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Some fixed point results on S-metric spaces

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Abstract. In this paper, a general form of the Suzuki type function is considered on S- metric space, to get a fixed point. Then we show that our results generalize some old results.

1 Introduction and preliminaries

In 1922, Banach [1] proposed a theorem, which is well-known as Banach's Fixed Point Theorem (or Banach's Contraction Principle, BCP for short) to establish the existence of solutions for nonlinear operator equations and integral equations. Since then, because of simplicity and usefulness, it has become a very popular tool in solving a variety of problems such as control theory, economic theory, nonlinear analysis and global analysis. Later, a huge amount

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of literature is witnessed on applications, generalizations and extensions of this theorem. They are carried out by several authors in different directions, e.g., by weakening the hypothesis, using different setups.

Many mathematics problems require one to find a distance between tow or more objects which is not easy to measure precisely in general. There exist different approaches to obtaining the appropriate concept of a metric structure. Due to the need to construct a suitable framework to model several distinguished problems of practical nature, the study of metric spaces has attracted and continues to attract the interest of many authors. Over last few decades, a numbers of generalizations of metric spaces have thus appeared in several papers, such as 2-metric spaces, G-metric spaces, D^*-metric spaces, partial metric spaces and cone metric spaces. These generalizations were then used to extend the scope of the study of fixed point theory. For more discussions of such generalizations, we refer to [3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 20, 21, 22, 23]. Sedghi et al [17] have introduced the notion of an S-metric space and proved that this notion is a generalization of a G-metric space and a D^*-metric space. Also, they have proved properties of S-metric spaces and some fixed point theorems for a self-map on an S-metric space.

The Banach contraction principle is the most powerful tool in the history of fixed point theory. Boyd and Wong [2] extended the Banach contraction principle to the nonlinear contraction mappings. We begin by briefly recalling some basic definitions and results for S-metric spaces that will be needed in the sequel. For more details please see [1, 14, 18].

Definition 1 [17] *Let X be a (nonempty) set, an S-metric on X is a function $S : X^3 \longrightarrow [0, +\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,*

(1). $S(x, y, z) \geq 0$,

(2). $S(x, y, z) = 0$ *if and only if* $x = y = z$,

(3). $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$,

for all $x, y, z, a \in X$.
The pair (X, S) is called an S-metric space.

Immediate examples of such S-metric spaces are:

Example 1 [15, 18] *Let $X = \mathbb{R}^n$ and $\| . \|$ a norm on X, then*

$$S(x, y, z) = \| y + z - 2x \| + \| y - z \|$$

is an S-metric on X.
Let X *be a nonempty set, d is ordinary metric on* X*, then*

$$S(x, y, z) = d(x, z) + d(y, z)$$

is an S-metric on X*. This S-metric is called the usual S-metric on* X*.*

Definition 2 [16] *Let* (X, S) *be an S-metric space.*

(i) *A sequence* $\{x_n\} \subset X$ *converges to* $x \in X$ *if* $S(x_n, x_n, x) \to 0$ *as* $n \to +\infty$*. That is, for each* $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *we have* $S(x_n, x_n, x) < \varepsilon$*. We write* $x_n \to x$ *for brevity.*

(ii) *A sequence* $\{x_n\} \subset X$ *is a Cauchy sequence if* $S(x_n, x_n, x_m) \to 0$ *as* $n, m \to +\infty$*.*
That is, for each $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $n, m \geq n_0$ *we have* $S(x_n, x_n, x_m) < \varepsilon$*.*

(iii) *The S-metric space* (X, S) *is compelet if every Cauchy sequence is a convergent sequence.*

Definition 3 [15] *Let* (X, S) *be an S-metric space. For* $r > 0$ *and* $x \in X$ *we define the open ball* $B_s(x, r)$ *and closed ball* $B_s[x, r]$ *with center* x *and radius* r *as follows respectively:*

$$\begin{aligned} B_s(x, r) &= \{y \in X : S(y, y, x) < r\}, \\ B_s[x, r] &= \{y \in X : S(x, x, y) \leq r\}. \end{aligned}$$

Example 2 [15] *Let* $X = \mathbb{R}$ *and* $S(x, y, z) = |y + z - 2x| + |y - z|$ *for all* $x, y, z \in \mathbb{R}$*. Then*

$$\begin{aligned} B_s(1, 2) &= \{y \in \mathbb{R} : S(y, y, 1) < 2\} = \{y \in \mathbb{R} : |y - 1| < 1\} \\ &= \{y \in \mathbb{R} : 0 < y < 2\} = (0, 2). \end{aligned}$$

Lemma 1 [16] *Let* (X, S) *be an S-metric space. If* $r > 0$ *and* $x \in X$*, then the ball* $B_s(x, r)$ *is open subset of* X*.*

