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The L_p-mixed quermassintegrals for $0 < p < 1$*

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Abstract. In the paper, L_p-harmonic addition, p-harmonic Blaschke addition and L_p-dual mixed volume are improved. A new p-harmonic Blaschke mixed quermassintegral is introduced. The relationship between p-harmonic Blaschke mixed volume and L_p-dual mixed volume is shown.

1 Notation and preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the subset of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. We write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the volume of K. Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin and contains the origin, its radial function is $\rho(K,\cdot) : S^{n-1} \to [0,\infty)$, defined by (see e. g. [1] and [2])

$$\rho(K,u) = \max\{\lambda \geq 0 : \lambda u \in K\}.$$

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If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies in $\mathbb{R}^n$. We write $S(K)$ for the surface area of star body K. If $k > 0$, then for all $u \in \mathbb{R}^n \backslash \{0\}$

$$\rho(kK,u) = k\rho(K,u). \tag{1}$$

Let $\tilde{\delta}$ denote the radial Hausdorff metric, as follows, if $K, L \in \mathcal{S}^n$, then (see e. g. [1])

$$\tilde{\delta}(K,L) = |\rho(K,u) - \rho(L,u)|_\infty.$$

1.1 Dual mixed volume

The radial Minkowski linear combination, $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$, defined by (see [3])

$$\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i,\ i = 1, \ldots, r\},$$

for $K_1, \ldots, K_r \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$. It has the following important property:

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda\rho(K,\cdot) + \mu\rho(L,\cdot),$$

for $K, L \in \mathcal{S}^n$ and $\lambda, \mu \geq 0$.

If $K_i \in \mathcal{S}^n$ $(i = 1, 2, \ldots, r)$ and λ_i $(i = 1, 2, \ldots, r)$ are nonnegative real numbers, then of fundamental importance is the fact that the dual volume of $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$ is a homogeneous polynomial in the λ_i given by (see e. g. [3])

$$V(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r) = \sum_{i_1,\ldots,i_n} \lambda_{i_1} \ldots \lambda_{i_n} \widetilde{V}_{i_1 \ldots i_n}, \tag{2}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$ and is uniquely determined by (2), it is called the dual mixed volume of $K_{i_1}, \ldots, K_{i_n}$, and is written as $\widetilde{V}(K_{i_1}, \ldots, K_{i_n})$. Let $K_1 = \ldots = K_{n-i} = K$ and $K_{n-i+1} = \ldots = K_n = L$, then the mixed volume $\widetilde{V}(K_1 \ldots K_n)$ is written as $\widetilde{V}_i(K,L)$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$, then the mixed volumes $V_i(K,B)$ is written as $\widetilde{W}_i(K)$ and is called the dual quermassintegral of star body K and $(n-i)\widetilde{W}_{i+1}$ is written as $S_i(K)$ and called the mixed surface area of K. The dual quermassintegral of star body K, defined as an integral by (see [4])

$$\widetilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i} dS(u). \tag{3}$$

It is convenient to write relation (2) in the form (see [5, p.137])

$$\widetilde{V}(\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_s K_s)$$
$$= \sum_{p_1+\cdots+p_r=n} \sum_{1\le i_1<\cdots<i_r\le s} \frac{n!}{p_1!\cdots p_r!} \lambda_{i_1}^{p_1}\cdots\lambda_{i_r}^{p_r} \widetilde{V}(\underbrace{K_{i_1},\ldots,K_{i_1}}_{p_1},\ldots,\underbrace{K_{i_r},\ldots,K_{i_r}}_{p_r}). \tag{4}$$

Let $s = 2, \lambda_1 = 1, K_1 = K, K_2 = B$, we have

$$V(K\widetilde{+}\lambda B) = \sum_{i=0}^{n} \binom{n}{i} \lambda^i \widetilde{W}_i(K),$$

known as formula "Steiner decomposition". Moreover, for star bodies K and L, (4) can show the following special case:

$$\widetilde{W}_i(K\widetilde{+}\lambda L) = \sum_{j=0}^{n-i} \binom{n-i}{j} \lambda^j \widetilde{V}(\underbrace{K,\ldots,K}_{n-i-j},\underbrace{B,\ldots,B}_{i},\underbrace{L,\ldots,L}_{j}). \tag{5}$$

1.2 The p-radial addition and p-dual mixed volume

For any $p \neq 0$, the p-radial addition $K\widetilde{+}_p L$ defined by (see [6] and [7])

$$\rho(K\widetilde{+}_p L, u)^p = \rho(K,u)^p + \rho(L,u)^p, \tag{6}$$

for $u \in S^{n-1}$ and $K, L \in \mathcal{S}^n$. When $p = \infty$ or $-\infty$, the p-radial addition is interpreted as $\rho(K\widetilde{+}_\infty L, u) = K \cup L$ or $\rho(K\widetilde{+}_{-\infty} L, u) = K \cap L$ (see e. g. [8]).

