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The complexity of an exotic edge coloring of graphs

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Abstract. Coloring the nodes of a graph is a commonly used preprocessing method to speed up clique search procedures. For the very same purpose we propose coloring the edges of the graph. It will be shown that the recommended type of edge coloring leads to an NP-complete problem. Therefore in practical computations we should rely on some approximate algorithm.

1 Introduction

Let $G = (V, E)$ be a finite simple graph. In other words G has finitely many nodes and G does not have any double edge or loop. In this situation an edge of G can be identified with a two element subset of V. Consequently the set of edges E of G forms a family of two element subsets of V. A subgraph Δ of G is a clique if each two distinct nodes in Δ are adjacent. A clique with k nodes is called a k-clique. The number of the nodes of a clique sometimes referred as the size of the clique. A k-clique in G is a maximal clique if it is not a subgraph of any $(k+1)$-clique in G. A k-clique Δ in G is a maximum clique if G does not have any $(k+1)$-clique. The size of a maximum clique in G is called the clique size of G and it is denoted by $\omega(G)$.

Computing the clique size of a given graph has many important applications inside and outside of mathematics. Many of these applications are described in [1]. It was pointed out in [2] that the performance of their algorithms to

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determine the clique size of a given graph is critically depend on efficiently computable upper estimates of clique sizes. The most commonly used method to estimate clique size is via coloring the nodes of the graph. Although there is an interesting additional technique based on dynamic programming presented in [5] and further developed in [4] in this paper we will restrict our attention to the coloring idea.

Suppose that the nodes of a finite simple graph G are colored using k given colors such that

(1) each node of G receives exactly one color,

(2) adjacent nodes never receive the same color.

This type of coloring is the most commonly encountered coloring of the nodes of a graph. We will refer to it saying that the vertices of G have an L type coloring with k colors. The letter L stands for the expression legal coloring. The connection between the coloring and the clique size is the following. If the nodes of the graph G have an L type coloring with k colors then $\omega(G) \leq k$.

It should not came to us as a surprise that coloring the edges of a graph can provide upper estimates for the clique size. We color the edges of a graph G with k colors such that

(1) each edge of G receives exactly one color,

(2) if x, y, z are nodes of a 3-clique in G, then the edges $\{x, y\}$, $\{x, z\}$, $\{y, z\}$ receive three distinct colors.

For the sake of easier reference we call this type of coloring of the edges of a graph an S type edge coloring. The coloring could be called a rainbow triangle coloring. (The letter S stands for the initial letter of the word "rainbow" in Hungarian.) The minimum number of colors k for which the edges of an n-clique have an S type coloring is denoted by $\chi_S(n)$. A possible connection between the edge coloring and the clique size of a graph is the following. If the edges of a graph G have an S type coloring with k colors and $\omega(G) = t$, then $\chi_S(t) \leq k$ must hold. In other words if the edges of G have an S type coloring with k colors and $\chi_S(t) > k$, then $\omega(G) < t$.

It is not hard to construct an S type coloring of the edges of a given graph in greedy fashion. A greedy S type edge coloring together with the next lemma provide a practical way to estimate the clique size.

Lemma 1 $\chi_S(n) \geq n - 1$ *for each positive integer* n*.*

Proof. Let $\Delta = (V, E)$ be an n-clique and suppose that the edges of Δ have an S type coloring with k colors. Let $f : E \to \{1, \ldots, k\}$ be the coloring of the edges of Δ. Finally let $v_1, \ldots, v_n$ be all the vertices of Δ. Note that $f(\{v_1, v_i\}) \neq f(\{v_1, v_j\})$ holds for each i, j, $1 \leq i < j \leq n$ since the edges $\{v_1, v_i\}$, $\{v_1, v_j\}$, $\{v_i, v_j\}$ receive three distinct colors. In particular the edges $\{v_1, v_i\}$, $2 \leq i \leq n$ receive distinct colors. It follows that $\chi_S(n) \geq k \geq n - 1$, as required. □

The reader will notice that the exact value of $\chi_S(n)$ can be determined. Namely,

$$\chi_S(n) = \begin{cases} n, & \text{if } n \text{ is odd,} \\ n-1, & \text{if } n \text{ is even} \end{cases}$$

holds for each $n \geq 2$. Suppose that the edges of an n-clique Δ have an S type coloring. The edges of Δ receiving color c form the c-color class of the coloring. Notice that the edges in the c-color class must form a matching in Δ. A maximum matching is a 1-factor of Δ. It is a known result that an n-clique can be decomposed into $n-1$ 1-factors if n is even. Further, an n-clique cannot be decomposed into $n-1$ 1-factors when n is odd. In the next section we will see that the cruder result stated in Lemma 1 will suffice for our purposes.

