Acta Univ. Sapientiae, Mathematica, 4, 2 (2012) 154–167

Spaces of entire functions represented by vector valued Dirichlet series of slow growth

G. S. Srivastava
Indian Institute of Technology Roorkee
Department of Mathematics
Roorkee, India
email: girssfma@iitr.ernet.in

Archna Sharma
Indian Institute of Technology Roorkee
Department of Mathematics
Roorkee, India
email: gs91490@gmail.com

Abstract. Spaces of all entire functions f represented by vector valued Dirichlet series and having slow growth have been considered. These are endowed with a certain topology under which they become a Frechet space. On this space the form of linear continuous transformations is characterized. Proper bases have also been characterized in terms of growth parameters.

1 Introduction

Let

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \;\; s = \sigma + it \; (\sigma, t \text{ are real variables}), \tag{1}$$

where $\{a_n\}$ is a sequence of complex numbers and the sequence $\{\lambda_n\}$ satisfies the conditions $0 < \lambda_1 < \lambda_2 < \lambda_3 \ldots < \lambda_n \ldots, \lambda_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \sup \frac{n}{\lambda_n} = D < \infty, \tag{2}$$

$$\lim_{n\to\infty} \sup(\lambda_{n+1} - \lambda_n) = h > 0, \tag{3}$$

2010 Mathematics Subject Classification: 30D15, 46A35
Key words and phrases: vector valued Dirichlet series, entire functions, spaces, bases

and

$$\lim_{n\to\infty} \sup \frac{\log |a_n|}{\lambda_n} = -\infty. \tag{4}$$

By giving different topologies on the set of entire functions represented by the Dirichlet series, Kamthan and Hussain [2] have studied various properties of this space.

Now let $a_n \in E, n = 1, 2, \ldots$, where $(E, \|\cdot\|)$ is a complex Banach space and (4) is replaced by the condition

$$\lim_{n\to\infty} \sup \frac{\log \|a_n\|}{\lambda_n} = -\infty. \tag{5}$$

Then the series in (1) is called a vector valued Dirichlet series and represents an entire function $f(s)$. In what follows, the series in (1) will represent a Vector valued entire Dirichlet series.

Let for entire functions defined as above by (1) and satisfying (2), (3) and (5),

$$M(\sigma, f) = M(\sigma) = \sup_{-\infty<t<\infty} \|f(\sigma + it)\|.$$

Then $M(\sigma)$ is called the maximum modulus of $f(s)$. The order ρ of $f(s)$ is defined as [1]

$$\rho = \lim_{\sigma\to\infty} \sup \frac{\log\log M(\sigma)}{\sigma}, \qquad 0 \le \rho \le \infty \tag{6}$$

Also, for $0 < \rho < \infty$ the type T of $f(s)$ is defined by [1]

$$T = \lim_{\sigma\to\infty} \sup \frac{\log M(\sigma)}{e^{\sigma\rho}}, \qquad 0 \le T \le \infty.$$

It was proved by Srivastava [1] that if $f(s)$ is of order ρ $(0 < \rho < \infty)$ and (2) holds then $f(s)$ is of type T if and only if

$$T = \lim_{n\to\infty} \sup \frac{\lambda_n}{\rho e} \|a_n\|^{\rho/\lambda_n}.$$

This implies

$$\lim_{n\to\infty} \sup \lambda_n^{1/\rho} \|a_n\|^{1/\lambda_n} = (T\rho e)^{1/\rho}. \tag{7}$$

We now denote by X the set of all vector valued entire functions $f(s)$ given by (1) and satisfying (2), (3) and (5) for which

$$\lim_{\sigma\to\infty} \sup \frac{\log M(\sigma)}{e^{\sigma\rho}} \le T < \infty, \qquad 0 < \rho < \infty.$$

Then from (7), we have

$$\lim_{n\to\infty} \sup \lambda_n^{1/\rho} \|a_n\|^{1/\lambda_n} \le (T\rho e)^{1/\rho} \ . \tag{8}$$

From (8), for arbitrary $\varepsilon > 0$ and all $n > n_0(\varepsilon)$,

$$\|a_n\|. \left[\frac{\lambda_n}{(T+\varepsilon)\, e\rho}\right]^{\lambda_n/\rho} < 1.$$

Hence, if we put

$$\|f\|_q = \sum_{n\ge 1} \|a_n\| \left[\frac{\lambda_n}{(T+q^{-1})\, e\rho}\right]^{\lambda_n/\rho} \qquad q \ge 1,$$

then $\|f\|_q$ is well defined and for $q_1 \le q_2$, $\|f\|_{q_1} \le \|f\|_{q_2}$. This norm induces a metric topology on X. We define

$$\lambda(f,g) = \sum_{q\ge 1} \frac{1}{2^q} \cdot \frac{\|f-g\|_q}{1+\|f-g\|_q}$$

We denote the space X with the above metric λ by X_λ. Various properties of bases of the space X_λ using the growth properties of the entire vector valued Dirichlet series have been obtained in [3]. These results obviously do not hold if the order ρ of the entire function $f(s)$ is zero. In this paper we have introduced a metric on the space of entire function of zero order represented by vector valued Dirichlet series thereby obtaining various properties of this space.

