Acta Univ. Sapientiae, Mathematica, **13**, 2 (2021) 494–505

DOI: 10.2478/ausm-2021-0031

Some classes of sequence spaces defined by a Musielak-Orlicz function

Sunil K. Sharma
Department of Mathematics,
Cluster University of Jammu,
Jammu, J&K 180001, India
email: sunilksharma42@gmail.com

Abstract. In the present paper we introduce the sequence spaces $c_0\{\mathcal{M}, \Lambda, p, q\}$, $c\{\mathcal{M}, \Lambda, p, q\}$ and $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ defined by a Musielak-Orlicz function $\mathcal{M} = (M_k)$. We study some topological properties and prove some inclusion relations between these spaces.

1 Introduction and preliminaries

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [3] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

2010 Mathematics Subject Classification: 40A05, 46A45, 46E30
Key words and phrases: Orlicz function, Musielak-Orlicz function, paranorm space, sequence space

Also, it was shown in [3] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t) dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function (see [4], [8]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf \Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf \Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Let w, l_∞, c and c_0 denote the spaces of all, bounded, convergent and null sequences $x = (x_k)$ with complex terms respectively. The zero sequence (0,0,...) is denoted by θ and $p = (p_k)$ is a sequence of strictly positive real numbers. Further the sequence (p_k^{-1}) will be represented by (t_k).

Mursaleen and Noman [6] introduced the notion of λ-convergent and λ-bounded sequences as follows :
Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity i.e.

$$0 < \lambda_0 < \lambda_1 < \cdots \quad \text{and} \quad \lambda_k \to \infty \quad \text{as} \quad k \to \infty$$

and said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x if $\Lambda_m(x) \longrightarrow L$ as $m \to \infty$, where

$$\lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) x_k.$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [6] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \left(\frac{1}{\lambda_m} \left(\sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) |x_k - a| \right) = 0.$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m \left| \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1})(x_k - a) \right| = 0$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [14], Theorem 10.4.2, pp. 183). For more details about sequence spaces (see [1], [2], [5], [7], [9], [10], [11], [12], [13]) and references therein.

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers and let (X, q) be a seminormed space seminormed by q. In the present paper, we define the following sequence spaces:

$$c_0\{\mathcal{M}, \Lambda, p, q\} = \Big\{ x = (x_k) \in w : \Big[M_k \Big(\frac{q\big(\Lambda_k(x)\big)}{\rho} \Big) \Big]^{p_k} t_k \to 0, \text{ as } k \to \infty, \text{ for some } \rho > 0 \Big\},$$

$$c\{\mathcal{M}, \Lambda, p, q\} = \Big\{ x = (x_k) \in w : \Big[M_k \Big(\frac{q\big(\Lambda_k(x)\big)}{\rho} \Big) \Big]^{p_k} t_k \to 0, \text{ as } k \to \infty, \text{ for some } L \in X \text{ and for some } \rho > 0 \Big\}$$

and

$$l_\infty\{\mathcal{M}, \Lambda, p, q\} = \Big\{ x = (x_k) \in w : \sup_k \Big[M_k \Big(\frac{q\big(\Lambda_k(x)\big)}{\rho} \Big) \Big]^{p_k} t_k < \infty, \text{ for some } \rho > 0 \Big\}.$$

If we take $p = (p_k) = 1$, we have

$$c_0\{\mathcal{M}, \Lambda, q\} = \Big\{ x = (x_k) \in w : \Big[M_k \Big(\frac{q\big(\Lambda_k(x)\big)}{\rho} \Big) \Big] \to 0, \text{ as } k \to \infty, \text{ for some } \rho > 0 \Big\},$$

$$c\{\mathcal{M}, \Lambda, q\} = \Big\{ x = (x_k) \in w : \Big[M_k \Big(\frac{q\big(\Lambda_k(x) - L\big)}{\rho} \Big) \Big] \to 0, \text{ as } k \to \infty, \text{ for some } L \in X \text{ and for some } \rho > 0 \Big\}$$

and

$$l_\infty\{\mathcal{M}, \Lambda, q\} = \Big\{ x = (x_k) \in w : \sup_k \Big[M_k \Big(\frac{q\big(\Lambda_k(x)\big)}{\rho} \Big) \Big] < \infty, \text{ for some } \rho > 0 \Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = K$, $D = \max(1, 2^{K-1})$ then

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^K)$ for all $a \in \mathbb{C}$.