By Lemma 1, if the edges of a given graph G have an S type coloring with k colors and $n - 1 > k$, then $\omega(G) < n$. By the main result of this note the problem to decide if the edges of a given graph have an S type coloring with k colors is an NP-complete problem for $k \geq 3$. This result loosely can be interpreted such that determining the minimum value of k for which the edges of G have an S type coloring with k colors is a computationally demanding problem.

2 Numerical experiments

The main motivation of this paper is to explore the possibility of utilizing edge coloring in clique search algorithms. It is relatively straightforward to construct S type edge coloring for a given graph in a greedy fashion. The greedy algorithm does not provide the optimum number of colors but it is computationally feasible.

Let $G = (V, E)$ be a given graph and suppose that we want to find an S type coloring of the edges of G. We locate a clique Δ in G. The clique Δ is not necessarily a largest clique in G. For our purposes any suboptimal clique is suitable. Let $e_1, \ldots, e_m$ be a fixed list of the edges of the given graph G such that we list first the edges of Δ then we list the remaining edges of G. Decomposing Δ into 1-factors and using the 1-factors as color classes we can

Name	$\|V\|$	$\|E\|$	L	S
MON03	27	189	6	9
MON04	64	1296	12	20
MON05	125	5500	20	35
MON06	216	17550	30	57
MON07	343	46305	42	79
MON08	512	106624	56	108
MON09	729	221616	72	141
MON10	1000	425250	90	178
MON11	1331	765325	110	218
MON12	1728	1306800	132	261
MON13	2197	2135484	156	309
MON14	2744	3362086	182	361
MON15	3375	5126625	210	418

Table 1: Graphs associated with monotonic matrices.

color the edges of Δ and we end up with a partial coloring of the edges of G. Suppose $C_1, \ldots, C_r$ are the existing color classes and e_i is the first uncolored edge of G. The edge e_i can be placed into the colors class C_1 if C_1 does not contain any edge e_j such that e_i and e_j are edges of a 3-clique in G. If e_i can be placed into C_1, then we put e_i into C_1. If e_i does not fit into C_1, then we try to place it into C_2. Continuing in this way either e_i fits into one of the colors classes $C_1, \ldots, C_r$ or we open a new color class C_{r+1} for e_i. When all the edges on the list $e_1, \ldots, e_m$ are colored, then we have an S type coloring of the edges of G.

We carried out a large scale numerical experiment to compare the upper estimates for the clique size of the given graph G provided by the ordinary L type node coloring and the proposed S type edge coloring of G. The results are summarized in Tables 1, 2, and 3. We considered $13 + 10 + 13 = 36$ graphs. These graphs are coming from coding theory. They are related to monotonic matrices, deletion error detecting, and error correcting codes, respectively. Using sequential greedy coloring algorithms we constructed an L type coloring of the nodes and an S type coloring of the edges for each graph. In the tables we listed the number of colors, the number of nodes and the number of edges of the graphs. From the results it is fairly clear that the greedy node coloring provides tighter estimates for the clique sizes of the graphs than the edge coloring does. Therefore in a clique search algorithm we do not recommend to

Name	|V|	|E|	L	S
DEL03	8	9	2	1
DEL04	16	57	4	4
DEL05	32	305	8	11
DEL06	64	1473	14	24
DEL07	128	6657	26	53
DEL08	256	28801	50	114
DEL09	512	121089	101	236
DEL10	1024	499713	199	492
DEL11	2048	2037761	395	995
DEL12	4096	8247297	782	2024

Table 2: Graphs associated with deletion error correcting codes.

Name	|V|	|E|	L	S
JOHNSON06	15	45	4	3
JOHNSON07	35	385	10	11
JOHNSON08	70	1855	20	26
JOHNSON09	126	6615	35	52
JOHNSON10	210	19425	56	85
JOHNSON11	330	49665	84	131
JOHNSON12	495	114345	120	197
JOHNSON13	715	242385	165	279
JOHNSON14	1001	480480	220	377
JOHNSON15	1365	900900	286	496
JOHNSON16	1820	1611610	364	646
JOHNSON17	2380	2769130	455	813
JOHNSON18	3060	4594590	560	1008

Table 3: Graphs associated with Johnson error correcting codes.

replace greedy sequential L type coloring of the nodes by greedy sequential S type coloring of the edges. We suggest to use the edge coloring in a different fashion. It can be used as a preconditioning tool.

