of positive integers with $1a_1 < a_2 < \dots < a_p$. Then there exists a bipartite graph $B = (V_1, V_2, E)$ such that $\delta(B) = \delta$. Furthermore, $B(V_1, V_2, E)$ is a connected, bipartite graph such that $\delta(V_1) = \delta(V_2)$ if and only if its minimum order is $2a_p$ so that $|V_1| = |V_2| = a_p$.*

Corollary 40 *Let $\delta_1 = \{a_1, a_2, \dots, a_p\}$ be a nonempty set of positive integers with $a_1 < a_2 < \dots < a_p$. Then there exists a connected bipartite graph $B(V_1, V_2, E)$ of order $2a_p$, such that $\delta(V_1) = \delta(V_2) = \delta$, where $|V_1| = |V_2| = a_p$.*

It is worth to mention, that this corollary is not true: e.g. if $\delta_1 = \{-5, -3\}$, then "order $2a_n$ " is meaningless, therefore in the lemma we substituted "the set of integers" with "a set of positive integers".

Manoussakis and Patil finished their paper so: "For arbitrary sets δ_1 and δ_2 of positive integers with **different** cardinalities, the problem of determining a bipartite graph that holds the property in Theorem 36 is open."

Our next theorem shows, that the existence of a bipartite graph having prescribed distributed degree set does not require the condition $|\delta_1| = |\delta_2|$.

Theorem 41 *Let $\delta_1 = \{a_1, a_2, \dots, a_p\}$ and $\delta_2 = \{b_1, b_2, \dots, b_q\}$ be nonempty sets of nonnegative integers. Then there exists a bipartite graph $B = (V_1, V_2, E)$ such that $\delta(V_1) = \delta_1$ and $\delta(V_2) = \delta_2$.*

Proof. If $a_1 = 0$, then we delete a_1 from δ_1 resulting $\delta'_1 = \{c_1, c_2, \dots, c_{p-1}\}$. If $b_1 = 0$, then we delete b_1 from δ_2 resulting $\delta'_2 = \{d_1, d_2, \dots, d_{q-1}\}$.

Let $\alpha = \sum_{i=1}^p a_i$, $\beta = \sum_{j=1}^q b_j$, $\mu = \alpha\beta$, $V_1 = \{u_1, u_2, \dots, u_\mu\}$ and $V_2 = \{v_1, v_2, \dots, v_\mu\}$. Let the multiplication factor of V_1 be $m_1 = \mu/p$, the multiplication factor of V_2 be $m_2 = \mu/q$, the degree sequence of V_1 be $\sigma_1 = [c_1^{n_1}]$

Now let consider the bipartite graph $B(V_1, V_2, E)$, where $|V_1| = p\beta$, and $|V_2| = q\alpha$, the degree sequence of V_1 is $\sigma_1 = (a_1^\beta, a_2^\beta, \dots, a_p^\beta) = (e_1, e_2, \dots, e_\mu)$, and the degree sequence of V_2 is $\sigma_2 = (b_1^\alpha, b_2^\alpha, \dots, b_q^\alpha) = (f_1, f_2, \dots, f_\mu)$. Let $V_1 = \{u_1, u_2, \dots, u_\mu\}$ and $V_2 = \{v_1, v_2, \dots, v_\mu\}$.

We construct the edge set of B as follows. We connect in cyclical order u_1 with the next e_1 vertices in V_2 (that is with $v_1, v_2, \dots, v_{e_1}$), then connect u_2 with the next e_2 vertices in V_2 (that is with $v_{e_1+1}, v_{e_1+2}, \dots, v_{e_1+e_2}$) and so on. $\square$

It is a simple observation, that if $0 \in \delta_1 \cup \delta_2$, then the graphs with distributed degree set (δ_1, δ_2) are never connected.

The following example shows that the absence of zero in the prescribed distributed degree set is not sufficient to guarantee the existence of a corresponding connected bipartite graph.

Example 42 *Let $\delta_1 = \{1\}$ and $\delta_2 = \{1, 2\}$. In all construction we have to connect vertices whose degree is 1, and so this pair of vertices will not be connected with the remaining part of the constructed graph.*

The following program constructs a bipartite graph having a prescribed distributed degree set.

Input. p : the number of elements in the prescribed degree set δ_1 of V_1 ;
 q : the number of elements in the prescribed degree set δ_2 of V_2 ;

$\delta_1 = \{a_1, a_2, \dots, a_p\}$: prescribed degree set for V_1

$\delta_2 = \{b_1, b_2, \dots, b_q\}$: prescribed degree set for V_2 .

Output. $\mu = \alpha\beta$: the number of rows and columns of M , that is the common length of the degree sequence of V_1 and V_2 ;

M : the incidence matrix of the constructed bipartite graph $B = (V_1, V_2, E)$.