2 Main results

The vector valued entire function $f(s)$ represented by (1), for which order ρ defined by (6) is equal to zero, we define the logarithmic order ρ^* by

$$\rho^* = \lim_{\sigma\to\infty} \sup \frac{\log\log M(\sigma)}{\log \sigma}, \qquad 1 \le \rho^* \le \infty.$$

For $1 < \rho^* < \infty$ the logarithmic type T^* is defined by

$$T^* = \lim_{\sigma\to\infty} \sup \frac{\log M(\sigma)}{\sigma^{\rho^*}}, \qquad 0 \le T^* \le \infty.$$

In [4] the authors have established that $f(s)$ is of logarithmic order ρ^*, $1 < \rho^* < \infty$, and logarithmic type T^*, $0 < T^* < \infty$, if and only if

$$\lim_{n\to\infty} \sup \frac{\lambda_n \phi(\lambda_n)}{\log \|a_n\|^{-1}} = \frac{\rho^*}{(\rho^* - 1)} (\rho^* T^*)^{1/(\rho^*-1)}, \tag{9}$$

where $\phi(t)$is the unique solution of the equation $t = \sigma^{\rho^*} - 1$. The above formula can be proved on the same lines as for ordinary Dirichlet series in [5]. Let Y denote the set of all entire functions $f(s)$ given by (1) and satisfying (2), (3) and (5), for which

$$\lim_{\sigma\to\infty} \sup \frac{\log M(\sigma)}{\sigma^\rho} \leq T^* < \infty, \qquad 0 < \rho^* < \infty.$$

Then from (9) we have

$$\lim_{n\to\infty} \sup \frac{\lambda_n \phi(\lambda_n)}{\log \|a_n\|^{-1}} \leq \frac{\rho^*}{(\rho^* - 1)} (\rho^* T^*)^{1/(\rho^*-1)}, \tag{10}$$

where $\phi(\lambda_n) = \lambda_n^{1/\rho^*-1}$. From (10), for arbitrary $\varepsilon > 0$ and all $n > n_0(\varepsilon)$,

$$\|a_n\| \leq \exp\left[-\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \varepsilon)\}^{1/(\rho^*-1)}} \right], \tag{11}$$

where $K = \{\rho^*/(\rho^* - 1)\}^{(\rho^*-1)}$ be a constant. For each $f \in Y$, we define the norm

$$\|f\|_\alpha = \sum_{n\geq 1} \|a_n\| \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}} \right], \qquad \alpha \geq 1$$

then $\|f\|_\alpha$ is well defined and for $\alpha_1 \leq \alpha_2$, $\|f\|_{\alpha_1} \leq \|f\|_{\alpha_2}$. This norm induces a metric topology on Y defined by

$$d(f, g) = \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha} \cdot \frac{\|f - g\|_\alpha}{1 + \|f - g\|_\alpha}.$$

We denote the space Y with the above metric d by Y_d. Now we prove

Theorem 1 *The space* Y_d *is a Frechet space.*

Proof. Here, Y_d is a normed linear metric space. For showing that Y_d is a Frechet space, we need to show that Y_d is complete. Hence, let $\{f_p\}$ be a Cauchy sequence in Y_d. Therefore, for any given $\varepsilon > 0$ there exists an integer $n_0 = n_0(\varepsilon)$ such that

$$d(f_p, f_q) < \varepsilon \ \forall\ p, q > n_0.$$

Hence $\|f_p - f_q\|_\alpha < \varepsilon \ \forall\ p, q > n_0,\ \alpha \geq 1$.
Denoting by $f_p(s) = \sum_{n=1}^{\infty} a_n^{(p)} e^{s\cdot\lambda_n}$, $f_q(s) = \sum_{n=1}^{\infty} a_n^{(q)} e^{s\cdot\lambda_n}$, we have therefore

$$\sum_{n=1}^{\infty} \|a_n^{(p)} - a_n^{(q)}\| \cdot \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}}\right] < \varepsilon \tag{12}$$

for all $p, q > n_0$, $\alpha \geq 1$. Since $\lambda_n \to \infty$ as $n \to \infty$, therefore we have $\|a_n^{(p)} - a_n^{(q)}\| < \varepsilon\ \forall p, q \geq n_0$, and $n = 1, 2, \ldots$, i.e. for each fixed $n = 1, 2, \ldots$, $\left\{a_n^{(p)}\right\}$ is a Cauchy sequence in the Banach space E.