We color the edges of the given graph G before the clique search starts. One can store the colors of the edges of G in an n by n matrix M conveniently. Here n is the number of the nodes of G. The rows and columns of M are labeled by the nodes of G and $m_{u,v}$ is the entry of M in the row labeled by node u and column labeled by node v. If c is the color of the edge $\{u, v\}$, then we set $m_{u,v}$ to be c. In the course of a clique search we can read off the colors of the edges from the matrix M with relatively low cost. Let H be a subgraph of G and suppose we are looking for a k-clique Δ in H. Note that if the edges of G have an S type coloring, then by inheritance the edges of H have an S type coloring too. The edges joining to a node v of Δ must have pair-wise distinct colors. Therefore if the edges of H joining to the node v are colored with less than $k-1$ colors, then v can be deleted from H. Deleting nodes from H reduces the size of the search space and might help in speeding up the computation.

3 A complexity result

Let $\Gamma = (V, E)$ be a finite simple graph. Using Γ we construct a new graph $G' = (V', E')$. We try to establish the following facts.

(1) If the nodes of Γ have an L type coloring with 3 colors, then the edges of G' have an S type coloring with 3 colors.

(2) If the edges of G' have an S type coloring with 3 colors, then the nodes of Γ have an L type coloring with 3 colors.

Let $v_1, \ldots, v_n$ be all the nodes of Γ. We assign a graph H_i to v_i for each i, $1 \leq i \leq n$. The constructions of H_i and G' are guided by the structure of the incidence matrix of Γ. The incidence matrix of Γ has $n = |V|$ rows and $m = |E|$ columns. The rows are labeled by the nodes $v_1, \ldots, v_n$ and the columns are labeled by the edges of Γ. If $e_k = \{v_i, v_j\}$ is an edge of Γ, then the two cells at the intersection of rows v_i, v_j and column e_k both contain a bullet.

We illustrate the construction working out the details in connection with a toy example. The graph Γ in the example can be seen in Figure 1 and the incidence matrix of this graph is in Table 4.

To vertex v_i of Γ we assign a graph H_i which has $4m$ nodes, where $m = |E|$. Let $K = (V'', E'')$ be a 4-clique such that $V'' = \{a, b, c, d\}$. We take m isomorphic copies $K_{i,1}, \ldots, K_{i,m}$ of K. We choose the notation such that $K_{i,j} =$

	e_1	e_2	e_3	e_4	e_5
v_1	•	•	•		
v_2	•			•	
v_3		•		•	•
v_4			•		•

Table 4: The node edge incidence matrix of of the graph Γ in the toy example.

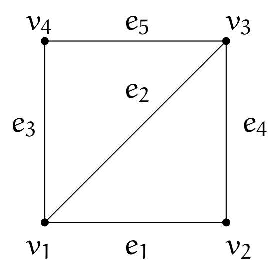

Figure 1: A geometric representation of the graph Γ in the toy example.

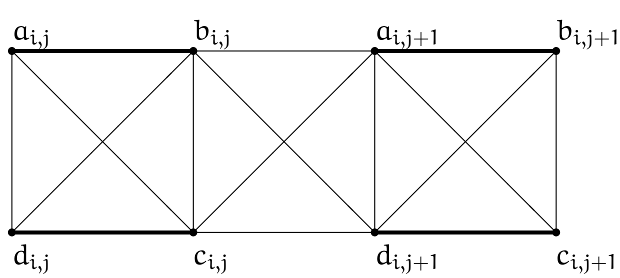

Figure 2: A step of the construction of H_i. The 1st square is $K_{i,j}$ and the 3rd square is $K_{i,j+1}$.