Lemma 2 [15, 16, 18] *In an S-metric space, we have* $S(x, x, y) = S(y, y, x)$*.*

Proof. By third condition of S-metric, we have

$$\begin{aligned} S(x, x, y) &\leq S(x, x, x) + S(x, x, x) + S(y, y, x) \\ &= S(y, y, x) \end{aligned} \tag{1}$$

$$\begin{aligned} S(y,y,x) &\le S(y,y,y) + S(y,y,y) + S(x,x,y) \\ &= S(x,x,y), \end{aligned} \tag{2}$$

hence by (1) and (2), we get $S(x,x,y) = S(y,y,x)$. □

Lemma 3 [18] *Let* (X,S) *be an S-metric space. If sequence* $\{x_n\}$ *in converges to* x*, then* x *is unique.*

Lemma 4 [18] *Let* (X,S) *be an S-metric space. If sequence* $\{x_n\}$ *in* X *is converges to* x*, then* $\{x_n\}$ *is a Cauchy sequence.*

Lemma 5 [15, 16, 18] *Let* (X,S) *be an S-metric space. If there exist sequences* $\{x_n\}$ *and* $\{y_n\}$ *such that* $\lim_{n\to+\infty} x_n = x$ *and* $\lim_{n\to+\infty} y_n = y$*, then* $\lim_{n\to+\infty} S(x_n,x_n,y_n) = S(x,x,y)$.

Definition 4 [15, 19] *Let* X *be a (nonempty) set, a b-metric on* X *is a function* $d : X^2 \longrightarrow [0,+\infty)$ *if there exists a real number* $b \ge 1$ *such that the following conditions hold for all* $x,y,z \in X$*,*

(1) $d(x,y) = 0$ *if and only if* $x = y$*,*
(2) $d(x,y) = d(y,x)$*,*
(3) $d(x,z) \le b[d(x,y) + d(y,z)]$.

The pair (X,d) *is called a b-metric space.*

Proposition 1 [16] *Let* (X,S) *be an S-metric space and let*

$$d(x,y) = S(x,x,y),$$

for all $x,y \in X$*. Then we have*

(1) d *is a b-metric on* X*;*
(2) $x_n \to x$ *in* (X,S) *if and only if* $x_n \to x$ *in* (X,d)*;*
(3) $\{x_n\}$ *is a Cauchy sequence in* (X,S) *if and only if* $\{x_n\}$ *is a Cauchy sequence in* (X,d).

Definition 5 *Let* £ *be the set of all continuous functions* $g : [0,\infty)^4 \to [0,+\infty)$*, satisfying the conditions:*

(i) $g(1,1,1,1) < 1$*,*
(ii) g *is subhomogeneous,i.e.,* $g(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4) \le \alpha g(x_1,x_2,x_3,x_4)$*, for all* $\alpha \ge 0$*,*

(iii) *if* $x_i, y_i \in [0, +\infty), x_i \leq y_i$ *for* $i = 1, \ldots, 4$ *we have* $g(x_1, x_2, x_3, x_4) \leq g(y_1, y_2, y_3, y_4)$.

Example 3 *The function* $g(x_1, x_2, x_3, x_4) = k \max\{x_i\}_{i=0}^{4}$ *for* $k \in (0, 1)$ *is in class £.*

Example 4 *The function* $g(x_1, x_2, x_3, x_4) = k \max\{x_1, x_2, \frac{x_3+x_4}{2}\}$ *for* $k \in (0, 1)$ *is in class £.*

Proposition 2 *If* $g \in £$ *and* $u, v \in [0, +\infty]$ *are such that* $u \leq g(v, v, v, u)$, *then* $u \leq hv$, *where* $h = g(1, 1, 1, 1)$.

Proof. If $v < u$, then

$$u \leq g(v, v, v, u) \leq g(u, u, u, u) < ug(1, 1, 1, 1) = hu < u,$$

which is a contradiction. Thus $u \leq v$, which implies

$$u \leq g(v, v, v, u) \leq g(v, v, v, v) < vg(1, 1, 1, 1) = hv.$$

□

Corollary 1 [15] *Let* (X, S) *be a complete* S*-metric space and* $T : X \to X$ *a function such that for, all* $x, y, z, a \in X$,

$$S(Tx, Ty, Tz) \leq LS(x, y, z),$$

where $L \in (0, 1/2)$. *Then there exists a unique point* $u \in X$ *such that* $Tu = u$.

2 Results

Now, we give our main result.

