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The sharp version of a strongly starlikeness condition

Olga Engel
Babeş-Bolyai University,
Cluj-Napoca, Romania
email: engel_olga@hotmail.com

Abdul Rahman S. Juma
University of Anbar,
Ramadi, Iraq
email: dr_juma@hotmail.com

Abstract. In this paper we give the best form of a strongly starlikeness condition. Some consequences of this result are deduced. The basic tool of the research is the method of differential subordinations.

1 Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane. Let $\mathcal{A}$ be the class of analytic functions f, which are defined on the unit disk $\mathbb{U}$ and have the properties $f(0) = f'(0) - 1 = 0$. The subclass of $\mathcal{A}$, consisting of functions for which the domain $f(\mathbb{U})$ is starlike with respect to 0 is denoted by S^*. An analytic characterization of S^* is given by

$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{U} \right\}.$$

In connection with the starlike functions has been introduced the following class

$$SS^*(\alpha) = \left\{ f \in \mathcal{A} : \left| \arg \frac{zf'(z)}{f(z)} \right| < \alpha \frac{\pi}{2},\ \alpha \in (0, 1],\ z \in \mathbb{U} \right\},$$

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which is the class of strongly starlike functions of order α.
Another subclass of $\mathcal{A}$ we deal with is the following

$$\mathcal{G}_b = \left\{ f \in \mathcal{A} : \left| \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1 \right| < b, \quad z \in \mathbb{U} \right\}, \tag{1}$$

where $b > 0$.
The authors of [3] proved the following result:

Theorem 1 *If the function* f *belongs to the class* $\mathcal{G}_{b(\beta)}$ *with*

$$b(\beta) = \frac{\beta}{\sqrt{(1-\beta)^{1-\beta}(1+\beta)^{1+\beta}}},$$

where $0 < \beta \leq 1$*, then* $f \in SS^*(\beta)$.

Let $-1 \leq B < A \leq 1$. The class $S^*(A, B)$ is defined by the equality

$$S^*(A,B) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \ z \in \mathbb{U} \right\}.$$

An other result regarding the class $\mathcal{G}_b$ is the following theorem published in [4].

Theorem 2 *Assume that* $-1 \leq B < A \leq 1$ *and* $b(1+|A|)^2 \leq |A - B|$*. If* $f \in \mathcal{G}_b$*, then* $f \in S^*(A, B)$.

The aim of this paper is to prove the sharp version of Theorem 1, and an improvement of Theorem 2.

In our work we need the following results.

2 Preliminaries

Let f and g be analytic functions in $\mathbb{U}$. The function f is said to be subordinate to g, written $f \prec g$, if there is a function w analytic in $\mathbb{U}$, with $w(0) = 0$, $|w(z)| < 1$, $z \in \mathbb{U}$ and $f(z) = g(w(z))$, $z \in \mathbb{U}$. Recall that if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Lemma 1 [1] *Let* $p(z) = a + \sum_{k=n}^{\infty} a_k z^k$ *be analytic in* $\mathbb{U}$ *with* $p(z) \not\equiv a$, $n \geq 1$ *and let* $q : \mathbb{U} \to \mathbb{C}$ *be an analytic and univalent function with* $q(0) = a$. *If* p *is not subordinate to* q, *then there are two points* $z_0 \in \mathbb{U}$, $|z_0| = r_0$ *and* $\zeta_0 \in \partial\mathbb{U}$ *and a real number* $m \in [n, \infty)$, *so that* q *is defined in* ζ_0, $p(\mathbb{U}(0, r_0)) \subset q(\mathbb{U})$, *and:*

(i) $p(z_0) = q(\zeta_0)$,

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$,

(iii) $\operatorname{Re}\left(1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right) \geq m \operatorname{Re}\left(1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right)$.

We note that $z_0 p'(z_0)$ *is the outward normal to the curve* $p(\partial\mathbb{U}(0, r_0))$ *at the point* $p(z_0)$, *while* $\partial\mathbb{U}(0, r_0)$ *denotes the border of the disc* $\mathbb{U}(0, r_0)$.

A basic result we need in our research is the following:

Lemma 2 *If* $f \in \mathcal{A}$, $b \in [0, 1)$, *and* $p(z) = \dfrac{zf'(z)}{f(z)}$, *then the inequality*

$$\left|\frac{zp'(z)}{p^2(z)}\right| < b, \quad z \in \mathbb{U}, \tag{2}$$

implies that

$$p(z) \prec \frac{1}{1 - bz}.$$

The result is sharp.

