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# Some notes on drawing twofolds in 4-dimensional Euclidean space

*Dedicated to the memory of Professor Elemér Kiss (1929-2006), who would be 80 this year.*

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**Abstract.** In the present paper we give an elementary and illustrative proof that in $\mathbf{E}^4$, the complete surfaces with constant positive curvature are not isomorphic. It is well-known, if two surfaces in $\mathbf{E}^3$ are complete with the same positive curvature they are global isomorphic. The same statement is not true in $\mathbf{E}^4$, although these surfaces remain global isometric. We will illustrate our proof with some nice examples.

## 1 Introduction

Many articles [9, 17, 18] presented the techniques of drawing objects in higher dimension than three, and they also highlighted the educational importance of them.

The subject of this paper is related to these drawing techniques and we want to state that the drawings should reflect the fact that the closed, compact,

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smooth surfaces in three dimensions will remain closed, compact and smooth in four dimensions too.

In the 4-dimensional Euclidean space there is an unsolved problem, namely whether there exists a complete analytical hyperbolic plane in $\mathbf{E}^4$. We know that there is no surface having this property in $\mathbf{E}^3$ (Hilbert theorem), but in $\mathbf{E}^5$ [6, 10, 15] and in $\mathbf{E}^6$ [1] it is possible. Thus the question is more exciting in $\mathbf{E}^4$ and proving either the existence or the non-existence would be an important result.

There is another interesting question: what is the graphical image in $\mathbf{E}^4$ for a compact, complete analytical surface which has negative constant curvature? Such a surface exists because it was given by Ōtsuki [13]. The surface constructed by him in $\mathbf{E}^4$ has negative curvature but it is not constant. On the other hand, with the constructed surface Ōtsuki [12] demonstrated that there are compact and complete surfaces with negative curvature in $\mathbf{E}^4$.

Furthermore, we are studying only the surfaces with positive constant curvature. It is well-known from Cohn-Vossen and Herglotz theorem [3, 8] that if two surfaces are complete with the same positive curvature, they are global isomorphic. Our aim is to give an elementary proof that in higher dimension than three, the complete surfaces with constant positive curvature will not remain rigid. Here rigid means that a complete surface with constant positive curvature could not be transformed into itself by one parameter movement. We also illustrate the proof with some examples using different drawing techniques.

## 2 The basic idea

It is well-known that a surface of revolution is a surface generated by rotating a plane curve about an axis. By definition, the axis of the surface of revolution is a straight line, although the axis of rotation can be imagined as a space curve. In the latter case we find a generalization of the surface of revolution, called canal surface. In other words, the canal surface is a surface formed as the envelope of a family of spheres whose centers lie on a space curve. If the sphere centers lie on a straight line, the channel surface is a surface of revolution. For example, the sphere is a special canal surface, whose axis is a straight line.

In the next part we use a simple mathematical deduction to prove that complete surfaces with constant positive curvature are not global isomorphic.

Let $p(u) = (x(u), y(u))$ be a planar curve, parameterized by arc length.

The corresponding Frenet formulas have the following form:

$$\begin{aligned} p'(u) &= e(u), \\ e'(u) &= \kappa(u)n(u), \\ n'(u) &= -\kappa(u)e(u), \end{aligned}$$

where the tangent vector for the curve $p$ is $e = e(u)$, the normal vector is $n = n(u)$ and the binormal vector is $b = b(u)$. We suppose $b'(u) = 0$, which means that $p$ is a planar curve. The canal surface of the planar curve $p$ has the following form:

$$f(u, v) = p(u) + r(u)(n(u)cos(v) + b(u)sin(v)),$$

where $r(u)$ is the radius of the spheres from the definition of the canal surface.