Hence there exists a sequence $\{a_n\} \subseteq E$ such that

$$\lim_{p\to\infty} a_n^{(p)} = a_n, \quad n \geq 1.$$

Now letting $q \to \infty$ in (12), we have for $p \geq n_0$,

$$\sum_{n=1}^{\infty} \|a_n^{(p)} - a_n\| \cdot \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}}\right] < \varepsilon \tag{13}$$

Taking $p = n_0$, we get for a fixed α in (12)

$$\|a_n\| \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}}\right] <$$
$$\|a_n^{(n_0)}\| \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}}\right] + \varepsilon$$

Now $f^{(n_0)} = \sum_{n=1}^{\infty} a_n^{(n_0)} e^{s.\lambda_n} \in Y_d$, hence the condition (11) is satisfied. For arbitrary $\alpha < \beta$, we have, $\|a_n^{(n_0)}\| < \exp\left[\frac{-\lambda_n \phi(\lambda_n)}{\{K\cdot\rho^*(T^*+\beta^{-1})\}^{1/(\rho^*-1)}}\right]$ for arbitrarily large n. Hence we have,

$$\|a_n\| \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}}\right] <$$
$$\exp\left[\frac{\lambda_n \phi(\lambda_n)}{(K \cdot \rho^*)^{1/(\rho^*-1)}} \left\{\frac{1}{(T^* + \alpha^{-1})^{1/(\rho^*-1)}} - \frac{1}{(T^* + \beta^{-1})^{1/(\rho^*-1)}}\right\}\right] + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary and the first term on the right hand side $\to 0$ as $n \to \infty$, we find that the sequence $\{a_n\}$ satisfies (11) and therefore $f(s) = \sum_{n=1}^{\infty} a_n e^{s \cdot \lambda_n}$ belongs to Y_d. Using (13) again, we have for α=1, 2...,

$$\|f_p - f\|_\alpha < \varepsilon.$$

Hence

$$d\left(f_p, f\right) = \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha} \frac{\|f_p - f\|_\alpha}{1 + \|f_p - f\|_\alpha} \leq \frac{\varepsilon}{1+\varepsilon} \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha} < \varepsilon.$$

Since the above inequality holds for all $p > n_0$, we finally get $f_p \to f$ as $p \to \infty$ with respect to the metric d, where $f \in Y_d$. Hence Y_d is complete. This proves Theorem 1. □

Next we prove

Theorem 2 *A continuous linear transformation* $\psi : Y_d \to E$ *is of the form*

$$\psi\left(f\right) = \sum_{n=1}^{\infty} a_n C_n$$

if and only if

$$|C_n| \leq A \cdot \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}}\right] \qquad \textit{for all } n \geq 1, \alpha \geq 1, \qquad (14)$$

where A *is a finite, positive number,* $f = f\left(s\right) = \sum_{n=1}^{\infty} a_n e^{s \cdot \lambda_n}$ *and* λ_1 *is sufficiently large.*

Proof. Let $\psi : Y_d \to E$ be a continuous linear transformation then for any sequence $\{f_m\} \subseteq Y_d$ such that $f_m \to f$, we have $\psi\left(f_m\right) \to \psi\left(f\right)$ as $m \to \infty$. Now, let $f\left(s\right) = \sum_{n=1}^{\infty} a_n e^{s . \lambda_n}$ where $a_n's \in E$ satisfy (11). Then $f \in Y_d$. Also, let $f_k\left(s\right) = \sum_{n=1}^{k} a_n \, e^{s\lambda_n}$. Then $f_k \in Y_d$ for $k = 1, 2 \ldots$. Let α be any fixed positive integer and let $0 < \varepsilon < \alpha^{-1}$. From (11) we can find an integer m such that

$$\|a_n\| < \exp\left[\frac{-\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \varepsilon)\}^{1/(\rho^*-1)}}\right], \;\; \forall n > m.$$

Then

$$\left\| f - \sum_{n=1}^{m} a_n e^{s\cdot\lambda_n} \right\|_\alpha = \left\| \sum_{n=m+1}^{\infty} a_n e^{s\cdot\lambda_n} \right\|_\alpha$$
$$= \sum_{n=m+1}^{\infty} \|a_n\| \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}} \right]$$
$$< \sum_{n=m+1}^{\infty} \exp\left[\frac{\lambda_n \phi(\lambda_n)}{(K \cdot \rho^*)^{1/(\rho^*-1)}} \left\{ (T^* + \alpha^{-1})^{-1/(\rho^*-1)} - (T^* + \varepsilon)^{-1/(\rho^*-1)} \right\} \right] < \varepsilon,$$

for sufficiently large values of m.