Work variables. i, j : cycle variables;

α : the sum of the elements of δ_1 ;

β : the sum of the elements of δ_2 ;

ν : the common length of the degree sequence of V_1 and V_2 , and so the common number of rows and columns in the incidence matrix M ;

$n_1 = \nu/p$: the multiplication factor of the degree sequence of V_1 ;

$n_2 = \nu/q$: the multiplication factor of the degree sequence of V_2 .

DISTRIBUTED-BIPARTITE(p, q, δ_1, δ_2)

```

01  $\alpha = 0$  // lines 01–03: computation of  $\alpha$ 
02 for  $i = 1$  to  $p$ 
03      $\alpha = \alpha + a_i$ 
04  $\beta = 0$  // lines 04–06: computation of  $\beta$ 
05 for  $i = 1$  to  $q$ 
06      $\beta = \beta + b_i$ 
07  $\nu = \alpha\beta$  // lines 07–09: computation of  $\beta, n_1$  and  $n_2$ 
08  $n_1 = \nu/p$ 
09  $n_2 = \nu/q$ 
10 for  $i = 1$  to  $\nu$  // lines 10–12: initialization of  $M$ 
11     for  $j = 1$  to  $\nu$ 
12          $M_{ij} = 0$ 
13  $i = 1$  // lines 13–16: initialization of  $i, j, x$ , and  $y$ 
14  $j = 1$ 
15  $x = n_1$ 
16  $y = n_2$ 
17 while  $j \leq \nu$  // lines 17–27: connecting of the vertices
18     while  $x > 0$  and  $y > 0$ 
19          $M_{ij} = 1$ 
20          $x = x - 1$ 
21          $y = y - 1$ 
22          $j = j + 1$ 
23         if  $x == 0$ 
24              $x = n_1$ 
25              $i = i + 1$ 

```

```

26         if y == 0
27             y = n2
28 return  $\mu, M$ 

```

Theorem 43 *The running time of DISTRIBUTED-BIPARTITE is in all cases $\Theta(v)$.*

Proof. The deciding parts of the running time are required by lines 10–12 and by lines 17–27 and both parts requires $\Theta(v)$ time. $\square$

Comparing with the algorithm proposed by Manoussakis and Patil disadvantage of DISTRIBUTED-BIPARTITE is that it constructs usually larger solution.

4 Tripartite graphs with prescribed global degrees

A graph $T(V, E)$ is said to be tripartite graph (or trigraph or 3-partite graph) if its vertex set V can be partitioned into three disjoint sets V_1 , V_2 , and V_3 with $V = V_1 \cup V_2 \cup V_3$ such that if $uv \in E$, $u \in V_i$ and $v \in V_j$, then $i \neq j$. A tripartite graph is *complete* if there is edge from each $u \in V_i$ to every $v \in V_j$ with $i \neq j$, $1 \leq i, j \leq 3$. If $|V_1| = n_1$, $|V_2| = n_2$ and $|V_3| = n_3$, then the complete bipartite graph is denoted by K_{n_1, n_2, n_3} .

The set of distinct degrees of T is called its *global degree set* and is denoted by $\gamma(T)$ (or simply γ).

In 2007 Pirzada, Naikoo and Dar proved the following assertions.

Theorem 44 (Pirzada, Naikoo, Dar [48]) *Let $\gamma\{g_1, g_2, \dots, g_p\}$ be a nonempty set of nonnegative integers. Then there exists a tripartite graph $T = (V_1, V_2, V_3, E)$ with global degree set γ .*

Proof. See [48] $\square$

Theorem 45 (Pirzada, Naikoo, Dar [48]). *Let $\gamma = \{g_1, g_2, \dots, g_p\}$ be a nonempty set of nonnegative integers with $g_2 g_3 \cdots g_p > 0$. Then there exists a tripartite graph whose global degree set is $\gamma = \{g_1, \sum_{i=1}^2 g_i, \dots, \sum_{i=1}^p g_i\}$.*

Proof. To prove the existence of such tripartite graphs, we consider two cases, (a) when $g_1 = 0$ and (b) $g_1 > 0$.

Case (a) Let $g_1 = 0$. We consider three subcases.

(i) Suppose $n = 1$. We choose the null tripartite graph $G(V_1, V_2, V_3, E)$ with $|V_1| = |V_2| = |V_3| = 1$. Here the degrees of the vertices $v_1 \in V_1$, $v_2 \in V_2$

and $v_3 \in V_3$ are $d_{v_1} = d_{v_2} = d_{v_3} = 0 = g_1$. So the global degree set of $G(V_1, V_2, V_3, E)$ is $\gamma = \{g_1\}$.

(ii) We now take $n = 2$. Consider a tripartite graph $G(V_1, V_2, V_3, E)$ with $|V_1| = 1, |V_2| = |V_3| = d_2$. Suppose there is an edge from each vertex of V_2 to every vertex of V_3 . We observe that the degrees of the vertices of $G(V_1, V_2, V_3, E)$ are as under.

