Acta Universitatis Sapientiae
Electrical and Mechanical Engineering, 13 (2021) 68–81

DOI: 10.2478/auseme-2021-0006

Theoretical Study of the Unconstrained and Constrained Nonlinear Optimal Discrete Time State Feedback Control

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Manuscript received October 10, 2021; revised November 19, 2021

Abstract: Majority of the optimal control techniques can only be applied successfully if the model of the controlled process is known and it is linear. If the system model is nonlinear, then this nonlinear model can be approximated with different simple, linear models. However, these models are valid only in the neighbourhood of the operating points. The success of the control algorithms is highly dependent on the used linearization methods. The aim of the paper is to compare different optimal control algorithms and linearization methods. The presented optimal control algorithms have been also tested in constrained and unconstrained versions.

Keywords: LQR control, SDRE control, nonlinear system, cost function, constrained control, parameter estimation

1. Introduction

The nonlinear optimal control, namely constrained optimal control of nonlinear dynamical systems, still remains nowadays a very interesting, useful and constantly evolving field of science. Its origin has been laid down in the 1950's with the introduction of dynamic programming, the Hamilton-Jacobi-Bellman partial differential equations, and the Pontryagin maximum-minimum principle. From these beginnings, numerous design methodologies have been developed, from the direct solution through numerical computations, generalizations of the classical Lyapunov theory with the control Lyapunov function (CLF), and extending the linear optimal control theory with linearization and the state dependent Riccati equation (SDRE) based techniques.

This paper briefly discusses the possible applications of nonlinear optimal controllers. There are presented some control algorithms that require to know the state space mathematical model of the controlled process, and an adequate criteria function. The known optimal control theories can be applied successfully to control of processes characterized by linear system models. The question arises, how can these approaches be applied to processes characterized by nonlinear system models? There exists a well-known method of linearization around the operation point, which requires the definition of the Jacobian matrix, which in many cases is disadvantageous. In this paper two linearization methods are presented, one of which requires knowledge of the analytical model of the system, and the other does not. This second version considers a black box model of the process to be controlled. It is important that all state variables have to be accessible. In both cases, the result will be a so-called linearized mathematical model, where the specified model matrices are state dependent. Two types of controllers will be introduced, where the optimal criteria function is quadratic: Discrete Linear Quadratic Regulator (DLQR) and the advanced Model Predictive Control (MPC) algorithm, where some constraints can also be introduced [1].

The present paper has the following structure. Section 2 briefly discusses different linearization methods, and section 3 introduces two control algorithms. Section 4 demonstrates the applicability of the presented methods using the nonlinear dynamical model of the inverted pendulum. The short conclusions are outlined in section 5.

2. Nonlinear optimal control theory

In this section, we present the nonlinear discrete time control, and its corresponding implementation theory. Consider the following discrete nonlinear mathematical model [2], [3]

$$\begin{aligned} x_{k+1} &= F(x_k, u_k) \\ y_k &= G(x_k, u_k) \end{aligned} \tag{1}$$

where x_k is the n dimensional state vector, u_k is the m dimensional input vector and y_k is the p dimensional output vector. The functions $F: R^{n+m} \rightarrow R^n$ and $G: R^{n+m} \rightarrow R^p$ are continuous, nonlinear vector functions. Thereinafter the state feedback control theory will be presented. The investigation of a tracking control problem is similar, but in the tracking control case, there appear some extra terms. Since this paper is only about the state feedback control, in the following the output equation will not be considered.

For linearization, this mathematical model (1) has to be transformed to the following mathematical form:

$$x_{k+1} = A(x_k)\cdot x_k + B(x_k)\cdot u_k \tag{2}$$

where the state-space system matrices (A and B) are state-dependent.