The following result follows immediately from (6).

$$\frac{p}{n} \lim_{\varepsilon\to 0^+} \frac{V(K\widetilde{+}_p \varepsilon\cdot L) - V(L)}{\varepsilon} = \frac{1}{n}\int_{S^{n-1}} \rho(K.u)^{n-p}\rho(L.u)^p dS(u).$$

Let $K, L \in \mathcal{S}^n$ and $p \neq 0$, the p-dual mixed volume of star K and L, $\widetilde{V}_p(K,L)$, defined by

$$\widetilde{V}_p(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K.u)^{n-p}\rho(L.u)^p dS(u). \tag{7}$$

The Minkowski inequality for the p-radial addition stated that: If $K, L \in \mathcal{S}^n$ and $0 < p \le n$, then (see [7])

$$\widetilde{V}_p(K,L)^n \le V(K)^{n-p}V(L)^p, \tag{8}$$

with equality if and only if K and L are dilates.

The inequality is reversed for $p > n$ or $p < 0$

2 The L_p-dual mixed volume for $0 < p < 1$

For $p \geq 1$, Lutwak defined the L_p-harmonic addition of star bodies K and L, $K \check{+}_p \varepsilon \diamond L$, defined by (see [9])

$$\rho(K \check{+}_p \varepsilon \diamond L, \cdot)^{-p} = \rho(K, \cdot)^{-p} + \varepsilon \rho(L, \cdot)^{-p}. \tag{9}$$

As defined in (9), $K \check{+}_p \varepsilon \diamond L$ has a constant coefficient p restricted to $p \geq 1$. We now extend the definition so that $K \check{+}_p L$ is defined for $0 < p < 1$.

Definition 1 *(The L_p-harmonic addition for $0 < p < 1$) If $K, L \in \mathcal{S}^n$ and $0 < p < 1$, the L_p-harmonic addition of star bodies K and L, $K \check{+}_p \varepsilon \diamond L$, defined by*

$$\rho(K \check{+}_p \varepsilon \diamond L, \cdot)^{-p} = \rho(K, \cdot)^{-p} + \varepsilon \rho(L, \cdot)^{-p}. \tag{10}$$

From (10), it is easy that for $0 < p < 1$ (and $p \geq 1$)

$$-\frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K \check{+}_p \varepsilon \diamond L) - V(K)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n+p} \rho(L, u)^{-p} dS(u).$$

Definition 2 *If $K, L \in \mathcal{S}^n$ and $0 < p < 1$, the L_p-dual mixed quermassintegral of K and L, $\widetilde{V}_{-p}(K, L)$, defined by*

$$\widetilde{V}_{-p}(K, L) := \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n+p} \rho(L, u)^{-p} dS(u). \tag{11}$$

Theorem 1 (L_p-*Minkowski inequality*) *If $K, L \in \mathcal{S}^n$ and $0 < p < 1$, then*

$$\widetilde{V}_{-p}(K, L)^n \geq V(K)^{n+p} V(L)^{-p}, \tag{12}$$

with equality if and only if K and L are dilates.

Proof. This integral representation (11) and together with Hölder integral inequality, this yields (12). □

The case $p \geq 1$, please see literatures [10] and [11].

Theorem 2 (L_p-*Brunn-Minkowski inequality*) *If $K, L \in \mathcal{S}^n$ and $0 < p < 1$, then*

$$\widetilde{V}(K \check{+}_p \varepsilon \diamond L)^{-p/n} \geq V(K)^{-p/n} + V(L)^{-p/n}, \tag{13}$$

with equality if and only if K and L are dilates.