For $v_1 \in V_1$, $d_{v_1} = 0 = g_1$; for all $v_2 \in V_2$, $d_{v_2} = |V_3| = g_2 = g_1 + g_2$; and for all $v_3 \in V_3$, $d_{v_3} = |V_2| = g_2 = g_1 + g_2$.

Thus the degree set of $G(V_1, V_2, V_3, E)$ is $D = \{g_1, g_1 + g_2\}$.

(iii) Now let $p \geq 3$. We construct a tripartite graph $G(V_1, V_2, V_3, E)$ whose

$$V_1 = \bigcup_{i=1}^n X_i, V_2 = \bigcup_{i=1}^{n-2} Y_i \text{ and } V_3 = Z_1 \bigcup \left(\bigcup_{i=2}^{n-1} Z_i \right) \bigcup \left(\bigcup_{i=2}^{p-1} Z'_i \right),$$

where for all $i \neq j$, $X_i \cap X_j = \emptyset$, $Y_i \cap Y_j = \emptyset$, $Z_i \cap Z_j = \emptyset$, $Z_i \cap Z'_j = \emptyset$, $Z'_i \cap Z'_j = \emptyset$;

for all $i, 2 \leq i \leq p$, $|X_1| = 1, |X_i| = |Z_{i-1}| = d_i$;

for all $i, 1 \leq i \leq p-2$, $|Y_i| = |Z'_{i+1}| = \sum_{r=2}^{i+1} g_r = g_2 + g_3 + \cdots + g_{i+1}$.

We choose an edge from each vertex of X_i to every vertex of Z_j whenever $i \geq j$; an edge from each vertex of Y_i to every vertex of Z_{i+1} for all $i, 1 \leq i \leq p-2$; and an edge from each vertex of Y_i to every vertex of Z'_{i+1} for all $i, 1 \leq i \leq p-2$.

The degrees of the vertices of the tripartite graph $G(V_1, V_2, V_3, E)$ constructed above are as follows.

For $x_1 \in X_1$, $d_{x_1} = 0 = g_1$;

and for $2 \leq i \leq n$, for all $x_i \in X_i$,

$$\begin{aligned} d_{x_i} &= \sum_{j=1}^{i-1} |Z_j| = \sum_{j=1}^{i-1} |g_{j+1}| \\ &= g_2 + g_3 + \cdots + g_i = g_1 + g_2 + \cdots + g_i; \end{aligned}$$

for $1 \leq i \leq n-2$, for all $y_i \in Y_i$,

$$\begin{aligned} d(y_i) &= |Z_{i+1}| + |Z'_{i+1}| \\ &= g_{i+2} + g_2 + g_3 + \cdots + g_{i+1} \\ &= g_1 + g_2 + g_3 + \cdots + g_{i+2}; \end{aligned}$$

for all $z_1 \in Z_1$,

$$\begin{aligned} d(z_1) &= \sum_{i=2}^n |X_i| = \sum_{i=2}^n g_i \\ &= g_2 + g_3 + \cdots + g_n \\ &= g_1 + g_2 + g_3 + \cdots + g_n; \end{aligned}$$

for $2 \leq i \leq n-1$, for all $z_i \in Z_i$

$$\begin{aligned} d(z_i) &= \sum_{j=i+1}^n |X_j| + |Y_{i-1}| = \sum_{j=i+1}^n g_j + (g_2 + \cdots + g_i) \\ &= g_{i+1} + g_{i+2} + \cdots + g_n + g_2 + g_3 + \cdots + g_i \\ &= g_1 + g_2 + \cdots + g_n; \end{aligned}$$

for $2 \leq i \leq n-1$, for all $z'_i \in Z'_i$

$$\begin{aligned} d(z'_i) &= |Y'_{i-1}| = g_2 + g_3 + \cdots + g_i \\ &= g_1 + g_2 + g_3 + \cdots + g_i. \end{aligned}$$

Thus we see that the degree set of $G(V_1, V_2, V_3)$ is

$$D = \{g_1, \sum_{i=1}^2 g_i, \dots, \sum_{i=1}^p g_i\}.$$

Case (b) Assume $g_1 > 0$. We have two subcases.

(i) Let $p = 1$. We consider the tripartite graph $G(V_1, V_2, V_3, E)$ with $V_1 = X_1$, $V_2 = Y_1$, $V_3 = Z_1 \cup Z_2$ and $Z_1 \cap Z_2 = \emptyset$, $|X_1| = |X_2| = |X_3| = 1$. In this graph, let there be an edge from each vertex of X_1 to every vertex of Z_1 , and from each each vertex of Y_1 to every vertex of Z_2 . Then the degrees of the vertices of $G(V_1, V_2)$ are given as $d(v_1) = |V_2| = g_1$ and $d(v_2) = |V_1| = g_1$, for all $v_1 \in V_1, v_2 \in V_2$. Thus the degree set of $G(V_1, V_2, V_3, E)$ is $\gamma = g_1$.