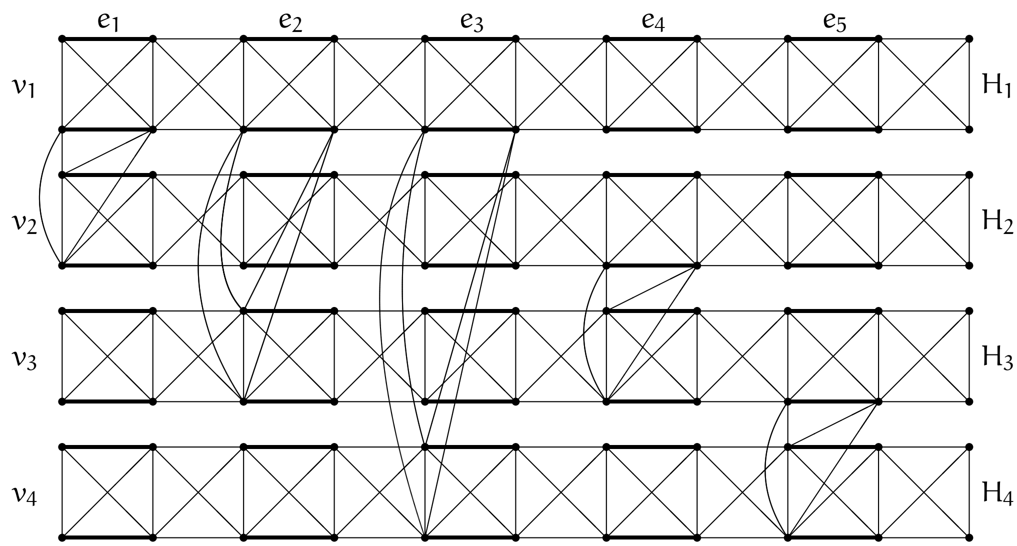

Figure 3: The graphs $H_1, \ldots, H_4$ in the toy example.

$(V''_{i,j}, E''_{i,j})$ and $V''_{i,j} = \{a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}\}$. We add the edges

$$\{b_{i,j}, a_{i,j+1}\}, \quad \{b_{i,j}, d_{i,j+1}\}, \quad \{c_{i,j}, a_{i,j+1}\}, \quad \{c_{i,j}, d_{i,j+1}\},$$

for each j, $1 \leq j \leq m-1$. This step of the construction is depicted in Figure 2. Finally we add the edges

$$\{b_{i,m}, a_{i,1}\}, \quad \{b_{i,m}, d_{i,1}\}, \quad \{c_{i,m}, a_{i,1}\}, \quad \{c_{i,m}, d_{i,1}\}.$$

We encourage the reader to visualize H_i as a long narrow paper strip divided into $2m$ squares. The two opposite short sides of the rectangle are united to form a closed strip. However, we draw H_i as an open flattened strip in Figure 3 in order not to clutter the diagram. Figure 3 exhibits the geometric representations of the graphs $H_1, \ldots, H_4$ associated with the vertices $v_1, \ldots, v_4$ of the toy example Γ.

If v_i and v_j are adjacent edges in Γ such that $i < j$, then to represent the edge $e_k = \{v_i, v_j\}$ of Γ in G' we add the edges

$$\{a_{j,k}, c_{i,k}\}, \quad \{a_{j,k}, d_{i,k}\}, \quad \{d_{j,k}, c_{i,k}\}, \quad \{d_{j,k}, d_{i,k}\}$$

to G'. (The reader may follow the flow of the argument in Figure 6.) If v_i and v_j are not adjacent edges in Γ, then we do not add any extra edges to G'. The

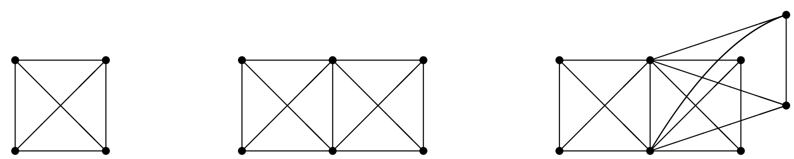

Figure 4: The subgraphs spanned by $N(u, v)$ in the proof of Lemma 2.

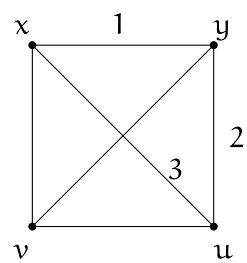

Figure 5: Coloring the edges of a 4-clique in the proof of Lemma 3.

toy example Γ has five edges

$$e_1 = \{v_1, v_2\}, \quad e_2 = \{v_1, v_3\}, \quad e_3 = \{v_1, v_4\},$$
$$e_4 = \{v_2, v_3\}, \quad e_5 = \{v_3, v_4\}.$$

The reader can spot five modifications corresponding to these edges in Figure 3.

When we analyze the graph G' we will use the following two lemmas.