According to our aim, we put the condition that the surface has positive Gaussian curvature. This means that

$$G(u) = +1 \tag{1}$$

equation must hold, where $G$ is the curvature of the surface. In order to solve this equation first we try to calculate the curvature of the canal surface using the next fundamental forms of it:

$$\begin{aligned} g_{11} &= r'^2(u) + (1 - \kappa(u)r(u)sin(v))^2, \\ g_{12} &= 0, \\ g_{22} &= r^2(u). \end{aligned}$$

Furthermore, if we put the condition $\kappa(u) = 0$, then we get the Gauss curvature:

$$G(u) = -\frac{r''(u)}{r(u)(1 + r'^2(u))^2}. \tag{2}$$

By replacing the found expression into formula (1), we get the following equation:

$$r''(u) = -r(u)(1 + r'^2(u))^2. \tag{3}$$

Equation (3) can be solved by integrating elementary, but we are interested in a result, which gives us the sphere as solution, so we get the next particular result: $r(u) = \sqrt{2u - u^2}$, where $u \in [0, 2]$.

Furthermore, we repeat the previous sequence of ideas in 4 dimensions, and we choose a space curve in $\mathbf{E}^3$ with this form: $p(u) = (x(u), y(u), z(u))$. Then we get by Frenet formulas in $\mathbf{E}^4$:

$$\begin{aligned} p'(u) &= e_1(u), \\ e_1'(u) &= \kappa(u)e_2(u), \\ e_2'(u) &= -\kappa(u)e_1(u) + \tau(u)e_3(u), \\ e_3'(u) &= -\tau(u)e_2, \\ e_4'(u) &= 0, \end{aligned}$$

where $\{e_1, e_2, e_3, e_4\}$ is the Frenet orthonormal basis. The canal surfaces in $\mathbf{E}^4$ have the following form:

$$f(u, v) = p(u) + r(u)(e_3(u)\cos(v) + e_4(u)\sin(v))$$

The Gauss fundamental forms for the surface $f$ are the following equations:

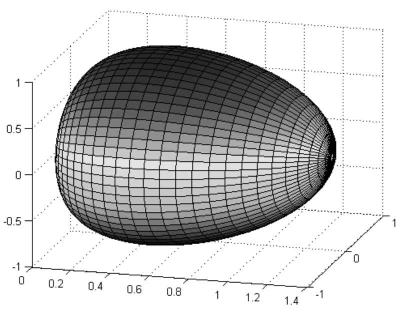
$$\begin{aligned} g_{11} &= r'^2(u) + 1 + r^2(u)\tau^2(u)\cos^2(v), \\ g_{12} &= 0, \\ g_{22} &= r^2(u). \end{aligned}$$

Furthermore, we put the condition that the torsion of the space curve has null value. This means that the space curve is a plane curve. By continuing the calculations, we get the curvature formula (2) for the surface and we are again interested in those solutions of equation (1), which give us complete surfaces.

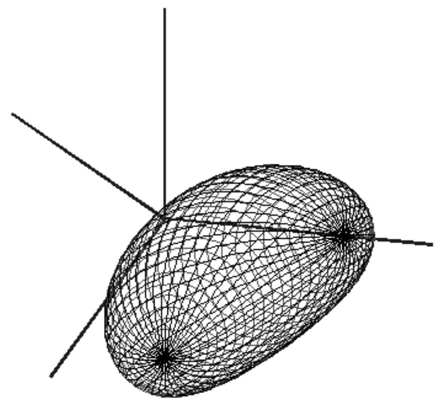
The calculations reflect the fact that the curvature for these surfaces is independent of the form of the planar curve in $\mathbf{E}^4$, which is the axis of the surface. In other words, the axes of the canal surface can be chosen in many ways, hence there is an infinite number of surfaces with positive constant curvatures. To summarize our results, we state the following theorem:

**Theorem 1** *For each surface of revolution with positive constant curvature in $E^3$ there are corresponding infinite number of canal surfaces with positive constant curvature in $E^4$.*

These results have a geometric interpretation, too. For example, let us consider the 3-dimensional sphere. It is well-known that we get it from the rotation of the circle around its diameter. If we take the sphere by its north and