Now, let $p = 1$ and $g_1 > 1$. Consider the tripartite graph $G(V_1, V_2, V_3, E)$ with $|V_1| = 1$, $|V_2| = g_1 - 1$, $|V_3| = g_1$, and let there be an edge from each vertex of V_1 to every vertex of V_3 , and from each vertex of V_2 to every vertex of V_3 . The degrees of the vertices of this graph are as follows.

For $v_1 \in V_1$, we have $d(v_1) = |V_3| = g_1$; for all $v_2 \in V_2$, we have $d(v_2) = |V_3| = g_1$; and for all $v_3 \in V_3$, we have $d(v_3) = |V_1| + |V_2| = 1 + g_1 - 1 = g_1$. Therefore the degree set of $G(V_1, V_2, V_3, E)$ is $\gamma = g_1$.

(ii) Let $p \geq 2$. Consider the tripartite graph $G(V_1, V_2, V_3, E)$ whose

$$V_1 = \bigcup_{i=1}^n X_i, V_2 = \bigcup_{i=1}^{n-1} Y_i, \text{ and } V_3 = Z_1 \bigcup \left(\bigcup_{i=2}^p Z_i \right) \bigcup \left(\bigcup_{i=2}^p Z'_i \right),$$

where for all $i \neq j$, $X_i \cap X_j = \emptyset$, $Y_i \cap Y_j = \emptyset$, $Z_i \cap Z_j = \emptyset$, $Z_i \cap Z'_j = \emptyset$, $Z'_i \cap Z'_j = \emptyset$ for all i , $1 \leq i \leq p$, $|X_i| = |Z_i| = g_i$. Also for all i , $1 \leq i \leq p-1$, we have $|Y_i| = |Z_{i+1}| = g_1 + g_2 + \cdots + g_i$.

Take an edge from each vertex of X_i to every vertex of Z_j whenever $i \geq j$; an edge from each vertex of Y_i to every vertex of Z_{i+1} for all i , $2 \leq i \leq n-1$, and an edge from each vertex of Y_i to every vertex of Z'_{i+1} for all i , $1 \leq i \leq p-1$. The following are the degrees of the vertices of the tripartite graph $G(V_1, V_2, V_3, E)$ constructed above.

For $1 \leq i \leq p$, for all $x_i \in X_i$,

$$d(x_i) = \sum_{j=1}^i |Z_j| = \sum_{j=1}^i v_j = v_1 + v_2 + \cdots + v_i.$$

For $1 \leq i \leq n-1$, for all $y_i \in Y_i$,

$$d(y_i) = |Z_{i+1}| + |Z'_{i+1}| = g_{i+1} + (g_1 + g_2 + \cdots + g_{i+1}) = g_1 + g_2 + \cdots + g_{i+1}.$$

For all $z_1 \in Z_1$, we have

$$d(z_1) = \sum_{i=1}^p |X_i| = \sum_{i=1}^p g_i = g_1 + g_2 + \cdots + g_p.$$

For $2 \leq i \leq p$, for all $z_i \in Z_i$,

$$\begin{aligned} d(z_i) &= \sum_{j=1}^p |X_j| + |Y_{i-1}| \\ &= \left(\sum_{j=1}^p d_j \right) + (g_1 + g_2 + \cdots + g_{i-1}) \\ &= (g_i + g_{i+1} + \cdots + g_p) + (g_1 + g_2 + \cdots + g_{i-1}) \\ &= g_1 + g_2 + \cdots + g_p. \end{aligned}$$

For $2 \leq i \leq p$, for all $z'_i \in Z'_i$,

$$d(z'_i) = |Y_{i-1}| = g_1 + g_2 + \cdots + g_{i-1}.$$

Therefore the degree set of the tripartite graph $G(V_1, V_2, V_3, E)$ constructed above is $\gamma = \{g_1, \sum_{i=1}^2 g_i, \dots, \sum_{i=1}^p g_i\}$. $\square$

Corollary 46 (Pirzada, Naikoo, Dar [48]). *Every nonempty set of positive integers, except $\{1\}$, is the global degree set of some connected tripartite graph.*

Proof. Here we give a new, correct proof. In case $g_1 > 0$ in the proof of Theorem 44, the construction gives a connected tripartite graph.

If the global degree set of a tripartite graph is $\gamma = \{1\}$, then let u and v be two connected vertices. If we connect one of these vertices with any other vertex, then the degree of this vertex will be at least 2. $\square$

The following algorithm GLOBAL-TRIPARTITE is based on Theorem 44. It constructs a tripartite graph having prescribed global degree set.

Input. p : the number of elements in the prescribed degree set γ ;

$\gamma = \{g_1, g_2, \dots, g_p\}$: the prescribed degree set for $B(V_1, V_2, E)$.