Lemma 2 *If Γ has at least one edge, then the clique number of G' is equal to 4. In symbols $\omega(G') = 4$.*

Proof. Since Γ has an edge, it follows that G' contains a 4-clique. Consequently $\omega(G') \geq 4$. It remains to show that $\omega(G') \leq 4$.

Let Δ be a maximum clique in G' and let $\{u, v\}$ be an edge in Δ. Let $N(u, v)$ be the set of the next nodes of G'.

(1) The nodes u and v.

(2) All the nodes adjacent to both u and v.

We call $N(u, v)$ the neighborhood set of the edge $\{u, v\}$.

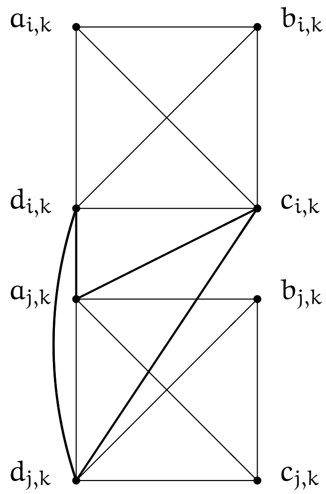

Figure 6: The connecting device in the first construction.

An inspection shows that the subgraph of G' spanned by the neighborhood set $N(u, v)$ can only be one of the three graphs shown in Figure 4. Since Δ must be a subgraph of the graph spanned by $N(u, v)$, it follows that $\omega(G') \leq 4$. □

Lemma 3 *Let Δ be a 4-clique with nodes x, y, u, v. If the edges of Δ have an S type coloring with 3 colors, then the "opposite" edges $\{x, y\}$ and $\{u, v\}$ must receive the same color.*

Proof. The edges of the 3-clique whose nodes are x, y, u must receive three distinct colors. We may assume that the edges $\{x, y\}$, $\{y, u\}$, $\{x, u\}$ receive colors 1, 2, 3 respectively since this is only a matter of rearranging the colors 1, 2, 3 among each other. Edge $\{x, v\}$ cannot receive color 1 because $\{x, v\}$ and $\{x, y\}$ are edges of the 3-clique with nodes x, y, v. Edge $\{x, v\}$ cannot receive color 3 since $\{x, v\}$ and $\{x, u\}$ are edges of the 3-clique with nodes x, u, v. Thus edge $\{x, v\}$ must receive color 2. Finally, edge $\{y, v\}$ has to be colored with color 3 and edge $\{u, v\}$ must be colored with color 1. (The reasoning can be followed in Figure 5.) □

Suppose now that the nodes of Γ have an L type coloring with 3 colors. Let $f : V \to \{1, 2, 3\}$ be the coloring. Let us consider the subgraph H_i of G' assigned to node v_i of Γ. We color the edge $\{a_{i,1}, d_{i,1}\}$ of G' with color $f(v_i)$. We know from Lemma 3 that the edge $\{b_{i,1}, c_{i,1}\}$ must be colored with color $f(v_i)$ in order to define an S type coloring of the edges of G' with 3 colors. Therefore we color all the "vertical" edges

$$\{a_{i,j}, d_{i,j}\}, \quad \{b_{i,j}, c_{i,j}\}, \quad 1 \leq j \leq m$$

of H_i with color $f(v_i)$. In a fixed 4-clique in H_i we color the opposite "horizontal" edges with the same color. Similarly in a fixed 4-clique in H_i we color the "diagonal" edges with the same color. If the colors used for the vertical, horizontal, and diagonal edges are pair-wise distinct, then the edges of H_i have an S type coloring with 3 colors.

There are further edges in G' which play the role of connecting devices between H_i and H_j when v_i and v_j are adjacent nodes of Γ. Let $e_k = \{v_i, v_j\}$ be the edge of Γ connecting the vertices v_i and v_j. The color of the edge $\{a_{j,k}, d_{j,k}\}$ has already been assigned to be $f(v_j)$. This forces us to color the edge $\{d_{i,k}, c_{i,k}\}$ with color $f(v_j)$. But in the 4-clique $K_{i,k}$ with edges $a_{i,k}$, $b_{i,k}$, $c_{i,k}$, $d_{i,k}$ only the color of the vertical edges are fixed to be $f(v_i)$ and so we have a freedom to choose the color of the horizontal edges.