Theorem 1 *Let* (X, S) *be a S- metric space and* $T : X \to X$ *be a function. Suppose that there exist* $g \in £$ *and* $\alpha \in (0, 1)$, *such that* $\alpha(h + 2) \leq 1$ *where* $h = g(1, 1, 1, 1)$. *Suppose also that* $\alpha S(x, x, Tx) \leq S(x, y, z)$ *implies*

$$S(Tx, Ty, Tz) \leq g(S(x, y, z), S(x, x, Tx), S(y, y, Ty), S(z, z, Tz)),$$

for all $x, y, z \in X$. *Then* $F(T)$ *is non-empty set.*

Proof. Fix arbitrary $x_0 \in X$ and let $Tx_0 = x_1$. Since

$$\alpha S(x_0, x_0, Tx_0) < S(x_0, x_0, x_1),$$

then by the hypothesis of the theorem and condition (iii) Definition 5, respectively, we have

$$\begin{aligned} S(x_1, x_1, Tx_1) &= S(Tx_0, Tx_0, Tx_1) \\ &\leq g(S(x_0, x_0, x_1), S(x_0, x_0, Tx_0), S(x_0, x_0, Tx_0), S(x_1, x_1, Tx_1)) \\ &= g(S(x_0, x_0, x_1), S(x_0, x_0, x_1), S(x_0, x_0, x_1), S(x_1, x_1, Tx_1)) \end{aligned}$$

Then, by Proposition 2, we have $S(x_1, x_1, Tx_1) \leq hS(x_0, x_0, x_1)$.
Now let $Tx_1 = x_2$. Since $\alpha S(x_1, x_1, Tx_1) < S(x_1, x_1, x_2)$, by using and the properties of the function g we have

$$\begin{aligned} S(x_2, x_2, Tx_2) &= S(Tx_1, Tx_1, Tx_2) \\ &\leq g(S(x_1, x_1, x_2), S(x_1, x_1, Tx_1), S(x_1, x_1, Tx_1), S(x_2, x_2, Tx_2)) \\ &= g(S(x_1, x_1, x_2), S(x_1, x_1, x_2), S(x_1, x_1, x_2), S(x_2, x_2, Tx_2)). \end{aligned}$$

Then, by Proposition 2, we have $S(x_2, x_2, Tx_2) \leq hS(x_1, x_1, x_2)$.
In a similar way, we can let $Tx_2 = x_3$. So we have

$$S(x_2, x_2, x_3) < hS(x_1, x_1, x_2) < h^2 S(x_0, x_0, x_1).$$

By continuing this process, we obtain a sequence $\{x_n\}_{n \geq 1}$ in X such that $x_{n+1} = Tx_n$, which satisfies $S(x_n, x_n, Tx_n) \leq hS(x_{n-1}, x_{n-1}, x_n)$ and

$$S(x_n, x_n, x_{n+1}) \leq h^n S(x_0, x_0, x_1).$$

If $x_m = x_{m+1}$ for some $m \geq 1$, then
Then T has a fixed point.
Suppose that $x_n \neq x_{n+1}$, for all $n \geq 1$. Repeated application of the triangle inequality implies

$$\begin{aligned} S(x_n, x_n, x_{n+m}) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_{n+m}, x_{n+m}, x_{n+1}) \\ &= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+m}) \\ &\leq 2S(x_n, x_n, x_{n+1}) + 2S(x_{n+1}, x_{n+1}, x_{n+2}) \\ &+ S(x_{n+m}, x_{n+m}, x_{n+2}) \\ &\leq 2[S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}) \\ &+ \cdots + S(x_{n+m-1}, x_{n+m-1}, x_{n+m})] \end{aligned}$$

$$\leq \quad 2\sum_{k=0}^{k=m-1} h^{k+n}S(x_0,x_0,x_1) \leq \frac{2h^n}{1-h}S(x_0,x_0,x_1).$$

So we get

$$\lim_{n\to+\infty} S(x_n,x_n,x_{n+m}) \to 0$$

and hence $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X,S). Regarding Definition 2, $\{x_n\}_{n\geq 1}$ is also a Cauchy sequence in (X,S).
Since (X,S) is a complete S- metric space, by Definition 2, (X,S) is also complete.
Thus $\{x_n\}_{n\geq 1}$ converges to a limit, say, $x \in X$, that is,

$$\lim_{n\to+\infty} S(x_n,x_n,x) = 0.$$

It is easy to see that $\lim_{n\to\infty} S(x_n,x_{n+1},x) = 0$. Now, we claim that for each $n \geq 1$ one of the relations

$$\alpha S(x_n,x_n,Tx_n) \leq S(x_n,x_n,x)$$

or

$$\alpha S(x_{n+1},x_{n+1},Tx_{n+1}) \leq S(x_n,x_n,x)$$

holds. If for some $n \geq 1$ we have

$$\alpha S(x_n,x_n,Tx_n) > S(x_n,x_n,x) \text{ and } \alpha S(x_{n+1},x_{n+1},Tx_{n+1}) > S(x_{n+1},x_{n+1},x),$$

then

$$\begin{array}{rcl} S(x_n,x_n,x_{n+1}) & \leq & 2S(x_n,x_n,x) + S(x_{n+1},x_{n+1},x) \\ & < & 2\alpha S(x_n,x_n,Tx_n) + \alpha S(x_{n+1},x_{n+1},Tx_{n+1}) \\ & = & 2\alpha S(x_n,x_n,x_{n+1}) + \alpha h S(x_n,x_n,x_{n+1}). \end{array}$$