The main aim of this paper is to study some toplogical properties and prove some inclusion relation between these spaces.

2 Main results

Theorem 1 *If $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers, then the spaces $c_0\{\mathcal{M}, \Lambda, p, q\}$, $c\{\mathcal{M}, \Lambda, p, q\}$ and $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ are linear spaces over the field of complex numbers $\mathbb{C}$.*

Proof. Let $x = (x_k),\ y = (y_k) \in c\{\mathcal{M}, \Lambda, p, q\}$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\left[M_k\left(\frac{q\big(\Lambda_k(x) - L\big)}{\rho_1}\right)\right]^{p_k} t_k \to 0, \ \text{ as } \ k \to \infty$$

and

$$\left[M_k\left(\frac{q\big(\Lambda_k(y) - L\big)}{\rho_2}\right)\right]^{p_k} t_k \to 0, \ \text{ as } \ k \to \infty.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since (M_k) is non-decreasing and convex by using inequality (1.1), we have

$$\begin{aligned}
&\left[M_k\left(\frac{q\big((\alpha\Lambda_k(x) + \beta\Lambda_k(y)) - 2L\big)}{\rho_3}\right)\right]^{p_k} t_k \\
&\leq \left[M_k\left(\frac{q\big(\alpha\Lambda_k(x) - L\big)}{\rho_3} + \frac{q\big(\beta\Lambda_k(y) - L\big)}{\rho_3}\right)\right]^{p_k} t_k \\
&\leq D\frac{1}{2^{p_k}}\left[M_k\left(\frac{q\big(\Lambda_k(x) - L\big)}{\rho_1}\right)\right]^{p_k} t_k + D\frac{1}{2^{p_k}}\left[M_k\left(\frac{q\big(\Lambda_k(y) - L\big)}{\rho_2}\right)\right]^{p_k} t_k \\
&\leq D\left[M_k\left(\frac{q\big(\Lambda_k(x) - L\big)}{\rho_1}\right)\right]^{p_k} t_k + D\left[M_k\left(\frac{q\big(\Lambda_k(y) - L\big)}{\rho_2}\right)\right]^{p_k} t_k \\
&\to 0 \ \text{ as } \ k \to \infty.
\end{aligned}$$

Thus, $\alpha x + \beta y \in c\{\mathcal{M}, \Lambda, p, q\}$. Hence $c\{\mathcal{M}, \Lambda, p, q\}$ is a linear space. Similarly, we can prove $c_0\{\mathcal{M}, \Lambda, p, q\}$ and $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ are linear spaces over the field of complex numbers $\mathbb{C}$. □

Theorem 2 *$\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers, then $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ is a paranormed space with the paranorm defined by*

$$g(x) = q(x_1) + \inf\left\{\rho^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)t_k{}^{\frac{1}{p_k}}\right\} \leq 1, \quad \rho > 0 \mathrm{bigg}\right\},$$

where $H = \max(1, K)$.

Proof. (i) Clearly, $g(x) \geq 0$ for $x = (x_k) \in l_\infty\{\mathcal{M}, \Lambda, p, q\}$. Since $M_k(0) = 0$, we get $g(\theta) = 0$.
(ii) $g(-x) = g(x)$
(iii) Let $x = (x_k),\ y = (y_k) \in l_\infty\{\mathcal{M}, \Lambda, p, q\}$, then there exist $\rho_1, \rho_2 > 0$ such that

$$\sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_1}\right)t_k{}^{\frac{1}{p_k}}\right\} \leq 1$$

and

$$\sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda(y)\big)}{\rho_2}\right)t_k{}^{\frac{1}{p_k}}\right\} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$, then by Minkowski's inequality, we have

$$\begin{aligned}\sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x+y)\big)}{\rho}\right)t_k{}^{\frac{1}{p_k}}\right\} &= \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x+y)\big)}{\rho_1+\rho_2}\right)t_k{}^{\frac{1}{p_k}}\right\} \\ &\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_{k\geq 1}\left[M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_1}\right)t_k{}^{\frac{1}{p_k}}\right] \\ &+ \left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_{k\geq 1}\left[M_k\left(\frac{q\big(\Lambda_k(y)\big)}{\rho_2}\right)t_k{}^{\frac{1}{p_k}}\right] \\ &\leq 1\end{aligned}$$