Summing up our considerations we may say that the edges of the graph G' have an S type coloring with 3 colors provided that the nodes of Γ have an L type coloring with 3 colors.

Suppose now that the edges of G' have an S type coloring with 3 colors. Let $f' : E' \to \{1, 2, 3\}$ be this coloring. In particular the edges of the subgraph H_i of G' have an S type coloring with 3 colors for each i, $1 \leq i \leq n$. By Lemma 3, in an S type coloring of the edges of H_i the vertical edges must receive the same color. This colors is $f'(\{a_{i,1}, d_{i,1}\})$. We color the node v_i of Γ with this color. In other words we define a map $f : V \to \{1, 2, 3\}$ by setting $f(v_i)$ to be $f'(\{a_{i,1}, d_{i,1}\})$.

We claim that $f(v_i) = f(v_j)$ implies that v_i and v_j are not adjacent nodes of Γ.

In order to verify the claim assume on the contrary that v_i and v_j are adjacent nodes of Γ and $f(v_i) = f(v_j)$ holds. Let $e_k = \{v_i, v_j\}$ be the edge of Γ that connects the nodes v_i and v_j. Let us consider the 4-clique $K_{i,j,k}$ of G' whose vertices are $c_{i,k}$, $d_{i,k}$, $a_{j,k}$, $d_{j,k}$. Since the edges of G' have an S type coloring with 3 colors, it follows that the edges of the 4-clique $K_{i,j,k}$ have an S type coloring with 3 colors. Lemma 3 is applicable to $K_{i,j,k}$ and gives that the edge $\{a_{j,k}, d_{j,k}\}$ of H_j and the edge $\{d_{i,k}, c_{i,k}\}$ of H_i are colored with the same color. This common color is $f(v_j)$. The vertical edge $\{a_{i,k}, d_{i,k}\}$ of H_i is colored with color $f(v_i)$. This implies $f(v_i) = f(v_j)$. From $f(v_i) = f(v_j)$, it follows that two edges of the 3-clique with nodes $a_{i,k}$, $d_{i,k}$, $c_{i,k}$ are colored with the same

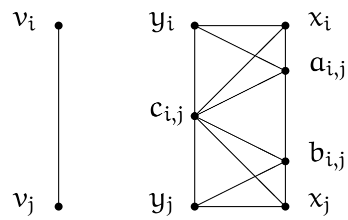

Figure 7: The graph assigned to the edge $\{v_i, v_j\}$.

color. Namely the edges $\{a_{i,k}, d_{i,k}\}$ and $\{c_{i,k}, d_{i,k}\}$ are receiving the same color. This contradiction proves our claim.

Theorem 4 *The problem to decide if the edges of a finite simple graph have an S type coloring with 3 colors is an NP-complete problem.*

Proof. For the proof we should recall the known result that the problem of deciding if the nodes of a finite simple graph have an L type coloring with 3 colors is an NP-complete problem. The result on the coloring of the nodes can be found for example in [3] or [6]. □

4 An alternative construction

In this section we give a second proof for Theorem 4 using a new construction.

Proof. Let $\Gamma = (V, E)$ be a finite simple graph. Using Γ we construct a new graph $G' = (V', E')$. We try to show that the following requirements hold.

(1) If the nodes of Γ have an L type coloring with 3 colors, then the edges of G' have an S type coloring with 3 colors.

(2) If the edges of G' have an S type coloring with 3 colors, then the nodes of Γ have an L type coloring with 3 colors.

Let $v_1, \ldots, v_n$ be all the nodes of Γ. We assign two points x_i and y_i to node v_i for each i, $1 \leq i \leq n$. We choose the points $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ to be pair-wise distinct. We connect the nodes x_i and y_i in G' with an edge.