A. Linearization around the operating point

The nonlinear optimal control methods usually involve Jacobian linearization of the system model around each operation point. This method assumes that, the discrete mathematical model and the parameters are known and the nonlinear functions (in the model description) are continuous. The linearized matrices can be computed by the following relations.

$$A(x_k) \cong \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}_{x_k, u_k} \quad B(x_k) \cong \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \cdots & \frac{\partial F_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial u_1} & \cdots & \frac{\partial F_n}{\partial u_m} \end{bmatrix}_{x_k, u_k} \tag{3}$$

This linear approximation of the model may introduce errors, especially if the states are far from the equilibrium point. In addition, the numerical approach of the derivatives can also bring inaccuracies [2].

B. Discrete time SDRE method

According to the SDRE theory, this can be easily accomplished if the F function in (1) can be written as [8], [13].

$$x_{k+1} = F(x_k, u_k) = f(x_k) + g(x_k)\cdot u_k \tag{4}$$

by introducing the following formal notation:

$$A(x_k) = \frac{f(x_k)}{x_k}, \quad B(x_k) = g(x_k) \tag{5}$$

This method can be used if the plant is characterized by a known nonlinear mathematical model with known parameters and this mathematical model can be transformed in a special difference equation form (4).

C. Linearized model with parameter estimation

This solution does not require knowledge of the mathematical model of the system, but states must be measurable here as well. In this case we determine the linearized matrices based just on the measured values. For this method the model (2) will be transformed theoretically as follows:

$$x_k^T = \begin{bmatrix} x_{k-1}^T & u_{k-1}^T \end{bmatrix} \cdot \begin{bmatrix} A^T(x_{k-1}) \\ B^T(x_{k-1}) \end{bmatrix} = \varphi_k^T \cdot \theta_k \tag{6}$$

The well-known parametric estimation algorithm can be applied to this mathematical model form, where θ_k contains the unknown parameters and φ_k is the measurement vector. The following steps characterize the Least Square Estimation (LSE) recursive algorithm [10]:

$$\begin{aligned} &\varepsilon_k = x_k^T - \varphi_k^T \cdot \theta_k, \\ &K = \frac{F_k \cdot \varphi_k}{\lambda + \varphi_k^T \cdot F_k \cdot \varphi_k}, \\ &F_{k+1} = \frac{1}{\lambda} \cdot (F_k - K \cdot \varphi_k^T \cdot F_k), \\ &\theta_{k+1} = \theta_k + K \cdot \varepsilon_k, \end{aligned} \tag{7}$$

where ε_k is the estimation error, K is the estimation gain vector, F_k the covariance matrix, λ the forgetting factor and θ_k is the estimated parameter vector. This is a recursive algorithm, so we have to make some initialization first. There have to be chosen the initial value of the parameter vector θ_0 and the initial value of the covariance matrix F_0.

3. Control methods

This section briefly introduces two control approaches. The first method is the DLQR method, which is not a strictly constrained control algorithm, with this only the weak constraints can be specified. Here the idealized infinite horizon control and the finite horizon version can also be tested. The second method is the MPC algorithm, which takes into account also the strict constraints, when calculating the control signal.

A. DLQR

Here only the state feedback control is discussed, where the goal is to control all states to zero. This type of problem is characterized with the following discrete criteria function [2], [3]:

$$J(\Delta u_k) = \sum_{k=0}^{N} x_k^T \cdot Q_k \cdot x_k + \sum_{k=0}^{N-1} \Delta u_k^T \cdot R_k \cdot \Delta u_k \tag{8}$$

where N is the horizon value and here it is considered, that the weight matrices R and Q are not state dependent. The variation of control is defined as

$$\Delta u_k = u_k - u_{k-1} \cdot \tag{9}$$

In a similar way to the classical LQR design the variation of the state-feedback control is calculated as:

$$\Delta u_k = -\left(R + B_k^T \cdot P_k \cdot B_k\right)^{-1} \cdot \left(B_k^T \cdot P_k \cdot A_k \cdot x_k + B_k^T \cdot P_k \cdot u_{k-1}\right) \tag{10}$$

For simplicity the following notations have been introduced for the Riccati matrix and for the state dependent matrices: $P(x_k)=P_k$, $A(x_k)=A_k$ and $B(x_k)=B_k$.. The Riccati matrix is the unique solution of the following discrete time state dependent Riccati equation.