and thus

$$\begin{aligned}g(x+y) &= q(x_1+y_1) \\ &+ \inf\left\{(\rho_1+\rho_2)^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x)+\Lambda_k(y)\big)}{\rho}\right)\right\}t_k{}^{\frac{1}{p_k}} \leq 1,\ \rho > 0\right\} \\ &\leq q(x_1) + \inf\left\{(\rho_1)^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_1}\right)\right\}t_k{}^{\frac{1}{p_k}} \leq 1,\ \rho > 0\right\} \\ &+ q(y_1) + \inf\left\{(\rho_2)^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(y)\big)}{\rho_2}\right)\right\}t_k{}^{\frac{1}{p_k}} \leq 1,\ \rho > 0\right\}\end{aligned}$$

$$\leq g(x) + g(y)$$

(iv) Finally, we prove that the scalar multiplication is continuous. Let μ be any complex number. By definition,

$$g(\mu x) = q(\mu x_1) + \inf\left\{\rho^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\mu\Lambda_k(x)\big)}{\rho}\right)\right\}t_k{}^{\frac{1}{p_k}} \leq 1,\ \ \rho > 0\right\}$$

$$= |\mu| q(x_1) + \inf\left\{(|\lambda| r)^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x)\big)}{r}\right)\right\}t_k{}^{\frac{1}{p_k}} \leq 1,\ \ r > 0\right\},$$

where $r = \frac{\rho}{|\mu|}$. Hence $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ is a paranormed space. □

Theorem 3 *For any Musielak-Orlicz function* $\mathcal{M} = (M_k)$ *and* $p = (p_k) \in l_\infty$, *then the spaces* $c_0\{\mathcal{M}, \Lambda, p, q\}$, $c\{\mathcal{M}, \Lambda, p, q\}$ *and* $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ *are complete paranormed spaces paranormed by* g.

Proof. Suppose (x^n) is a Cauchy sequence in $l_\infty\{\mathcal{M}, \Lambda, p, q\}$, where $x^n = (x_k^n)_{k=1}^\infty$ for all $n \in \mathbb{N}$. So that $g(x^i - x^j) \to 0$ as $i, j \to \infty$. Suppose $\epsilon > 0$ is given and let s and x_0 be such that $\frac{\epsilon}{sx_0} > 0$ and $M_k\big(\frac{sx_0}{2}\big) \geq \sup_{k\geq 1}(p_k)^{t_k}$. Since $g(x^i - x^j) \to 0$, as $i, j \to \infty$ which means that there exists $n_0 \in \mathbb{N}$ such that

$$g(x^i - x^j) < \frac{\epsilon}{sx_0},\ \text{ for all }\ i, j \geq n_0.$$

This gives $g(x_1^i - x_1^j) < \frac{\epsilon}{sx_0}$ and

$$\inf\left\{\rho^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x^i - x^j)\big)}{\rho}\right)t_k{}^{\frac{1}{p_k}}\right\} \leq 1, \rho > 0\right\} < \frac{\epsilon}{sx_0}. \tag{2}$$

It shows that (x_1^i) is a Cauchy sequence in X. Therefore (x_1^i) is convergent in X because X is complete. Suppose $\lim_{i\to\infty} x_1^i = x_1$ then $\lim_{j\to\infty} g(x_1^i - x_1^j) < \dfrac{\epsilon}{sx_0}$, we get

$$g(x_1^i - x_1) < \frac{\epsilon}{sx_0}.$$

Thus, we have

$$M_k\left(\frac{q\big(\Lambda_k(x^i - x^j)\big)}{g(x^i - x^j)}\right)t_k{}^{\frac{1}{p_k}} \leq 1.$$

This implies that

$$M_k\left(\frac{q\big(\Lambda_k(x^i - x^j)\big)}{g(x^i - x^j)}\right) \leq (p_k)^{t_k} \leq M_k(\frac{sx_0}{2})$$

and thus

$$q(\Lambda_k(x^i - x^j)) < \frac{sx_0}{2}.\frac{\epsilon}{sx_0} < \frac{\epsilon}{2}$$

which shows that $(\Lambda_k(x^i))$ is a Cauchy sequence in X for all $k \in \mathbb{N}$. Therefore, $(\Lambda_k(x^i))$ converges in X. Suppose $\lim_{i\to\infty} \Lambda_k(x^i) = y$ for all $k \in \mathbb{N}$. Also, we have $\lim_{i\to\infty} \Lambda_k(x_2^i) = y_1 - x_1$. On repeating the same procedure, we obtain $\lim_{i\to\infty} \Lambda_k(x_{k+1}^i) = y_k - x_k$ for all $k \in \mathbb{N}$. Therefore by continuity of (M_k), we get