(a)



(b)

Figure 1: The Ōtsuki sphere.

south pole and if we "bend" the rotation axis towards the fourth dimension, the surface will still have a constant curvature which is reflected in the calculations, but we get spheres which will not be isomorphic in the 4-dimensional space. If the shape of the rotation axis is a quarter of the asteroid then we get the famous sphere of Ōtsuki [11] (see Fig. 1) represented by the (4)–(7) equations:

$$x_1(u,v) = \frac{4}{3}\cos^3\frac{u}{2}, \quad (4)$$

$$x_2(u,v) = \frac{4}{3}\sin^3\frac{u}{2}, \quad (5)$$

$$x_3(u,v) = \sin(u)\cos(v), \quad (6)$$

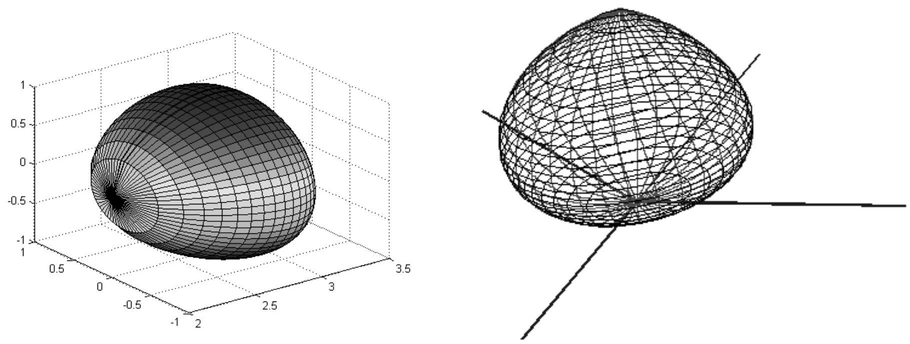
$$x_4(u,v) = \sin(u)\sin(v), \quad (7)$$

where $u \in [0,\pi], v \in [0,2\pi]$.

Furthermore, we give three other examples of complete surfaces with constant positive curvature in $\mathbf{E}^4$:

**Example 1**

$$\begin{aligned} x_1(u,v) &= x_1(u) = 2\arcsin(u/2) + \sqrt{4-u^2}, \\ x_2(u,v) &= x_2(u) = \sqrt{2}\sqrt{(2+u)u} - \sqrt{2}\ln(1+u+\sqrt{(2u+u^2)}), \\ x_3(u,v) &= \sqrt{u(2-u)}\sin v, \\ x_4(u,v) &= \sqrt{u(2-u)}\cos v, \quad u \in [0,2], v \in [0,2\pi]. \end{aligned}$$




(a) (b)

Figure 2: Drawings for Example 1: (a) Intersection with a hyperplane. (b) Axonometric mapping from $\mathbf{E}^4$ into $\mathbf{E}^2$.

**Example 2**

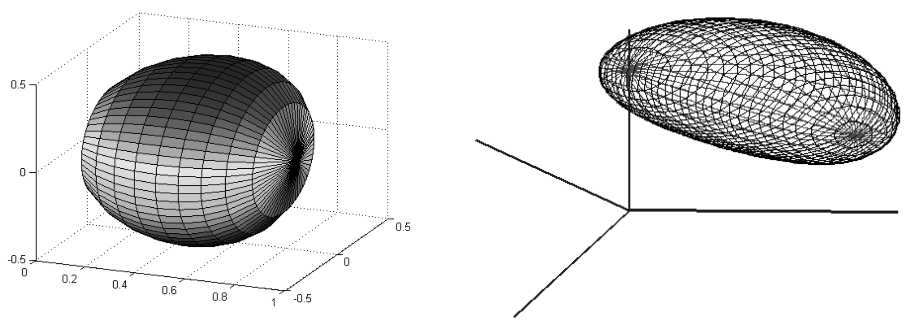
$$\begin{aligned}
x_1(u,v) &= x_1(u) = \sin u, \\
x_2(u,v) &= x_2(u) = \cos u, \\
x_3(u,v) &= \sqrt{u(1-u)}\sin v, \\
x_4(u,v) &= \sqrt{u(1-u)}\cos v, \quad u \in [0,1], v \in [0,2\pi].
\end{aligned}$$