$$P_k = Q + A_k^T \cdot (P_{k+1} - P_{k+1} \cdot B_k \cdot \left(R + B_k^T \cdot P_{k+1} \cdot B_k\right)^{-1} B_k^T \cdot P_{k+1}) \cdot A_k \tag{11}$$

The solution of SDRE (11) is only a sub-optimal solution, because there were neglected derivatives of the system matrices ($A(x_k)$ and $B(x_k)$) [8]. If the value of the horizon (N) is infinite, the solution of the difference Riccati equation approaches the solution of the algebraic equation ($P_k = P_{k+1}$) [2].

B. MPC

The model predictive algorithm looks for the vector ΔU_k that minimizes a cost function represented by the following relationship

$$J(\Delta U_k) = X_k^T \cdot Q^* \cdot X_k + \Delta U_k^T \cdot R^* \cdot \Delta U_k \tag{12}$$

where X_k is the vector with the predictions of the controlled state variables, ΔUk is a vector with future input changes, Q^* is a diagonal matrix with weights for the states, R^* is a diagonal matrix with weights for the control action changes [1], [7], [15]. If the prediction horizon is N and the control horizon is Nc, these vectors and matrices are:

$$X_k = \begin{bmatrix} x_{k+1/k} \\ \vdots \\ x_{k+N/k} \end{bmatrix} \quad \Delta U_k = \begin{bmatrix} \Delta u_{k/k} \\ \vdots \\ \Delta u_{k+N_c-1/k} \end{bmatrix} \tag{13}$$

$$Q^* = \begin{bmatrix} Q_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & Q_N \end{bmatrix} \quad R^* = \begin{bmatrix} R_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & R_{N_c-1} \end{bmatrix} \tag{14}$$

The following representation is obtained for the predictions:

$$X_k = A_k^* \cdot x_k + B_k^* \cdot u_{k-1} + G_k^* \cdot \Delta U_k \tag{15}$$

where

$$A_k^* = \begin{bmatrix} A_k \\ \vdots \\ A_k^N \end{bmatrix} \quad B_k^* = \begin{bmatrix} B_k \\ \vdots \\ \sum_{i=0}^{N-1} A_k^i \cdot B_k \end{bmatrix}, G_k^* = \begin{bmatrix} B_k & \dots & 0 \\ \vdots & \ddots & \vdots \\ \sum_{i=0}^{N-1} A_k^i \cdot B_k & \dots & \sum_{i=0}^{N-Nc} A_k^i \cdot B_k \end{bmatrix} \tag{16}$$

The cost function can be written as

$$J(\Delta U_k) = \frac{1}{2} \Delta U_k{}^T \cdot H \cdot \Delta U_k + f^T \cdot \Delta U_k + const \tag{17}$$

where

$$H = 2 \cdot \left[G_k{}^T \cdot Q^* \cdot G_k + R^* \right] \tag{18}$$

$$f = 2 \cdot G_k{}^T \cdot Q^* \cdot (A_k^* \cdot x_k + B_k^* \cdot u_{k-1}) \tag{19}$$

For the unconstrained problems, the model predictive control determines the vector ΔU_k:

$$\begin{aligned} \Delta U_k &= \frac{1}{2} \cdot \left(H + H^T \right)^{-1} \cdot f = \\ &= -\left(G_k^T \cdot Q^* \cdot G_k + R^* \right)^{-1} \cdot G_k^T \cdot Q^* \cdot (A_k^* \cdot x_k + B_k^* \cdot u_{k-1}) \end{aligned} \tag{20}$$

To the quadratic cost function (17) there can be assigned different linear constraint inequalities

$$A_c \cdot \Delta U_k \le B_c \tag{21}$$

These types of problems can be solved with different numerical quadratic programming algorithms (ex. interior point method, quadratic penalty method [4], [5], [6]).