Hence

$$d(f, f_m) = \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha} \frac{\|f - f_m\|_\alpha}{1 + \|f - f_m\|_\alpha} \leq \frac{\varepsilon}{1+\varepsilon} < \varepsilon,$$

i.e. $f_m \to f$ as $m \to \infty$ in Y_d. Since ψ is continuous, we have

$$\lim_{m\to\infty} \psi(f_m) = \psi(f).$$

Let us denote by $C_n = \psi\left(e^{s\cdot\lambda_n}\right)$. Then

$$\psi(f_m) = \sum_{n=1}^{m} a_n \psi\left(e^{s.\lambda_n}\right) = \sum_{n=1}^{m} a_n C_n.$$

Also $|C_n| = |\psi\left(e^{s\lambda_n}\right)|$. Since ψ is continuous on Y_d it is continuous on $Y_{\|\cdot\|_\alpha}$ for each $\alpha = 1, 2, 3 \ldots$. Hence there exists a positive constant A independent of α such that

$$|\psi\left(e^{s.\lambda_n}\right)| = |C_n| \leq A\|p\|_\alpha, \qquad \alpha \geq 1$$

where $p(s) = e^{s\cdot\lambda_n}$. Now using the definition of the norm for $p(s)$, we get

$$|C_n| \leq A \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}} \right], \qquad n \geq 1, \quad \alpha \geq 1.$$

Hence we get $\psi(f) = \sum_{n=1}^{\infty} a_n C_n$, where the sequence $\{C_n\}$ satisfies (14).

Conversely, suppose that $\psi(f) = \sum_{n=1}^{\infty} a_n C_n$ and $C_n's$ satisfy (14). Then for $\alpha \geq 1$,

$$\|\psi(f)\| \leq A \sum_{n=1}^{\infty} \|a_n\| \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}} \right]$$

i.e. $|\psi(f)| \leq A\|f\|_\alpha \ \alpha \geq 1$.

Now, since $d(f,g) = \sum_{\alpha\geq 1} \frac{1}{2^\alpha} \cdot \frac{\|f-g\|_\alpha}{1+\|f-g\|_\alpha}$, therefore ψ is continuous. This completes the proof of Theorem 2. □

3 Linear continuous transformations and proper bases

Following Kamthan and Hussain [2] we give some more definitions. A subspace X_0 of X is said to be spanned by a sequence $\{\alpha_n\} \subseteq X$ if X_0 consists of all linear combinations $\sum_{n=1}^{\infty} c_n\alpha_n$ such that $\sum_{n=1}^{\infty} c_n\alpha_n$ converges in X. A sequence $\{\alpha_n\} \subseteq X$ which is linearly independent and spans a subspace X_0 of X is said to be a base in X_0. In particular, if $e_n \in X$, $e_n(s) = e^{s\lambda_n}$, $n \geq 1$, then $\{e_n\}$ is a base in X. A sequence $\{\alpha_n\} \subseteq X$ will be called a 'proper base' if it is a base and it satisfies the condition:
"for all sequences $\{a_n\} \subseteq E$, convergence of $\sum_{n=1}^{\infty} \|a_n\|\,\alpha_n$ in X implies the convergence of $\sum_{n=1}^{\infty} a_n e_n$ in X". As defined above, for $f \in Y$, we put $\|f, T^* + \delta\| = \sum_{n\geq 1} \|a_n\| \exp\left[\frac{\lambda_n\varphi(\lambda_n)}{\{K\rho^*(T^*+\delta)\}^{1/(\rho^*-1)}}\right]$. We now prove

Theorem 3 *A necessary and sufficient condition that there exists a continuous linear transformation* $F : Y \to Y$ *with* $F(e_n) = \alpha_n$, $n = 1, 2, \ldots,$ *where* $\alpha_n \in Y$, *is that for each* $\delta > 0$

$$\lim_{n\to\infty} \sup \frac{\log \|\alpha_n : T^* + \delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} \leq \left(\frac{\rho^*-1}{\rho^*}\right)(\rho^* T^*)^{-1/\rho^*-1}. \tag{15}$$

Proof. Let F be a continuous linear transformation from Y into Y with $F(e_n) = \alpha_n$, $n = 1, 2, \ldots$ Then for any given $\delta > 0$, there exists a $\delta_1 > 0$ and a constant $K' = K'(\delta)$ depending on δ only, such that

$$\begin{aligned}
\|F(e_n); T^* + \delta\| \leq K'\|e_n; T^* + \delta_1\| &\Rightarrow \|\alpha_n; T^* + \delta\| \\
&\leq K' \exp\left\{\frac{(\rho^*-1)\lambda_n\varphi(\lambda_n)}{(T^*+\delta_1)^{1/\rho^*-1}\left((\rho^*)^{\rho^*/\rho^*-1}\right)}\right\} \\
&\Rightarrow \log \|\alpha_n; T^* + \delta\|^{1/\lambda_n} \\
&\leq o(1) + \frac{\varphi(\lambda_n)(\rho^*-1)}{(T^*+\delta_1)^{1/\rho^*-1}\left((\rho^*)^{\rho^*/\rho^*-1}\right)}, \\
&\Rightarrow \lim_{n\to\infty} \sup \frac{\log \|\alpha_n; T^* + \delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} \leq \frac{(\rho^*-1)}{\rho^*(\rho^* T^*)^{1/\rho^*-1}}.
\end{aligned}$$