Output. $M(B)$: the incidence matrix of the constructed tripartite graph (V_1, V_2, V_3, E) ;

n_1 : the number of lines of M , that is the size of the vertex set V_1 ;

n_2 : the number of columns of M , that is the size of the vertex set V_2 .

Work variables. i, j : cycle variables;

$n'_1 = \bigcup_i = 1^p |X_i|$;

$n'_1 = \bigcup_i = 1^p |X_i|$;

$n'_2 = \bigcup_{i=1}^{p-1} |X'_i|$;

$n''_2 = \bigcup_{i=3}^{p-1} |X''_i|$.

GLOBAL-TRIPARTITE($p, q, r, \delta_1, \delta_2, \delta_3$)

```

01 if  $a_1 == 0$                                      // lines 01–06: the case  $a_1 = 0$ 
02   if  $p == 1$                                      // lines 02–06: the subcase  $a_1 = 0$  and  $p = 1$ 
03      $n_1 = 1$ 
04      $n_2 = 1$ 
05      $M_{n_1, n_2} = 1$ 
06     return  $M$ 
07   if  $p == 2$                                      // lines 7–16: the subcase  $a_1 = 0$  and  $p = 2$ 
08      $n_1 = a_2 + 1$ 
09      $n_2 = a_2$ 
10     for  $i = 1$  to  $n_1$                              // lines 10–12: initialization of  $M$ 
11       for  $j = 1$  to  $n_2$ 
12          $M_{ij} = 0$ 
13     for  $i = 2$  to  $n_1$ 

```

```

14      for j = 1 to n2
15          Mij = 1
16      return M
17  if p == 3 // lines 17–33: the subcase a1 = 0 and p = 3
18      n'1 =  $\sum_{i=2}^p a_i$  // lines 18–23: computation of n1 and n2
19      n''1 =  $\sum_{i=3}^p \sum_{j=2}^i -1 a_i$ 
20      n1 = n'1 + n''1
21      n'2 =  $\sum_{i=1}^{n-1} a_i$ 
22      n''2 =  $\sum_{i=3}^n \sum_{j=2}^{i-1} a_j$ 
23      n2 = n'2 + n''2
24      for i = 1 to n1 // lines 24–26: initialization of M
25          for j = 1 to n2
26              Mij = 0
27      for i = 2 to n1 // lines 27–32: drawing of the edges of M
28          for j = 1 to i
29              Mij = 1
30      for i = n'1 + 1 to n1
31          for j = n'2 to n2
32              Mij = 1
33      return n1, n2, M // line 33: return of the result

```

Theorem 47 *The running time of GLOBAL-BIPARTITE is $\Theta(1)$ in best case and $\Theta(n_1 n_2)$ in worst case.*

Different authors proved that signed tripartite graphs have similar properties, then tripartite graphs.

Theorem 48 (Pirzada, Dar [38]) *Let $\gamma = \{g_1, g_2, \dots, g_p\}$ be a nonempty set of positive integers. Then there exists a connected signed tripartite graph with signed global degree set $\left\{ \sum_{i=1}^1 g_i, \sum_{i=1}^2 g_i, \dots, \sum_{i=1}^p g_i \right\}$.*

Proof. See [38]. □

The next result follows from Theorem 48 by interchanging positive edges with negative ones.

Corollary 49 (Pirzada, Dar [38]) *Every set of negative integers is the global degree set of some connected signed tripartite graph.*

Proof. See [38] □

Pirzada and Dar proved the following stronger assertion too.

Theorem 50 (Pirzada, Dar [38]) *Let $\gamma = \{g_1, g_2, \dots, g_p\}$ be a nonempty set of positive integers. Then there exists a connected signed tripartite graph with signed global degree set γ .*

Proof. See [38]. □

5 Tripartite graphs with prescribed distributed degree set

As the following theorem shows, the existence of a corresponding tripartite graph do not require the condition $|\gamma_1| = |\gamma_2|, |\gamma_3|$.

Theorem 51 *Let $\delta_1 = \{a_1, a_2, \dots, a_{n_1}\}$, $\delta_2 = \{b_1, b_2, \dots, b_{n_2}\}$, and $\delta_3 = \{c_1, c_2, \dots, c_{n_3}\}$ be nonempty sets of positive integers. positive integers with $a_1 < a_2 < \dots < a_{n_1}$, $b_1 < b_2 < \dots < b_{n_2}$ and $c_1 < c_2 < \dots < c_{n_3}$. Then there exists a tripartite graph $B = (V_1, V_2, E)$ such that $\delta(V_1) = \delta_1$, and $\delta_2(V_2) = \delta_2$ and $\delta_3(V_3) = \delta_3$.*

Proof. Let $A = \sum_{i=1}^{n_1} a_i$ and $B = \sum_{i=1}^{n_2} b_i$ and $C = \sum_{i=1}^{n_3} c_i$. □

The following program `DISTRIBUTED-TRIPARTITE` constructs a tripartite graph having a prescribed distributed degree set.