4. Example and numerical simulation

For testing the presented control methods, the well-known cart on inverted pendulum dynamical system can be used as an example. The control of this system is quite difficult due to the characteristics of the system: instability, nonlinearity of the model, with single input and four state variables. The schematic representation of the inverted pendulum system is shown in *Fig. 1*.

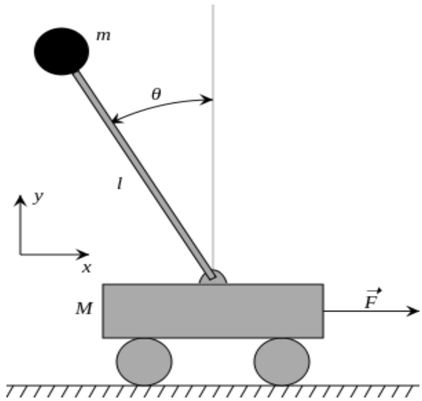

Figure 1: Inverted pendulum on cart

The Euler-Lagrange theory has been used to determine this mathematical model, where the inertia of the system has been neglected [9], [11], [12]. The nonlinear mathematical discrete time model is the following.

$$
\begin{aligned}
x_{1,k+1} &= x_{1,k} + T_s \cdot x_{2,k} \\
x_{2,k+1} &= x_{2,k} + T_s \cdot \frac{\left(l \cdot x_{4,k}^2 \cdot \sin(x_{3,k}) - g \cdot \sin(x_{3,k}) \cdot \cos(x_{3,k}) + \frac{1}{m} \cdot u_k\right)}{\left(\frac{M}{m} + \sin^2(x_{3,k})\right)} \\
x_{3,k+1} &= x_{3,k} + T_s \cdot x_{4,k} \\
x_{4,k+1} &= x_{4,k} + T_s \cdot \frac{\left(\frac{m+M}{ml} g \cdot \sin(x_{3,k}) - \frac{1}{2} x_{4,k}^2 \cdot \sin(2 \cdot x_{3,k}) - \frac{1}{ml} \cdot \cos(x_{3,k}) \cdot u_k\right)}{\left(\frac{M}{m} + \sin^2(x_{3,k})\right)}
\end{aligned}
\tag{22}
$$

The four state variables in the model are: $x_{1,k}$ displacement of the cart, $x_{2,k}$ speed of the car, $x_{3,k}$ pendulum angle and $x_{4,k}$ pendulum angular speed. The input signal u_k is the force acting horizontally on the car. The model parameters and their numeric values are shown in Table 1.

Table 1: Model parameters

System parameters	**Value**
M (mass of cart)	0.6 kg
m (mass of pendulum)	0.45 kg
l (length of pendulum)	0.35 m
g (acceleration of gravity)	9.81 m/s^2
Ts (time sampling)	0.01 s

In the following there will be listed some special values and relationships that will be used along with different linearization methods:

- For the linearization around the operating point it is necessary to calculate certain derivatives. This can be solved using a simple approximation method, where the perturbation values is set to 0.0001.
- For testing the SDRE method the following state dependent matrices can be used:

$$A(x_k)=\begin{bmatrix} 1 & T_s & 0 & 0 \\ 0 & 1 & T_s\cdot\dfrac{-g\cdot\sin(x_{3,k})\cdot\cos(x_{3,k})}{x_{3,k}\cdot(\frac{M}{m}+\sin^2(x_{3,k}))} & T_s\cdot\dfrac{l\cdot x_{4,k}\cdot\sin(x_{3,k})}{(\frac{M}{m}+\sin^2(x_{3,k}))} \\ 0 & 0 & 1 & T_s \\ 0 & 0 & T_s\cdot\dfrac{\frac{m+M}{ml}g\cdot\sin(x_{3,k})}{x_{3,k}\cdot(\frac{M}{m}+\sin^2(x_{3,k}))} & 1-T_s\cdot\dfrac{\frac{1}{2}x_{4,k}\cdot\sin(2\cdot x_{3,k})}{(\frac{M}{m}+\sin^2(x_{3,k}))} \end{bmatrix} \tag{23}$$

$$B(x_k)=\begin{bmatrix} 0 \\ \dfrac{\frac{1}{m}\cdot T_s}{(\frac{M}{m}+\sin^2(x_{3,k}))} \\ 0 \\ \dfrac{-T_s\cdot\frac{1}{ml}\cdot\cos(x_{3,k})}{(\frac{M}{m}+\sin^2(x_{3,k}))} \end{bmatrix} \tag{24}$$