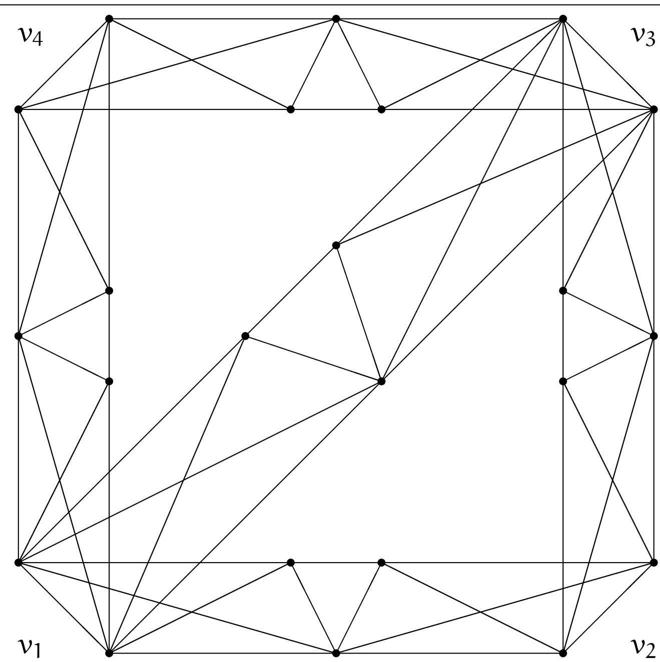

Figure 8: The graph G' associated with the toy example.

If v_i and v_j are adjacent nodes in Γ, then we add three new nodes $a_{i,j}$, $b_{i,j}$, $c_{i,j}$ and eleven new edges

$$\begin{array}{llll} \{x_i, a_{i,j}\}, & \{x_i, c_{i,j}\}, & \{y_i, a_{i,j}\}, & \{y_i, c_{i,j}\}, \\ \{x_j, b_{i,j}\}, & \{x_j, c_{i,j}\}, & \{y_j, b_{i,j}\}, & \{y_j, c_{i,j}\}, \\ \{a_{i,j}, b_{i,j}\}, & \{a_{i,j}, c_{i,j}\}, & \{b_{i,j}, c_{i,j}\}. & \end{array}$$

If v_i and v_j are not adjacent in Γ, then we do not add any new node or new edge to G'. This step of the construction is illustrated in Figure 7. The graph G' associated with the toy example is in Figure 8.

Suppose first that the nodes of Γ have an L type coloring with 3 colors. Let $f : V \to \{1, 2, 3\}$ be such a coloring. We define an edge coloring $f' : E' \to \{1, 2, 3\}$ of G'. To do so we set $f'(\{x_i, y_i\})$ to be $f(v_i)$ and we set

$$f'(\{a_{i,j}, c_{i,j}\}) = f(v_i), \quad f'(\{b_{i,j}, c_{i,j}\}) = f(v_j).$$

Let us consider the 3-clique Δ in G' whose nodes are $a_{i,j}$, $b_{i,j}$, $c_{i,j}$. Two edges of Δ has already been colored. So we color the edge $\{a_{i,j}, b_{i,j}\}$ with the only color in the set $\{1, 2, 3\} \setminus \{f(v_i), f(v_j)\}$. We color the edges $\{x_i, a_{i,j}\}$, $\{y_i, c_{i,j}\}$ with one of the colors in the set $\{1,2,3\} \setminus \{f(v_i)\}$ and we color the edges $\{x_i, c_{i,j}\}$, $\{y_i, a_{i,j}\}$ with the remaining last color. Similarly, we color the edges $\{x_j, b_{i,j}\}$, $\{y_j, c_{i,j}\}$ with one of the colors in the set $\{1,2,3\} \setminus \{f(v_j)\}$ and we color the edges $\{x_j, c_{i,j}\}$, $\{y_j, c_{i,j}\}$ with the remaining last color.

An inspection shows that the coloring $f' : E' \to \{1,2,3\}$ is an S type coloring of the edges of G'.

Next suppose that the edges of G' have an S type coloring with 3 colors. Let $f' : E' \to \{1,2,3\}$ be such a coloring. Using the edge coloring f' of G' we define a coloring $f : V \to \{1,2,3\}$ of the nodes of Γ by setting $f(v_i)$ to be $f'(\{x_i, y_i\})$. We claim that $f(v_i) = f(v_j)$ implies that v_i and v_j are not adjacent in Γ.

In order to prove the claim assume on the contrary that $f(v_i) = f(v_j)$ and the nodes v_i and v_j are adjacent in Γ. Since f' is an S type coloring of the edges of G', it follows that

$$\begin{aligned} f'(\{a_{i,j}, c_{i,j}\}) &= f'(\{x_i, y_i\}) = f(v_i), \\ f'(\{b_{i,j}, c_{i,j}\}) &= f'(\{x_j, y_j\}) = f(v_j). \end{aligned}$$

Let us watch the 3-clique Δ in G' whose nodes are $a_{i,j}$, $b_{i,j}$, $c_{i,j}$. (The reader may consult with Figure 7.) We get the contradiction that two edges of Δ are colored with the same color. □

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