Proof. This follows immediately from (10) and (12). □

3 The p-harmonic Blaschke addition for $0 < p < 1$

Let us recall the concept, the harmonic Blaschke addition, defined by Lutwak [12]. Suppose K and L are star bodies in $\mathbb{R}^n$, the harmonic Blaschke linear addition, $K\widehat{+}L$, by

$$\frac{\rho(K\widehat{+}L,\cdot)^{n+1}}{V(K\widehat{+}L)} = \frac{\rho(K,\cdot)^{n+1}}{V(K)} + \frac{\rho(L,\cdot)^{n+1}}{V(L)}. \tag{14}$$

Lutwak's Brunn-Minkowski inequality for the harmonic Blaschke addition was established (see [12]). If $K, L \in \mathcal{S}^n$, then

$$V(K\widehat{+}L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \tag{15}$$

with equality if and only if K and L are dilates. More generally, for any $p \geq 1$, the p-harmonic Blaschke addition $K\widehat{+}_pL$ defined by (see [13] and [14]).

$$\frac{\rho(K\widehat{+}_pL,\cdot)^{n+p}}{V(K\widehat{+}_pL)} = \frac{\rho(K,\cdot)^{n+p}}{V(K)} + \frac{\rho(L,\cdot)^{n+p}}{V(L)}. \tag{16}$$

The L_p Brunn-Minkowski inequality for the p-harmonic Blaschke addition was established (see [13]). If $K, L \in \mathcal{S}^n$ and $p \geq 1$, then

$$V(K\widehat{+}_pL)^{p/n} \geq V(K)^{p/n} + V(L)^{p/n}, \tag{17}$$

with equality if and only if K and L are dilates.

As defined in (16), $K\widehat{+}_pL$ has a constant coefficient p restricted to $p \geq 1$. We now extend the definition so that $K\widehat{+}_pL$ is defined for $0 < p < 1$.

Definition 3 (*The* p*-harmonic Blaschke addition for* $0 < p < 1$) *If* $K, L \in \mathcal{S}^n$, $0 \leq i < n$ *and* $0 < p < 1$, *the* p*-harmonic Blaschke addition of* K *and* L, $K\widehat{+}_pL$, *defined by*

$$\frac{\rho(K\widehat{+}_pL,\cdot)^{n-i+p}}{\widetilde{W}_i(K\widehat{+}_pL)} = \frac{\rho(K,\cdot)^{n-i+p}}{\widetilde{W}_i(K)} + \frac{\rho(L,\cdot)^{n-i+p}}{\widetilde{W}_i(L)}. \tag{18}$$

Obviously, the case $i = 0$ *and* $p \geq 1$, *is just* (16), *and the case of* $p = 1$ *and* $i = 0$, *is just* (14).

Definition 4 *Let* $K, L \in \mathcal{S}^n$, $0 \leq i < n$, $0 < p < 1$, *and* $\alpha, \beta \geq 0$ (*not both zero*), *the* p*-harmonic Blaschke liner combination of* K *and* L, $\alpha\blacklozenge K\widehat{+}_p\beta\blacklozenge L$, *defined by*

$$\frac{\rho(\alpha\blacklozenge K\widehat{+}_p\beta\blacklozenge L,u)^{n-i+p}}{\widetilde{W}_i(\alpha\blacklozenge K\widehat{+}_p\beta\blacklozenge L)} = \alpha\frac{\rho(K,u)^{n-i+p}}{\widetilde{W}_i(K)} + \beta\frac{\rho(L,u)^{n-i+p}}{\widetilde{W}_i(L)}. \tag{19}$$

From (19) *with* $\beta = 0$ *and* (1), *it is easy that*

$$\frac{\rho(\alpha\blacklozenge K,u)^{n-i+p}}{\widetilde{W}_i(\alpha\blacklozenge K)} = \alpha\frac{\rho(K,u)^{n-i+p}}{\widetilde{W}_i(K)} = \frac{\rho(\alpha^{1/p}K,u)^{n-i+p}}{\widetilde{W}_i(\alpha^{1/p}K)}.$$

Hence

$$\alpha\blacklozenge K = \alpha^{1/p}K. \tag{20}$$

4 Inequalities for p-harmonic Blaschke mixed quermassintegral for $0 < p < 1$

In order to define the p-harmonic Blaschke mixed quermassintegral for $0 < p < 1$ with respect to p-harmonic Blaschke addition, we need the following lemmas.