- For the parameter estimation algorithm, the choice of the initial parameter matrix and of the initial covariance matrix is important. For better results, the initial value of the parameter matrix is initialized with the initial values obtained by the SDRE method and the initial covariance matrix is a diagonal matrix $10\cdot I_5$.

For the DLQR and the MPC control methods, the following parameters and limits have been used: the weighting matrix (value) of the controls $R=0.1\ I_1$, the weighting matrix of the states $Q=100\ I_4$, the prediction horizon $N=50$, the control horizon $Nc=5$, and the limits of the control signals $u_{min}=-50$ and $u_{max}=50$. Here can be set also some limits for the variation of the control $\Delta u_{min}=-10$ and $\Delta u_{max}=10$. In accordance with (21) these limits can be given by the following matrices:

$$A_c=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \end{bmatrix} \quad B_c=\begin{bmatrix} u_{\max}-u_{k-1} \\ -u_{\min}+u_{k-1} \\ \Delta u_{\max} \\ -\Delta u_{\min} \end{bmatrix} \tag{25}$$

For numerical simulation the following abbreviations were used to refer to the presented linearization methods:

L_op – method of linearization around an operating point;

L_sdre - linearization using the SDRE approach;

L_est - linear parameter estimation of the discrete time linear model.

The following notations will be assigned to the type of control algorithm: DLQR1 - optimal quadratic control algorithm with infinite horizon, DLQR2 - optimal quadratic control algorithm with finite horizon and MPC- model predictive control.

The graphical results of the unrestricted control version (for the presented control methods and linearization methods) are presented in the following figures. *Fig. 2* shows the control result (variation of the states $x_1(t)$ and $x_3(t)$) achieved by the linearization method around the operation point. These figures show the control effects of the initial states and of the modified states during the simulation (at about 40 sec). In this linearization method (L_op) the weakest control result was obtained for the case of finite horizon method. The best result was obtained for the infinite horizon DLQR control, but this method can be used just for theoretical simulations. The MPC algorithm is also a good solution, but here is already used the control horizon value (N_c). All this is observed mainly in the controlling of the initial state. The perturbation of states (during the simulation) leads to similar results in all three variants.

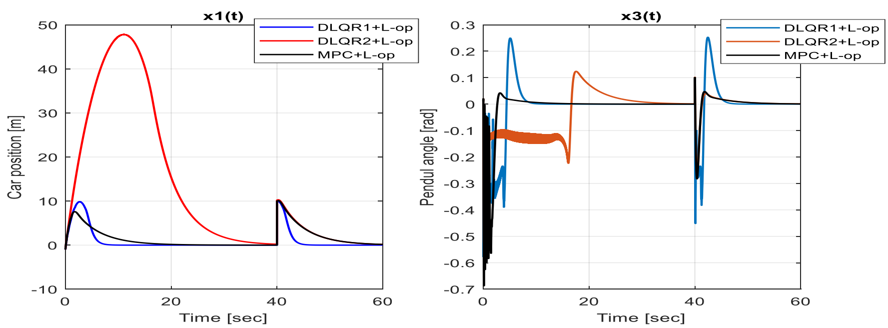

Figure 2: States controlled by different unconstrained control methods using linearization around operation points (L_op)