Conversely, let the sequence $\{\alpha_n\}$ satisfy (15) and let $\alpha = \sum_{n=1}^{\infty} a_n e_n$. Then we have

$$\lim_{n\to\infty} \sup \frac{\lambda_n \ \phi(\lambda_n)}{\log \|a_n\|^{-1}} \leq \frac{\rho^*(\rho^* T)^{1/\rho^*-1}}{(\rho^*-1)} .$$

Hence, given $\eta > 0$, there exists $N_0 = N_0(\eta)$, such that

$$\frac{\varphi(\lambda_n)}{\log \|a_n\|^{-1/\lambda_n}} \leq \frac{\rho^*}{(\rho^*-1)} \{\rho^*(T^*+\eta)\}^{1/\rho^*-1} \quad \forall n \geq N_0.$$

Further, for a given $\eta_1 > \eta$, from (15), we can find $N_1 = N_1(\eta_1)$ such that for $n \geq N_1$

$$\frac{\log \|\alpha_n; T^*+\delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} \leq \left(\frac{\rho^*-1}{\rho^*}\right) \{\rho^*(T^*+\eta_1)\}^{-1/(\rho^*-1)}.$$

Choose $n \geq \max(N_0, N_1)$. Then

$$\frac{\log \|\alpha_n; T^*+\delta\|^{1/\lambda_n}}{\log \|a_n\|^{-1/\lambda_n}} \leq \left(\frac{T^*+\eta}{T^*+\eta_1}\right)^{1/(\rho^*-1)}$$

$\Rightarrow \|a_n\| \, \|\alpha_n; T^*+\delta\| \leq \|a_n\|^{1-(T^*+\eta/T^*+\eta_1)^{1/(\rho^*-1)}} = \|a_n\|^{\beta}$ (say) where $\beta = 1-(T^*+\eta/T^*+\eta_1)^{1/(\rho^*-1)} > 0$. Now from (5) we can easily show that for any arbitrary large number $K > 0, \|a_n\| < e^{-k\lambda_n}$.

Hence we have for all large values of $n, \|a_n\| \, \|\alpha_n; T^*+\delta\| \leq e^{-K\beta\lambda_n}$.

Consequently the series $\sum_{n=1}^{\infty} \|a_n\| \, \|\alpha_n; T^*+\delta\|$ converges for each $\delta > 0$. Therefore $\sum_{n=1}^{\infty} \|a_n\| \, \alpha_n$ converges to an element of Y. For each $\alpha \in Y$, We define $F(\alpha) = \sum_{n=1}^{\infty} a_n \, \alpha_n$. Then $F(e_n) = \alpha_n$. Now, given $\delta > 0$, $\exists \, \delta_1 > 0$ such that

$$\frac{\log \|\alpha_n; T^*+\delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} \leq \left(\frac{\rho^*-1}{\rho^*}\right) \{\rho^*(T^*+\eta_1)\}^{-1/(\rho^*-1)}$$

for all $n \geq N = N(\delta, \delta_1)$. Hence

$$\Rightarrow \|\alpha_n; T^*+\delta\| \leq K' \exp\left\{\frac{(\rho^*-1)\,\lambda_n\,\varphi(\lambda_n)}{\rho^*\{\rho^*(T^*+\delta_1)\}^{1/\rho^*-1}}\right\}$$

where $K' = K'(\delta)$ and the inequality is true for all $n > 0$. Now

$$\|F(\alpha); T^*+\delta\| \leq \sum_{n=1}^{\infty} \|a_n\| \, \|\alpha_n; T^*+\delta\|$$
$$\leq K' \sum_{n=1}^{\infty} \|a_n\| \exp\left\{\frac{(\rho^*-1)\lambda_n\varphi(\lambda_n)}{\rho^*\{\rho^*(T^*+\delta_1)\}^{1/\rho^*-1}}\right\} = K' \, \|a_n; T^*+\delta\|.$$

Hence F is continuous. This proves Theorem 3. □

We now give some results characterizing the proper bases.

Lemma 1 *In the space* Y_d*, the following three conditions are equivalent:*

(i) *For each* $\delta > 0$, $\lim_{n\to\infty} \sup \frac{\log \|\alpha_n;T^*+\delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} \le \left(\frac{\rho^*-1}{\rho^*}\right) (\rho^* T^*)^{-1/(\rho^*-1)}$.

(ii) *For any sequence* $\{a_n\}$ *in* E*, the convergence of* $\sum_{n=1}^{\infty} a_n e_n$ *in* Y *implies that* $\lim_{n\to\infty} \|a_n\|\alpha_n = 0$ *in* Y.