Input. p : the number of elements in the prescribed degree set for V_1 ;
 q : the number of elements in the prescribed degree set for V_2 ;
 r : the number of elements in the prescribed degree set for V_3 ;
 $\delta_1 = \{a_1, a_2, \dots, a_p\}$: prescribed degree set for V_1 ;
 $\delta_2 = \{b_1, b_2, \dots, b_q\}$: prescribed degree set for V_2 ;
 $\delta_3 = \{c_1, c_2, \dots, c_r\}$: prescribed degree set for V_3 .

Output. n_1 : the size of vertex set V_1 ;
 n_2 : the size of vertex set V_2 ;
 n_3 : the size of vertex set V_3 ;
 M_1 : the incidence matrix of the constructed bipartite graph with distributed degree set δ_1, δ_2 ;
 M_2 : the incidence matrix of the constructed bipartite graph with distributed degree set δ_1, δ_3 ;
 M_3 : the incidence matrix of the constructed bipartite graph with distributed degree set δ_2, δ_3 ;
 M : the incidence matrix of the constructed tripartite graph $B = (V_1, V_2, V_3, E)$.

Work variables. i, j : cycle variables;
 α : the sum of the elements of δ_1 ;

β : the sum of the elements of δ_2 ;
 τ : the sum of the elements of δ_3 ;
 $\sigma_1\alpha$: the least common multiple of α and β ;
 $n_1 = \sigma/\alpha$: the number of rows of M , that is the size of V_1 ;
 $n_2 = \sigma\beta$: the number of columns of M , that is the size of V_2 ;;
 $\phi = (p_1, p_2, \dots, p_\beta)$: the degree vector of V_1 ;
 $\rho = (r_1, r_2, \dots, r_\alpha)$: the degree vector of V_2 .

DISTRIBUTED-TRIPARTITE($p, q, r, \delta_1, \delta_2, \delta_3$)

```

01 DISTRIBUTED-BIPARTITE( $p, q, \delta_1, \delta_2$ )  // lines 01–03: computation of  $M_1$ 
02  $N = M$ 
03  $\mu_2 = \mu$ 
04 DISTRIBUTED-BIPARTITE( $p, r, \delta_1, \delta_3$ )  // lines 04–06: computation of  $M_2$ 
05  $P = M$ 
06  $\mu_2 = \mu$ 
07 DISTRIBUTED-BIPARTITE( $q, r, \delta_2, \delta_3$ )  // lines 07–09: computation of  $M_3$ 
08  $Q = M$ 
09  $\mu_3 = \mu$ 
10 for  $i = 1$  to  $\mu_1 + \mu_2 + \mu_3$            // lines 10–11: initialization of  $M$ 
11    $M_{ij} = 0$ 
12 for  $i = 1$  to  $\mu_1$                          // lines 12–20: computation of  $M$ 
13   for  $j = 1$  to  $\mu_1$ 
14      $M_{ij} = N_{ij}$ 
15 for  $i = 1$  to  $\mu_2$ 
16   for  $j = 1$  to  $\mu_2$ 
17      $M_{\mu_1+i, \mu_1+j} = P_{ij}$ 
18 for  $i = 1$  to  $\mu_3$ 
19   for  $j = 1$  to  $\mu_3$ 
20      $M_{\mu_1+\mu_2+i, \mu_1+\mu_2+j} = Q_{ij}$ 
21  $\Sigma = \mu_1 + \mu_2 + \mu_3$                  // lines 21–21: computation of  $\Sigma$ 
22 return  $\Sigma, M$                            // lines 22–22: return of the results
  
```

Theorem 52 *The running time of DISTRIBUTED-TRIPARTITE is in all cases $\Theta(\Sigma^2)$.*

Proof. The deciding part of the running time is required by lines 10–12. $\square$

In 2007 Pirzada and Dar proved the following result on the global degree set of signed tripartite graphs.

Theorem 53 (Pirzada, Dar [38]) *Let $\gamma = \{g_1, g_2, \dots, g_p\}$ be a nonempty set of positive integers. Then there exists a connected signed tripartite graph G whose global degree set is $\sum_{i=1}^1 g_i, \sum_{i=1}^2 g_i, \dots, \sum_{i=1}^p g_i$.*

Proof. See [38]. □

Corollary 54 (Pirzada, Dar [38]) *Every set of negative integers is the global degree set of a connected tripartite signed graph.*

Proof. See [38]. □

Theorem 55 (Pirzada, Dar [38]) *Every set of integers is the global signed degree set of some connected signed tripartite graph.*

Proof. See [38]. □

6 Simple, bipartite and tripartite digraphs with prescribed global score sets

For directed graphs there are similar results as for undirected graphs.

6.1 Simple digraphs with prescribed score sets

The following papers contain the known results on the simple digraphs having a prescribed score set: [3, 12, 18, 19, 20, 23, 22, 35, 37, 40, 44, 50, 51, 63].