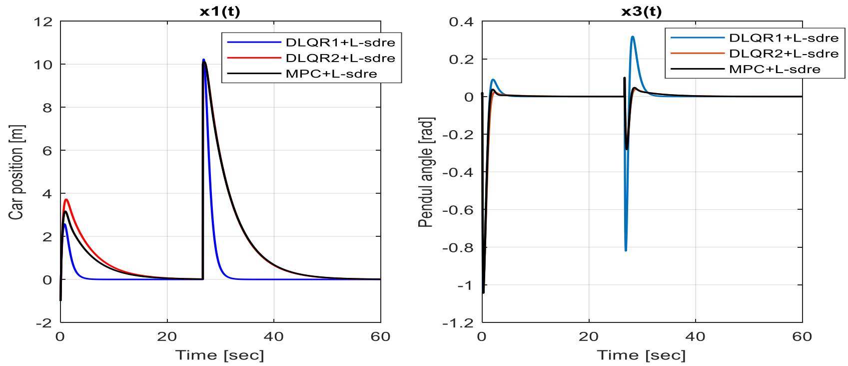

Figure 3: States controlled by different unconstrained control methods using linearization based on SDRE approach (L_sdre)

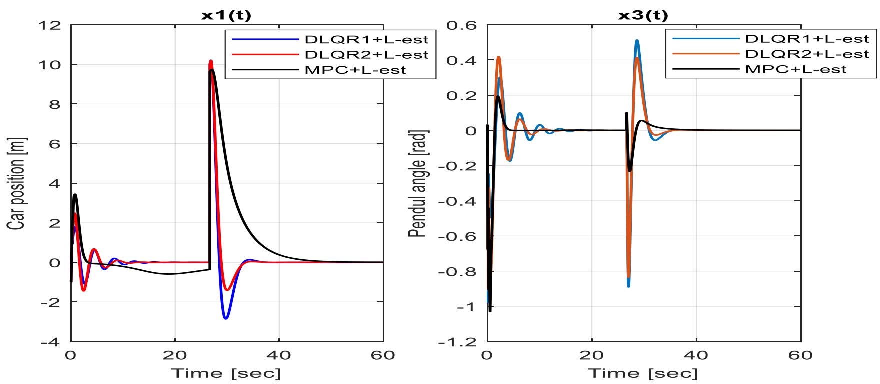

Figure 4: States controlled by the different unconstrained control methods using the linearization method with parameter estimation (L_est)

Fig. 3 and *Fig. 4* in principle the same process is shown, but here the linearization is made with SDRE approach and with parameter estimation algorithms. These plots show also the effect on the control process both of the initial state and of the perturbation at about 30 sec. Here the perturbation of states leads to similar results in all three variants. These simulation results have been obtained for unconstrained control signals. The variation of the control signals (for the first 30 sec) in case of using the third linearization method (L_est) is shown in *Fig. 5*.

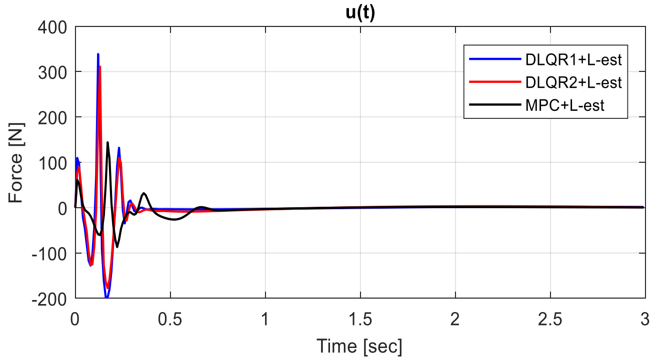

Figure 5: Variation of the unconstrained control signals using the linearization method with parameter estimation (L_est)

In the following we test the effect of the constrained control on these algorithms. Taking into account the limits of the control signal (and their variations), the results are shown in *Fig. 6* and *Fig. 7*. For the MPC algorithm, these constraints are already taken into account in the calculation of the control signal. In the other methods (LQR), the signal values are simply limited when the thresholds are exceeded. The constrained control results for the LQR1 algorithm are shown in *Fig. 6* and for MPC algorithm in *Fig. 7*. In both cases, the linearized model is determined by an on-line parametric estimation method. It can be seen from these figures that the MPC algorithm handles much better the constrained control problems.