(iii) *For any sequence*$\{a_n\}$ *in* E*, the convergence of* $\sum_{n=1}^{\infty} a_n e_n$ *in* Y *implies the convergence of* $\sum_{n=1}^{\infty} \|a_n\|\alpha_n$ *in* Y.

Proof. First suppose that (ii) holds. Then for any sequence $\{a_n\}$ $\sum_{n=1}^{\infty} a_n e_n$ converges in Y implies that $\sum_{n=1}^{\infty} \|a_n\|\, \alpha_n$ converges in Y which in turn implies that $\|a_n\|\alpha_n \to 0$ as $n \to \infty$. Hence (ii) $\Rightarrow$ (iii).

Now we assume that (iii) is true but (i) is false. Hence for some $\delta > 0$, there exists a sequence $\{n_k\}$ of positive integers such that $\forall\, n_k$, $k = 1, 2, \ldots,$

$$\frac{\log \|\alpha_{n_k};T^*+\delta\|^{1/\lambda_{n_k}}}{\varphi(\lambda_{n_k})} > \left(\frac{\rho^*-1}{\rho^*}\right) \left\{\rho^*(T^*+\frac{1}{k})\right\}^{-1/(\rho^*-1)}.$$

Define a sequence $\{a_n\}$ as follows:

$$\|a_n\| = \begin{cases} \|\alpha_n;T^*+\delta\|^{-1}, & n = n_k \\ 0; & n \ne n_k \end{cases} \tag{16}$$

Then, we have for all large values of k,

$$\frac{\varphi(\lambda_{n_k})}{\log \|a_{n_k}\|^{-1/\lambda_{n_k}}} = \frac{\varphi(\lambda_{n_k})}{\log \|\alpha_{n_k};T^*+\delta\|^{1/\lambda_{n_k}}} < \left(\frac{\rho^*}{\rho^*-1}\right) \left\{\rho^*(T^*+\frac{1}{k})\right\}^{1/(\rho^*-1)}.$$

Hence,

$$\lim_{k\to\infty} \sup \frac{\varphi(\lambda_{n_k})}{\log \|a_{n_k}\|^{-1/\lambda_{n_k}}} \le \left(\frac{\rho^*}{\rho^*-1}\right) (\rho^* T^*)^{1/(\rho^*-1)}.$$

Thus $\{a_n\}$ defined by (16) satisfies the condition

$$\lim_{n\to\infty} \sup \frac{\varphi(\lambda_n)}{\log \|a_n\|^{-1/\lambda_n}} \le \left(\frac{\rho^*}{\rho^*-1}\right) (\rho^* T^*)^{1/(\rho^*-1)}$$

which in view of Theorem 1 above is equivalent to the condition that $\sum a_n e_n$ converges in Y. Hence by (iii), $\lim_{n\to\infty} \|a_n\|\alpha_n = 0$. However

$$\|\, \|a_{n_k}\|\alpha_{n_k};T^*+\delta\| = \|a_{n_k}\| \cdot \|\alpha_{n_k};T^*+\delta\| = 1.$$

Hence $\lim_{n\to\infty} \|a_n\|\alpha_n \neq 0$ in $Y(\rho^*, T^*, \delta)$. This is a contradiction. Hence (iii)⇒(i). In the course of proof of Theorem 3 above, we have already proved that (i)⇒(ii). Thus the proof of Lemma 1 is complete. □

Next we prove

Lemma 2 *The following three properties are equivalent:*

(a) For all sequences $\{a_n\}$ *in* E, $\lim_{n\to\infty} a_n\alpha_n = 0$ *in* Y *implies that* $\sum_{n=1}^{\infty} a_n e_n$ *converges in* Y.

(b) For all sequences $\{a_n\}$ *in* E, *the convergence of* $\sum_{n=1}^{\infty} \|a_n\|\alpha_n$ *in* Y *implies the convergence of* $\sum_{n=1}^{\infty} a_n e_n$.

(c) $\lim_{\delta\to 0}\left\{\lim_{n\to\infty}\inf \frac{\log\|\alpha_n;T^*+\delta\|^{1/\lambda_n}}{\varphi(\lambda_n)}\right\} \geq \left(\frac{\rho^*-1}{\rho^*}\right)(\rho^* T^*)^{-1/(\rho^*-1)}$.