Lemma 1 ([15] *and* [16, p.51]) *If* $a, b \geq 0$ *and* $\lambda \geq 1$, *then*

$$a^\lambda + b^\lambda \leq (a+b)^\lambda \leq 2^{\lambda-1}(a^\lambda + b^\lambda). \tag{21}$$

Lemma 2 *Let* $0 < p < 1$, $0 \leq i < n$ *and* $\varepsilon > 0$. *If* $K, L \in \mathcal{S}^n$, *then*

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\frac{\rho(K\widehat{+}_p\varepsilon\blacklozenge L,u)^{n-i}-\rho(K,u)^{n-i}}{\varepsilon}\\ &\geq \frac{n-i}{n-i+p}\left(\frac{S_i(K)}{\widetilde{W}_i(K)}\rho(K,u)^{n-i}+\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\rho(K,u)^{-p}\rho(L,u)^{n-i+p}\right).\end{aligned} \tag{22}$$

Proof. From (19) and in view of the L'Hôpital's rule, we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\frac{\rho(K\widehat{+}_p\varepsilon\blacklozenge L,u)^{n-i}-\rho(K,u)^{n-i}}{\varepsilon}\\ &=\lim_{\varepsilon\to 0^+}\frac{\left(\left(\frac{\rho(K,u)^{n-i+p}}{\widetilde{W}_i(K)}+\varepsilon\frac{\rho(L,u)^{n-i+p}}{\widetilde{W}_i(L)}\right)\widetilde{W}_i(K\widehat{+}_p\varepsilon\blacklozenge L)\right)^{n-i/(n-i+p)}-\rho(K,u)^{n-i}}{\varepsilon}\\ &=\lim_{\varepsilon\to 0^+}\frac{n-i}{n-i+p}\left(\left(\frac{\rho(K,u)^{n-i+p}}{\widetilde{W}_i(K)}+\varepsilon\frac{\rho(L,u)^{n-i+p}}{\widetilde{W}_i(L)}\right)\widetilde{W}_i(K\widehat{+}_p\varepsilon\blacklozenge L)\right)^{-p/(n-i+p)}\\ &\times\left(\widetilde{W}_i(K\widehat{+}_p\varepsilon\blacklozenge L)'\left(\frac{\rho(K,u)^{n-i+p}}{\widetilde{W}_i(K)}+\varepsilon\frac{\rho(L,u)^{n-i+p}}{\widetilde{W}_i(L)}\right)+\widetilde{W}_i(K\widehat{+}_p\varepsilon\blacklozenge L)\frac{\rho(L,u)^{n-i+p}}{\widetilde{W}_i(L)}\right).\end{aligned} \tag{23}$$

In the following, we estimate the value of the derivative $\widetilde{W}_i(K\widehat{+}_p\varepsilon\blacklozenge L)'$. Let $f_i(t) = \widetilde{W}_i(K\widehat{+}_p t\blacklozenge L)$ and from (5), (20) and (21), we obtain

$$\begin{aligned}
f_i(t+\varepsilon) &= \widetilde{W}_i(K\widehat{+}_p(t+\varepsilon)\blacklozenge B)\\
&= \widetilde{W}_i(K\widehat{+}_p(t+\varepsilon)^{1/p}B)\\
&\geq \widetilde{W}_i(K\widehat{+}_p(t^{1/p}+\varepsilon^{1/p})B)\\
&\geq \widetilde{W}_i((K\widehat{+}_p t\blacklozenge B)+\varepsilon B)\\
&= \sum_{j=0}^{n-i}\tbinom{n-i}{j}\varepsilon^j\tilde{W}_{i+j}(K\widehat{+}_p t\blacklozenge B)\\
&= f_i(t)+\varepsilon(n-i)\tilde{W}_{i+1}(K\widehat{+}_p t\blacklozenge B)+o(\varepsilon^2).
\end{aligned}$$

Further

$$V(K\widehat{+}_p t\blacklozenge L)' = \lim_{\varepsilon\to 0^+}\frac{f(t+\varepsilon)-f(t)}{\varepsilon} \geq (n-i)\tilde{W}_{i+1}(K\widehat{+}_p t\blacklozenge B). \tag{24}$$

From (23) and (24) and in view of $(n-i)\tilde{W}_{i+1}(K) = S_i(K)$, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\frac{\rho(K\widehat{+}_p\varepsilon\blacklozenge L,u)^{n-i}-\rho(K,u)^{n-i}}{\varepsilon}\\
&\quad\geq \frac{n-i}{n-i+p}\left(\frac{S_i(K)}{\widetilde{W}_i(K)}\rho(K,u)^{n-i}+\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\rho(K,u)^{-p}\rho(L,u)^{n-i+p}\right).
\end{aligned}$$

□

Theorem 3 *Let $0 < p < 1$, $0 \leq i < n$ and $\varepsilon > 0$. If $K, L \in \mathcal{S}^n$, then*

$$\begin{aligned}
&\frac{n-i+p}{n-i}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widehat{+}_p\varepsilon\blacklozenge L,u)-\widetilde{W}_i(K)}{\varepsilon}\\
&\geq \left(S_i(K)+\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{-p}\rho(L,u)^{n-i+p}dS(u)\right).
\end{aligned}\tag{25}$$