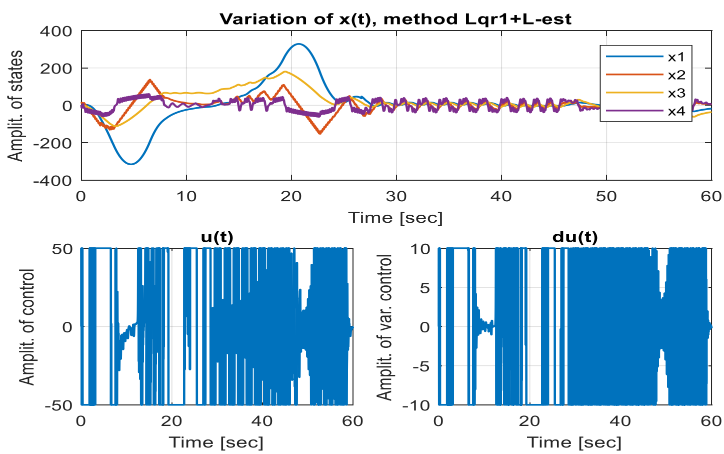

Figure 6: States controlled by constrained LQR method, where the linearized parameter estimation (L_est) was used

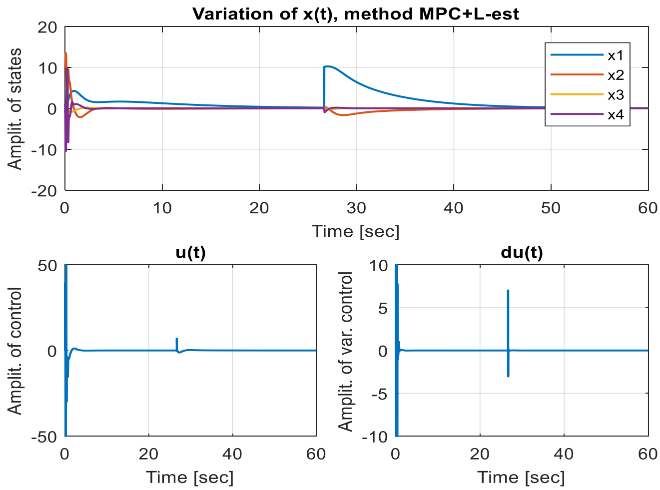

Figure 7: The evolution in time of the states controlled with constrained MPC method, with the linearized parameter estimation (L_est) method

The presented algorithms have been tested also for small changes of the system parameters. The first two linearization methods (L_op and L_sdre) do not have a good behavior in case of the parametric change. This is because these two methods require the knowledge of the analytical model of the process. However, the third linearization method (L_est) has a proper effect in eliminating the unpleasant effect of such a modification (adaptive control behavior). This is shown in *Fig. 8* where during the simulation (after about 50 sec) some of the system parameters have been modified. Here it is shown the variation of the controlled states, and the change of the estimated during the estimation algorithm can also be observed.

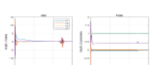

Figure 8: Controlled states and estimated parameters (MPC method with linearized parameter estimation)

5. Conclusion

In the paper a theoretical study has been presented regarding the implementation off different state feedback optimal control approaches in case of a nonlinear system model. Two types of optimal controller have been presented: the LQR controller and the modern MPC controller. Because the goal is to apply these controllers to a nonlinear system, there is need for linearization. To accomplish this, there have been presented three possible linearization methods: linearization around the operation point, linearization based on the SDRE approach, and finally a method using parameter estimation algorithm. These control algorithms and linearization methods have been tested on a nonlinear system, namely inverse pendulum. Applying these methods, it can be observed that the linearization around the operation point is very sensitive, and the convergence of the parametric algorithm depends on the correct choice of the initial values. The control algorithms are highly dependent on the control parameters, the value of the horizon and the predetermined limits. For the constrained control tasks, the MPC controller provides very good results (and it is valid, independently of the used linearization methods).

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