Proof. Obviously (a)⇒(b). We now prove that (b)⇒(c). To prove this, we suppose that (b) holds but (c) does not hold. Hence

$$\lim_{\delta\to 0}\left\{\lim_{n\to\infty}\inf \frac{\log\|\alpha_n;T^*+\delta\|^{1/\lambda_n}}{\varphi(\lambda_n)}\right\} < \left(\frac{\rho^*-1}{\rho^*}\right)(\rho^* T^*)^{-1/(\rho^*-1)}.$$

Since $\log\|\alpha_n;T+\delta\|$ increases as δ decreases, this implies that for each $\delta > 0$,

$$\left\{\lim_{n\to\infty}\inf \frac{\log\|\alpha_n;T^*+\delta\|^{1/\lambda_n}}{\varphi(\lambda_n)}\right\} < \left(\frac{\rho^*-1}{\rho^*}\right)(\rho^* T^*)^{-1/(\rho^*-1)}.$$

Hence, if $\eta > 0$ be a fixed small positive number, then for each $r > 0$, we can find a positive number n_r such that $\forall r$, we have $n_{r+1} > n_r$ and

$$\lim_{n\to\infty}\inf \frac{\log\|\alpha_{n_r};T^*+r^{-1}\|^{1/\lambda_{n_r}}}{\varphi(\lambda_n)} < \left(\frac{\rho^*-1}{\rho^*}\right)\{\rho^*(T^*+\eta)\}^{-1/(\rho^*-1)} \tag{17}$$

Now we choose a positive number $\eta_1 < \eta$, and define a sequence $\{a_n\}$ as

$$\|a_n\| = \begin{cases} \left(\frac{T^*+\eta_1}{T^*+\eta}\right)^{\lambda_n} \exp\left\{-\left(\frac{\rho^*-1}{\rho^*}\right)\frac{\lambda_n\varphi(\lambda_n)}{\{\rho^*(T^*+\eta)\}^{1/(\rho^*-1)}}\right\}, & n = n_r \\ 0, & n \neq n_r \end{cases}.$$

Then, for any $\delta > 0$

$$\sum_{n=1}^{\infty}\|a_n\|\cdot\|\alpha_n;T^*+\delta\| = \sum_{r=1}^{\infty}\|a_{n_r}\|\cdot\|\alpha_{n_r};T^*+\delta\|. \tag{18}$$

For any given $\delta > 0$, we omit from the above series those finite number of terms, which correspond to those number n_r for which $1/r$ is greater than δ. The remainder of the series in (18) is dominated by $\sum_{r=1}^{\infty} \|a_{n_r}\| \cdot \|\alpha_{n_r}; T^* + r^{-1}\|$. Now by (17) and (18), we find that

$$\sum_{r=1}^{\infty} \|a_{n_r}\| \cdot \|\alpha_{n_r}; T^* + r^{-1}\|$$
$$\leq \sum_{r=1}^{\infty} \left\{ \exp\left\{ -\left(\frac{\rho^*-1}{\rho^*}\right) \frac{\lambda_{n_r}\varphi(\lambda_{n_r})}{\{\rho^*(T^*+\eta)\}^{1/(\rho^*-1)}} \right\} \left(\frac{T^*+\eta_1}{T^*+\eta}\right)^{\lambda_{n_r}} \right\}$$
$$\times \exp\left\{ \left(\frac{\rho^*-1}{\rho^*}\right) \frac{\lambda_{n_r}\varphi(\lambda_{n_r})}{\{\rho^*(T^*+\eta)\}^{1/(\rho^*-1)}} \right\} \leq \sum_{r=1}^{\infty} \left(\frac{T^*+\eta_1}{T^*+\eta}\right)^{\lambda_{n_r}}.$$

Since $\eta_1 < \eta$, therefore the above series on the right hand side is convergent. For this sequence $\{a_n\}$, $\sum_{n=1}^{\infty} \|a_n\| \, \alpha_n$ converges in $Y(\rho^*, T^*, \delta)$ for each $\delta > 0$ and hence converges in Y.

But we have,

$$\lim_{n\to\infty} \sup \frac{\varphi(\lambda_n)}{\log \|a_n\|^{-1/\lambda_n}} = \left(\frac{\rho^*}{\rho^*-1}\right) \{\rho^*(T^*+\eta)\}^{1/(\rho^*-1)}$$

which contradicts (10). This proves (b) $\Rightarrow$(c).

Now we prove that (c)$\Rightarrow$(a). We assume (c) is true but (a) is not true. Then there exists a sequences $\{a_n\}$ of complex numbers for which $\|a_n\|\alpha_n \to 0$ in Y, but $\sum_{n=1}^{\infty} a_n e_n$ does not converge in Y. This implies that

$$\lim_{n\to\infty} \sup \frac{\varphi(\lambda_n)}{\log \|a_n\|^{-1/\lambda_n}} > \left(\frac{\rho^*}{\rho^*-1}\right) (\rho^* T^*)^{1/(\rho^*-1)}$$

Hence there exists a positive number ε and a sequence $\{n_k\}$ of positive integers such that

$$\frac{\varphi(\lambda_n)}{\log \|a_n\|^{-1/\lambda_n}} = \left(\frac{\rho^*}{\rho^*-1}\right) \{\rho^*(T^*+\varepsilon)\}^{1/(\rho^*-1)}, \quad \forall n = n_k$$