A directed graph is called *asymmetric* or *oriented*, if whenever a vertex $uv \in E$, then $vu \notin E$. A complete asymmetric graph is called *tournament*. The outdegrees of the vertices of a tournament are called *scores*, and the sequence of the scores is called *score sequence*, the set of scores is called *score set*.

Reid in 1978 proved the following sufficient conditions for a tournament T to have a prescribed score set.

Theorem 56 (Reid [50])

1. *Every singleton and doubleton set of positive integers is the score set of a tournament.*
2. *Let $a \geq 1$, $d \geq 2$ and $n \geq 0$ be integers and $\gamma = \{a, ad, ad^2, \dots, ad^n\}$. Then there exists a tournament T whose score set is γ .*
3. *Let $a \geq 1$, $d \geq 1$ and $n \geq 0$ be integers and $\gamma = \{a, a+d, a+2d, \dots, a+nd\}$. Then there exists a tournament T whose score set is γ .*

Proof. See [50] □

Since a single vertex is also a tournament, therefore $S = 0$ is also the score set of a tournament. If $a \geq 1$ and T is the union of T_1 , consisting of a single vertex and T_2 is such $(2a + 1)$ -regular tournament, that the elements of T_2 win against the players in T_1 , then the score set of T is $\{0, a\}$, that is the first part of Theorem 56 is true not only for positive, but also for nonnegative elements.

In the same paper [50] Reid formulated the conjecture, that any set of nonnegative integers is a score set of some tournament.

In 1986 Hager [12] continued the researches of Reid proving that any set of 3, 4 or 5 nonnegative elements are also the score sets of some tournament.

Finally in 1989 Yao [63] proved the conjecture of Reid.

Theorem 57 (Yao [63]) *Any set of nonnegative integers is the global degree set of some tournament.*

Proof. See [63]. □

Let $n \geq 1$ a positive integer and $\mu_o(\gamma)$ be the minimal order of oriented graphs having score set $\gamma = \{g_1, g_2, \dots, g_n\}$. In 1976 Chartrand, Lesniak and Roberts proved the following assertions.

Lemma 58 (Chartrand, Lesniak and Roberts [4]). *If a is a nonnegative integer, then $\mu_o(\{a\}) = 2a + 1$.*

Lemma 59 (Chartrand, Lesniak and Roberts [4]). *If γ is a finite, nonempty set of nonnegative integers and p is an integer such that $p \geq \mu_o(\gamma)$, then there exists an asymmetric digraph D of order p such that $\gamma(D) = \gamma$.*

As a simple consequence of Lemma 59, we have the following result.

Lemma 60 (Chartrand, Lesniak and Roberts [4]). *If γ is a finite, nonempty set of nonnegative integers and p is an integer such that $p \geq \mu_o(\gamma)$, then there exists an asymmetric digraph D of order p such that $\mathcal{D}(D) = \gamma$.*

Corollary 61 (Chartrand, Lesniak and Roberts [4]). *If p is a positive integer and $\gamma = \{a_1, a_2, \dots, a_p\}$ is a set of nonnegative integers with $a_1 < a_2 < \dots < a_p$ and $a_1 = 0$, then $\mu_o(\gamma) = a_p + 1$.*

Lemma 62 (Chartrand, Lesniak and Roberts [4]). *If $n \geq 2$ and $1 \leq a_1 < \dots < a_n$, then $\mu_o(a_1, a_2, \dots, a_n) \geq 2a_1 + t$, where $t > 1$ is the least integer for which $(n + t - 2)a_1 + \binom{t}{2} \geq \sum_{i=1}^p a_i$.*

The main result of Chartrand, Lesniak and Roberts is the following theorem.

Theorem 63 (Chartrand, Lesniak and Roberts [4]). *Let $p \geq 2$ be an integer $\gamma = (a_1, a_2, \dots, a_p)$ be a sequence of positive integers, and let t be the least integer exceeding one for which $(p + t - 2)a_1 + \binom{t}{2} \geq \sum_{i=1}^n a_i$. Then*

$$\mu_0(a_1, a_2, \dots, a_p) = \begin{cases} a_p + 1 & \text{if } a_p \geq \mu_0(a_1, a_2, \dots, a_{p-1}), \\ 2a_1 + 1 & \text{if } a_p < \mu_0(a_1, a_2, \dots, a_{p-1}). \end{cases}$$

Proof. The proofs of Lemma 60, Corollary 61, Lemma 62, Lemma ?? are in [4]. $\square$

In 1983 Harary and Harzheim [17] investigated the degree sets of infinite connected graphs.

In 2006 Pirzada and Naikoo proved the following assertion on the score sets of k -partite tournaments.