Proof. This follows immediately from Lemma 2 and (3). □

Definition 5 *Let $K, L \in \mathcal{S}^n, 0 \leq i < n$ and $0 < p < 1$, we define the p-ith harmonic Blaschke mixed quermassintegral of star bodies K and L, denoted by $\widehat{W}_{p,i}(K,L)$, defined by*

$$\widehat{W}_{p,i}(K,L) = \frac{n-i+p}{n-i}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widehat{+}_p\varepsilon\blacklozenge L,u)-\widetilde{W}_i(K)}{\varepsilon}. \tag{26}$$

When $i = 0$*, the* p*-harmonic Blaschke mixed quermassintegral* $\widehat{W}_{p,i}(K, L)$ *becomes the* p*-harmonic Blaschke mixed volume* $\widehat{V}_p(K, L)$ *and*

$$\widehat{V}_p(K, L) = \frac{n+p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K \widehat{+}_p \varepsilon \blacklozenge L, u)^n - V(K)^n}{\varepsilon}. \tag{27}$$

Theorem 4 (L_p*-Minkowski type inequality) If* $K, L \in \mathcal{S}^n$*,* $0 \leq i < n$ *and* $0 < p < 1$*, then*

$$(\widehat{W}_{p,i}(K, L) - S_i(K))^{n-i} \geq \widetilde{W}_i(K)^{n-i-p} \widetilde{W}_i(L)^p \tag{28}$$

Proof. This follows immediately from Theorem 3, (27) and Hölder integral inequality. □

Corollary 1 *If* $K, L \in \mathcal{S}^n$ *and* $0 < p < 1$*, then*

$$(\widehat{V}_p(K, L) - S(K))^n \geq V(K)^{n-p} V(L)^p. \tag{29}$$

Proof. This follows immediately from Theorem 4 with $i = 0$. □

5 The relationship between the two mixed volumes

In the following, we give a relationship between the p-harmonic Blaschke mixed volume $\widehat{V}_p(K, L)$ and the L_p-dual mixed volume $\widetilde{V}_{-p}(K, L)$.

Theorem 5 *If* $K, L \in \mathcal{S}^n$ *and* $0 < p < 1$*, then*

$$\frac{\widehat{V}_p(K, L)}{V(K)} \geq \frac{\widetilde{V}_{-p}(L, K)}{V(L)}. \tag{30}$$

Proof. This follows immediately from (11), (27) and Theorem 3 with $i = 0$. □

We give also a relationship between the p-harmonic Blaschke mixed volume $\widehat{V}_p(K, L)$ and the p-dual mixed volume $\widetilde{V}_p(K, L)$.

Theorem 6 *If* $K, L \in \mathcal{S}^n$ *and* $0 < p < 1$*, then*

$$\widehat{V}_p(K, L) \geq \widetilde{V}_p(K, L). \tag{31}$$

Proof. From (11), (12), (8), (25) and (27), we obtain

$$\begin{aligned}\widehat{V}_p(K, L) &\geq \frac{V(K)}{V(L)}\frac{1}{n}\int_{S^{n-1}} \rho(L, u)^{n+p}\rho(K, u)^{-p} dS(u) \\ &= \frac{V(K)}{V(L)}\tilde{V}_{-p}(L, K) \\ &\geq \frac{V(K)}{V(L)}V(L)^{(n+p)/n}V(K)^{-p/n} \\ &= V(K)^{(n-p)/n}V(L)^{p/n} \\ &\geq \tilde{V}_p(K, L).\end{aligned}$$

□

Finally, we establish the Brunn-Minkowski inequality for the p-ith harmonic Blaschke addition.

Theorem 7 *If* $K, L \in \mathcal{S}^n$, $0 \leq i < n$, $0 < p < 1$ *and* $\lambda, \mu \geq 0$, *then*

$$\widetilde{W}_i(\lambda\blacklozenge K\widehat{+}_p\mu\blacklozenge L)^{p/(n-i)} \geq \lambda\widetilde{W}_i(K)^{p/(n-i)} + \mu\widetilde{W}_i(L)^{p/(n-i)}, \tag{32}$$

with equality if and only if K *and* L *are dilates.*

Proof. This follows immediately from (3), (19) and Minkowski integral inequality. □

This case of $\lambda = \mu = 1$, $p \geq 1$ and $i = 0$ is just (17). This case of $p = 1$, $\lambda = \mu = 1$ and $i = 0$ is just (15).

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