We choose another positive number $\eta < \varepsilon/2$. By assumption we can find a positive number δ i.e. $\delta = \delta(\eta)$ such that

$$\lim_{n\to\infty} \inf \frac{\log \|\alpha_n, T^*+\delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} > \left(\frac{\rho^*-1}{\rho^*}\right) \{\rho^*(T^*+\eta)\}^{-1/(\rho^*-1)}.$$

Hence there exists $N = N(\eta)$, such that

$$\frac{\log \|\alpha_n, T^* + \delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} \geq \left(\frac{\rho^* - 1}{\rho^*}\right) \{\rho^*(T^* + 2\eta)\}^{-1/(\rho^*-1)}, \quad \forall n \geq N.$$

Therefore

$$\begin{aligned} \max \| \|a_n\| \alpha_n; T^* + \delta\| &= \max \{\|a_n\| \cdot \|\alpha_n; T^* + \delta\|\} \\ &\geq \max\{\|a_{n_k}\| \cdot \|\alpha_{n_k}; T^* + \delta\|\} \\ &\geq \exp\left\{\frac{-\lambda_{n_k}\varphi(\lambda_{n_k})(\rho^* - 1)}{\rho^*\{\rho^*(T^* + \varepsilon)\}^{1/(\rho^*-1)}}\right\} \\ &\quad \times \exp\left\{\frac{\lambda_{n_k}\varphi(\lambda_{n_k})(\rho^* - 1)}{\rho^*\{\rho^*(T^* + 2\eta)\}^{1/(\rho^*-1)}}\right\} > 1 \end{aligned}$$

for $n_k > N$ as $\varepsilon > 2\eta$.
Thus $\{\|a_n\| \alpha_n\}$ does not tend to zero in $Y(\rho^*, T^*, \delta)$ for the δ chosen above. Hence $\{\|a_n\| \alpha_n\}$ does not tend to 0 in Y and this is a contradiction. Thus (c)⇒(a) is proved. This proves Lemma 2. □

Lastly we prove:

Theorem 4 *A base $\{\alpha_n\}$ in a closed subspace Y_0 of Y is proper if and only if the conditions (i) and (c) stated above are satisfied.*

Proof. Let $\{\alpha_n\}$ be a proper base in a closed subspace Y_0 of Y. Hence for any sequence of complex number $\{a_n\}$ the convergence of $\sum_{n=1}^{\infty} \|a_n\| \alpha_n$ in Y_0 implies the convergence of $\sum_{n=1}^{\infty} a_n e_n$ in Y_0. Therefore (b) and hence (c) is satisfied. Further the convergence of $\sum_{n=1}^{\infty} a_n e_n$ in Y_0 is equivalent to the condition

$$\lim_{n\to\infty} \sup \frac{\varphi(\lambda_n)}{\log \|a_n\|^{-1/\lambda_n}} = \left(\frac{\rho^*}{\rho^* - 1}\right) (\rho^* T^*)^{1/(\rho^*-1)}.$$

Now let $\alpha = \sum_{n=1}^{\infty} a_n e_n$. Then proceeding as in second part of the proof of Theorem 1, we can prove that $\sum_{n=1}^{\infty} \|a_n\| \alpha_n$ converges to an element of Y_0 and thus (ii) is satisfied. But (ii) is equivalent to (i). Hence necessary part of the theorem is proved.
Conversely, suppose that conditions (i) and (c) are satisfied, with $\{\alpha_n\}$ being a base in a closed subspace Y_0 of Y. Then by Lemma 2, we find that for any sequence $\{a_n\}$ in E, convergence of $\sum_{n=1}^{\infty} \|a_n\| \alpha_n$ in Y_0 implies the convergence of $\sum_{n=1}^{\infty} a_n e_n$ in Y_0. Therefore $\{\alpha_n\}$ is a proper base of Y_0. This concludes the proof. □

Acknowledgements

The authors are grateful to the referee for his valuable comments and suggestions.

References

[1] B. L. Srivastava, *A study of spaces of certain classes of vector valued Dirichlet series*, Thesis, I. I. T Kanpur, 1983.

[2] P. K. Kamthan and T. Hussain, Spaces of entire functions represented by Dirichlet series, *Collec. Math.*, **19**, 3 (1968), 203–216.

[3] G. S. Srivastava, A. Sharma, Spaces of entire functions represented by vector valued Dirichlet series, *J. Math. Appl.*, **34**, (2011), 97–107.

[4] G. S. Srivastava, A. Sharma, On generalized order and generalized type of vector valued Dirichlet series of slow growth, *Int. J. Math. Archive*, **2**, 12 (2011), 2652–2659.

[5] G. S. Srivastava, A note on the proximate order of entire function represented by Dirichlet series, *Bull. De L'Academie Polonaise des Sciences*, **19**, 3 (1971), 199–202.

Received: February 4, 2012