Proof. If the subordination $p(z) \prec q(z) = \dfrac{1}{1 - bz}$ does not holds, then there are two points $z_0 \in \mathbb{U}$, $|z_0| = r_0 < 1$ and $\zeta_0 \in \partial\mathbb{U}$ and a real number $m \in [1, \infty)$, so that q is defined in ζ_0, $p(\mathbb{U}(0, r_0)) \subset q(\mathbb{U})$, and:

$$p(z_0) = q(\zeta_0) = \frac{1}{1 - b\zeta_0}$$

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) = m\frac{b\zeta_0}{(1 - b\zeta_0)^2}.$$

Thus we get

$$\frac{z_0 p'(z_0)}{p^2(z_0)} = mb\zeta_0. \tag{3}$$

Since $|mb\zeta_0| \geq b$, it follows that the equality (3) contradicts (2), and the proof is done. □

3 Main results

The following theorem is the sharp version of Theorem 1.

Theorem 3 *If* $\alpha \in (0,1)$, *and* $f \in \mathcal{G}_{b(\alpha)}$, *where* $b(\alpha) = \sin\left(\alpha\frac{\pi}{2}\right)$, *then* $f \in SS^*(\alpha)$. *The result is sharp.*

Proof. If we denote $p(z) = \frac{zg'(z)}{g(z)}$, then the condition $f \in \mathcal{G}_{b(\alpha)}$ becomes

$$\left|\frac{zp'(z)}{p^2(z)}\right| < b(\alpha), \quad z \in \mathbb{U}, \tag{4}$$

and according to Lemma 2 we get

$$p(z) \prec q(z) = \frac{1}{1 - b(\alpha)z}.$$

The domain $D = q\big(\mathbb{U}\big)$ is symmetric with respect to the real axis and the boundary of D is the curve

$$\Gamma = \begin{cases} x(\theta) = \mathrm{Re}\,\frac{1}{1-b(\alpha)e^{i\theta}} = \frac{1-b(\alpha)\cos\theta}{1+b^2(\alpha)-2b(\alpha)\cos\theta}, \\ y(\theta) = \mathrm{Im}\,\frac{1}{1-b(\alpha)e^{i\theta}} = \frac{b(\alpha)\sin\theta}{1+b^2(\alpha)-2b(\alpha)\cos\theta}, \end{cases} \quad \theta \in [-\pi, \pi].$$

The subordination $p(z) \prec q(z)$ implies that $|\arg(p(z))| \leq \arctan(M)$, where M is the slope of the tangent line to the curve Γ trough the origin.
The equation of the tangent line is

$$\frac{x - x(\theta)}{x'(\theta)} = \frac{y - y(\theta)}{y'(\theta)}.$$

This tangent line crosses the origin if and only if

$$\frac{x(\theta)}{x'(\theta)} = \frac{y(\theta)}{y'(\theta)},$$

and this equation is equivalent to

$$2b(\alpha)\cos^2\theta - (3b^2(\alpha)+1)\cos\theta + b(\alpha)(b^2(\alpha)+1) = 0.$$

After a short calculation we get $\cos\theta = b(\alpha)$ and this implies

$$M = \frac{y'(\theta)}{x'(\theta)} = \frac{y(\theta)}{x(\theta)} = \frac{b(\alpha)\sin\theta}{1 - b(\alpha)\cos\theta} = \frac{b(\alpha)}{\sqrt{1 - b^2(\alpha)}}.$$

Finally if we put $b(\alpha) = \sin\left(\alpha\frac{\pi}{2}\right)$, then it follows that $|\arg(p(z))|$ $< \arctan(M) = \arctan\frac{b(\alpha)}{\sqrt{1-b^2(\alpha)}} = \alpha\frac{\pi}{2},\ z \in \mathbb{U}$.
Thus we have proved the implication

$$\left|\frac{zp'(z)}{p^2(z)}\right| < \sin\left(\alpha\frac{\pi}{2}\right) \Rightarrow |\arg(p(z))| < \arctan(M) = \alpha\frac{\pi}{2},$$

and the proof is done. □

Putting $\alpha = 1$ in Theorem 3, we get the following starlikeness condition, which is the sharp version of Corollary 1 from [3].

Corollary 1 *If* $f \in \mathcal{A}$ *and*

$$\left|\frac{1+\dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} - 1\right| < 1, \quad z \in \mathbb{U},$$

then $f \in S^*$.

For $\alpha = \frac{1}{2}$, we get the sharp version of Corollary 2 from [3].

Corollary 2 *If* $f \in \mathcal{A}$ *and*

$$\left|\frac{1+\dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} - 1\right| < \frac{\sqrt{2}}{2}, \quad z \in \mathbb{U},$$

then $f \in SS^*\left(\frac{1}{2}\right)$.