Theorem 64 (Pirzada, Naikoo [43]) *Let $k \geq 1$, $d_1, d_2, \dots, d_k$ be nonnegative integers with $d_2 d_3 \dots d_k > 0$. Then there exists a tripartite tournament with global score set $\{\sum_{i=1}^1 d_i, \sum_{i=1}^2 d_i, \dots, \sum_{i=1}^k d_i\}$ except for $p = 1$, $d_1 = 0$, and $p = 2$, $d_1 = 0$, $d_2 = 2$.*

Proof. See [43]. $\square$

Theorem 65 (Pirzada, Naikoo [43]) *Let $d_1, d_2, \dots, d_p$ be nonnegative integers with $d_2, d_3, \dots, d_p > 0$. Then for every $p \geq k \geq 2$ then there exists a k -partite tournament with global score set $\{\sum_{i=1}^1 d_i, \sum_{i=1}^2 d_i, \dots, \sum_{i=1}^k d_i\}$.*

Proof. See [43]. $\square$

In 2006 Dziechcińska-Halamoda, Majcher, Michael, and Skupień [8] studied the properties of sets of pairs of scores in oriented graphs.

In 2006 Pirzada, Naikoo and Chishti proved the following conditions which are sufficient for an oriented graph to have special degree sets.

Theorem 66 (Pirzada, Naikoo, Chishti [45]) *If γ contains one, two or three positive integers, then there exists an oriented graph whose global degree set is γ .*

Proof. See [45]. $\square$

It is also a sufficient condition, if γ contains an arithmetical or geometrical sequence.

In 2008 Pirzada and Naikoo gave the following sufficient conditions for an oriented graph G to have the global degree set γ .

Theorem 67 (Pirzada, Naikoo [44]). *Let $a > 0$, $d > 1$ and $n \geq 0$ be integers and $A = \{a, ad, ad^2, \dots, ad^n\}$. Then there exists an oriented graph with degree set A except for $a = 1$, $d = 2$, $n > 0$ and for $a = 1$, $d = 3$, $n > 0$.*

Theorem 68 (Pirzada, Naikoo [44]) *Let If $n \geq 1$ and $a_1, a_2, \dots, a_n$ are non-negative integers with $a_1 < a_2 < \dots < a_n$, then there exists an asymmetric graph with $a_n + 1$ vertices and global degree set $a'_1, a'_2, \dots, a'_n$, where*

$$a'_i = \begin{cases} a_i & \text{for } i = 1, \\ a_{i-1} + a_i + 1 & \text{for } i > 1. \end{cases}$$

In 2014 Khan [27] proved, that the problem of construction of a tournament having prescribed imbalance set is weakly NP-complete.

6.2 Bipartite digraphs with prescribed score sets

A bipartite tournament is a complete asymmetric bipartite graph. Let $\delta_1 = \{a_1, a_2, \dots, a_p = a\}$ and $\delta_2 = \{b_1, b_2, \dots, b_q = b\}$ be finite, nonempty, increasingly ordered sets, containing nonnegative integers, whose elements are nonnegative integers with $a_1 + b_1 > 0$.

In 1983 Wayland proved the following assertion.

Theorem 69 (Wayland [60]). *There exists a bipartite tournament $T = (V_1, V_2, E)$ with distributed score set (δ_1, δ_2) , if and only if*

$$\sum_{i=1}^p s_i + (t - p + 1)q + \sum_{j=1}^q b_j + 1 - q(b_q + 1)$$

is positive.

Proof. See [60]. □

Corollary 70 *If $s > m + 1$, then there exists a bipartite tournament with distributed score set (δ_1, δ_2) .*

Proof. See [60]. □

Also in 1983 Petrović published the following assertion.

Theorem 71 (Petrović [36]) *The set of nonnegative integers $\delta_1 = \{a\}$ and $\delta_2 = \{b_1, b_2, \dots, b_n\}$ form a distributed score set for some bipartite tournament if and only if one of the following conditions are satisfied:*

- a) $b_1 + b + 2 + \dots + b_n = b(n - a - 1)b_n$;
- b) $b_1 + b + 2 + \dots + b_n > (n - a + 1)b_n$;
- c) $b_1 + b + 2 + \dots + b_n = (n - a + 1)b_n + d$, $1 \leq d \leq n - a - 1$.

Proof. See [36]. □

Corollary 72 (Petrović [36], Wayland [60]). *Any nonempty set of nonnegative integers except $\{0\}$ is the global degree set of some bipartite tournament.*

Proof. See Petrović [36], Wayland [60]. □

6.3 Tripartite digraphs with prescribed global score sets

Let k be a positive integer and $D = (V_1, V_2, \dots, V_k, E)$ be a k -partite oriented graph. In 2006 Pirzada, and Naikoo [42]—using an unusual definition of score sets—published sufficient conditions of the existence of 3-partite oriented graphs having special singleton sets, arithmetical and geometrical series as their prescribed global score set.

In 2007 Pirzada et al. [41] gave further sufficient conditions for the existence of oriented tripartite graphs having prescribed global score set.

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