**Example 3**

$$\begin{aligned}
x_1(u,v) &= x_1(u) = 2\sin(u/2), \\
x_2(u,v) &= x_2(u) = 2\cos(u/2), \\
x_3(u,v) &= \sqrt{u(2-u)}\sin v, \\
x_4(u,v) &= \sqrt{u(2-u)}\cos v, \quad u \in [0,2], v \in [0,2\pi].
\end{aligned}$$

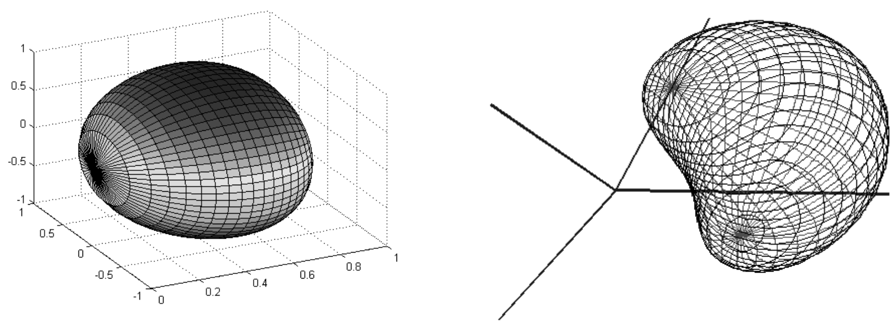
## 3 The used drawing techniques

We have drawn two kinds of figures using MATLAB programming language. The first type of drawings are intersections in $\mathbf{E}^4$ with a hyperplane. This means that we omit one of the four coordinates from the surface representation, and after that we apply an axonometry by mapping the three-dimensional figure onto the plane (see Fig. 1(a), 2(a), 3(a), 4(a)).




(a) (b)

Figure 3: Drawings for Example 2: (a) Intersection with a hyperplane. (b) Axonometric mapping from $\mathbf{E}^4$ into $\mathbf{E}^2$.




(a) (b)

Figure 4: Drawings for Example 3: (a) Intersection with a hyperplane. (b) Axonometric mapping from $\mathbf{E}^4$ into $\mathbf{E}^2$.

The second type of figures are axonometric mappings from $\mathbf{E}^4$ into $\mathbf{E}^2$ (see Fig. 1(b), 2(b), 3(b), 4(b)). The transformation has the following form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} \begin{pmatrix} x_1(u,v) \\ x_2(u,v) \\ x_3(u,v) \\ x_4(u,v) \end{pmatrix},$$

where the rank of the transformation matrix $[a_{ij}]_{2\times 4}$ is equal to 2. These kinds of mapping techniques were studied by Szabó [17, 18], and he proved that the objects can be also represented in $\mathbf{R}^n$, both in axonometric and perspective way. These mappings keep their straight lines and proportion in case of axonometry and in case of perspective, they keep their straight lines and cross-ratio.

On the other hand, Stiefel [16] showed that in $\mathbf{E}^4$ the Pohlke's theorem (i.e. the axonometric image of a shape is similar to the parallel projection of the shape) is not valid. Nevertheless, some properties remain valid. For example the close parameter lines on the surfaces in $\mathbf{E}^4$ are transformed into closed curves as you can see in the figures.

## 4 Conclusions

In this paper we considered constant positive curvature surfaces from the 4-dimensional Euclidean space. Many surfaces with constant positive curvature have the interesting property that they are not global isomorphic in $\mathbf{E}^4$, while in $\mathbf{E}^3$ the same property is not true. We have proved this property mathematically and also illustrated with some nice examples.

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