Theorem 4 *If* $f \in \mathcal{G}_b$ *and* $b(1 + A - B + |B|) < A - B$, *then* $f \in S^*(A, B)$.

Proof. Let $q, h : \mathbb{U} \to \mathbb{C}$ be the functions defined by

$$q(z) = \frac{1}{1-bz}, \quad h(z) = \frac{1+Az}{1+Bz}.$$

According to Lemma 2 we have $p(z) = \dfrac{zf'(z)}{f(z)} \prec q(z)$ which is equivalent to

$$p(\mathbb{U}) \subset q(\mathbb{U}). \tag{5}$$

We will prove that $q(\mathbb{U}) \subset h(\mathbb{U})$. A simple calculation shows that the domains $q(\mathbb{U})$ and $h(\mathbb{U})$ are convex.
The border of the domain $q(\mathbb{U})$ is the curve

$$\Gamma: \quad q(e^{i\theta}) = \frac{1}{1-be^{i\theta}}, \quad \theta \in [0, 2\pi],$$

and the border of $h(\mathbb{U})$ is the curve

$$\Delta: \quad h(e^{i\eta}) = \frac{1+Ae^{i\eta}}{1+Be^{i\eta}}, \quad \eta \in [0, 2\pi].$$

The inequality $b(1+A-B+|B|) < A-B$ is equivalent to $\frac{b}{1-b} < \frac{A-B}{1+|B|}$.
This inequality implies

$$|q(e^{i\theta})-1| = \frac{b}{|1-be^{i\theta}|} \leq \frac{b}{1-b} < \frac{A-B}{1+|B|} \leq \frac{A-B}{|1+Be^{i\eta}|} = |h(e^{i\eta})-1|.$$

Thus we get

$$|q(e^{i\theta})-1| < |h(e^{i\eta})-1|, \quad \text{for every } \theta, \eta \in [0, 2\pi]. \tag{6}$$

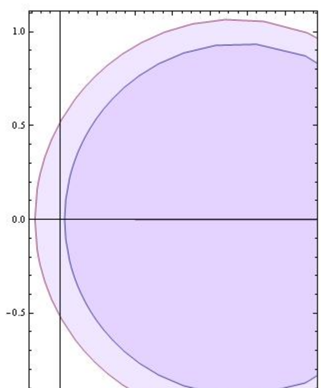
Since $1 \in q(\mathbb{U})$ and $1 \in h(\mathbb{U})$, the inequality (6) implies that the curve Γ is inside the curve Δ.
This means that

$$q(\mathbb{U}) \subset h(\mathbb{U}). \tag{7}$$

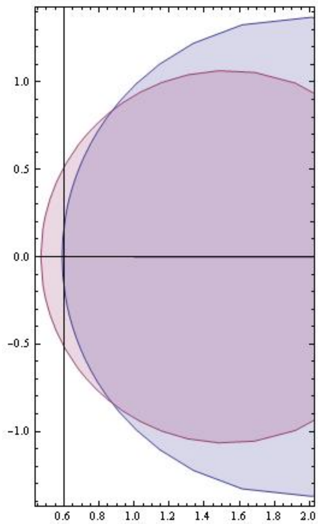
For example if we consider

$$q(z) = \frac{1}{1-0.6z} \text{ and } h(z) = \frac{1+0.3z}{1-0.5z}$$

and the inequality $b(1+A-B+|B|) < A-B$ is satisfied for $b = 0.6$, $A = 0.3$ and $B = -0.5$ then we obtain the following graphics:



which shows that $q(\mathbb{U}) \subset h(\mathbb{U})$. For $b = 0.7$, $A = 0.3$ and $B = -0.5$ the inequality $b(1 + A - B + |B|) < A - B$ is not satisfied and consequently we obtain the following image:



which shows that $q(\mathbb{U}) \not\subset h(\mathbb{U})$. Finally (5) and (7) implies $p(\mathbb{U}) \subset h(\mathbb{U})$ and since h is univalent we infer $\dfrac{zf'(z)}{f(z)} = p(z) \prec h(z), \quad z \in \mathbb{U}$.
This subordination is equivalent to $f \in S^*(A, B)$. □

If $0 \leq B < A \leq 1$, then we get the following corollary, which improvs the result of Theorem 2.

Corollary 3 *Let* $0 \leq B < A \leq 1$ *and* $b \in (0, +\infty)$ *such that* $b(1+A) \leq 1+B$. *If* $f \in \mathcal{G}_b$, *then* $f \in S^*(A, B)$.

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