This results in $\alpha(h+2) > 1$, which contradidts the intial assumption. Hence, our claim is proved.
Observe that by the assumption of the theorem, we have either

$$S(Tx_n,Tx_n,Tx) \leq g(S(x_n,x_n,x),S(Tx_n,x_n,x),S(Tx_n,x_n,x),S(Tx,x_n,x_n)),$$

or

$$\begin{array}{rcl} S(Tx_{n+1},Tx_{n+1},Tx) & \leq & g(S(x_{n+1},x_{n+1},x),S(Tx_{n+1},x_{n+1},x), \\ & & S(Tx_{n+1},x_n,x),S(Tx,x_{n+1},x_{n+1})). \end{array}$$

Therefore, one of the following cases holds.
Case (i). There exists an infinite subset $I \subseteq \mathbb{N}$ such that

$$\begin{aligned} S(x_{n+1}, x_{n+1}, Tx) &= S(Tx_n, Tx_n, Tx) \\ &\leq g(S(x_n, x_n, x), S(Tx_n, x_n, x), S(Tx_n, x_n, x), S(Tx, x_n, x_n)) \\ &= g(S(x_n, x_n, x), S(x_{n+1}, x_n, x), S(x_{n+1}, x_n, x), S(Tx, x_n, x_n)). \end{aligned}$$

for all $n \in I$.
Case (ii). There exists an infinite subset $J \subseteq \mathbb{N}$ such that

$$\begin{aligned} S(x_{n+2}, x_{n+2}, Tx) &= S(Tx_{n+1}, Tx_{n+1}, Tx) \\ &\leq g(S(x_{n+1}, x_{n+1}, x), S(Tx_{n+1}, x_{n+1}, x), \\ &\quad S(Tx_{n+1}, x_{n+1}, x), S(Tx, x_{n+1}, x_{n+1})) \\ &= g(S(x_{n+1}, x_{n+1}, x), S(x_{n+2}, x_{n+1}, x), \\ &\quad S(x_{n+2}, x_{n+1}, x), S(Tx, x_{n+1}, x_{n+1})). \end{aligned}$$

for all $n \in I$. In case (i), taking the limit as $n \to +\infty$ we obtain

$$S(x, x, Tx) \leq g(0, 0, 0, S(x, x, Tx))$$

Now by using Definition 5, Proposition 2, we have $S(x, x, Tx) = 0$, and thus $x = Tx$.
In case(ii), taking the limit as $n \to \infty$ we obtain

$$S(x, x, Tx) \leq g(0, 0, 0, S(x, x, Tx))$$

Now by using definition 5, propositions 2, we have $S(x, x, Tx) = 0$, and thus $x = Tx$. This completes the proof. □

Corollary 2 *Let* (X, S) *be a S- metric space and* $T : X \to X$ *be a function. Suppose that there exist* $g \in \pounds$ *and* $\alpha \in (0, 1)$*, such that* $\alpha(h + 2) \leq 1$ *where* $h = g(1, 1, 1, 1)$*. Suppose also that* $\alpha S(y, y, Ty) \leq S(x, y, z)$ *implies*

$$S(Tx, Ty, Tz) \leq g(S(x, y, z), S(x, x, Tx), S(y, y, Ty), S(z, z, Tz))$$

for all $x, y, z \in X$*. Then* $F(T)$ *is non-empty.*

Corollary 3 *Let* (X, S) *be a complete S-metric space and* $T : X \to X$ *a function such that for all* $x, y, z \in X$*,*

$$S(Tx, Ty, Tz) \leq LS(x, y, z),$$

where $L \in (0, 1)$*. Then there exists a unique point* $u \in X$ *such that* $Tu = u$*.*

Proof. Let $g(x_1, x_2, x_3, x_4) = Lx_1$. $\square$

Corollary 4 *Let* (X, S) *be a complete S-metric space and* $T : X \to X$ *a function such that for all* $x, y, z \in X$,

$$S(Tx, Ty, Tz) \leq L \max\{S(x, y, z), S(x, x, Tx), S(y, y, Ty), S(z, z, Tz)\}$$

where $L \in (0, 1)$. *Then there exists a unique point* $u \in X$ *such that* $Tu = u$.

Proof. Let $g(x_1, x_2, x_3, x_4) = L \max\{x_1, x_2, x_3, x_4\}$. $\square$

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