$$\lim_{j\to\infty} \sup_{k\geq 1} M_k\left(\frac{q\big(\Lambda_k(x^i - x^j)\big)}{\rho}\right) t_k^{\frac{1}{p_k}} \leq 1,$$

so that

$$\sup_{k\geq 1} M_k\left(\frac{q\big(\Lambda_k(x^i - x^j)\big)}{\rho}\right) t_k^{\frac{1}{p_k}} \leq 1.$$

Let $i \geq n_0$ and taking infimum of each ρ's, we have

$$g(x^i - x) < \epsilon.$$

So $(x^i - x) \in l_\infty\{\mathcal{M}, \Lambda, p, q\}$. Hence $x = x^i - (x^i - x) \in l_\infty\{\mathcal{M}, \Lambda, p, q\}$, since $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ is a linear space. Hence, $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ is a complete paranormed space. Similarly, we can prove the spaces $c_0\{\mathcal{M}, \Lambda, p, q\}$ and $c\{\mathcal{M}, \Lambda, p, q\}$ are complete paranormed spaces. □

Theorem 4 *If* $0 < p_k \leq r_k < \infty$ *for each* k*, then*

$$Z\{\mathcal{M}, \Lambda, p, q\} \subseteq Z\{\mathcal{M}, \Lambda, r, q\}$$

for $Z = c_0$ *and* c*.*

Proof. Let $x = (x_k) \in c\{\mathcal{M}, \Lambda, p, q\}$. Then there exists some $\rho > 0$ and $L \in X$ such that

$$M_k\left(\frac{q\big(\Lambda_k(x) - L\big)}{\rho}\right)^{p_k} t_k \to 0 \ \text{ as } \ k \to \infty.$$

This implies that

$$M_k\left(\frac{q\big(\Lambda_k(x) - L\big)}{\rho}\right) < \epsilon, \ \ (0 < \epsilon < 1)$$

for sufficiently large k. Hence we get

$$M_k\left(\frac{q\big(\Lambda_k(x)-L\big)}{\rho}\right)^{r_k} t_k \;\leq\; M_k\left(\frac{q\big(\Lambda_k(x)-L\big)}{\rho}\right)^{p_k} t_k \to 0 \;\text{ as }\; k \to \infty.$$

This implies that $x=(x_k)\in c\{\mathcal{M},\Lambda,r,q\}$. This completes the proof. Similarly, we can prove for the case $Z=c_0$. □

Theorem 5 *Suppose $\mathcal{M}'=(M_k')$ and $\mathcal{M}''=(M_k'')$ are Musielak-Orlicz functions satisfying the Δ_2-condition then we have the following results:*

(i) *if $(p_k)\in l_\infty$ then $Z\{\mathcal{M}',\Lambda,p,q\}\subseteq Z\{\mathcal{M}''\circ\mathcal{M}',\Lambda,p,q\}$ for $Z=c,c_0$ and l_∞.*

(ii) *$Z\{\mathcal{M}',\Lambda,p,q\}\cap Z\{\mathcal{M}'',\Lambda,p,q\}\subseteq Z\{\mathcal{M}'+\mathcal{M}'',\Lambda,p,q\}$ for $Z=c,c_0$ and l_∞.*

Proof. If $x=(x_k)\in c_0\{\mathcal{M},\Lambda,p,q\}$ then there exists some $\rho>0$ such that

$$\left\{M_k'\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)\right\}^{p_k} t_k \to 0 \;\text{ as }\; k\to\infty.$$

Suppose

$$y_k = M_k'\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right) \;\text{ for all }\; k\in\mathbb{N}.$$

Choose $\delta>0$ be such that $0<\delta<1$, then for $y_k\geq\delta$ we have $y_k<\frac{y_k}{\delta}<1+\frac{y_k}{\delta}$. Now (M_k'') satisfies Δ_2-condition so that there exists $J\geq 1$ such that

$$M_k''(y_k) < \frac{Jy_k}{2\delta}M_k''(2) + \frac{Jy_k}{2\delta}M_k''(2) = \frac{Jy_k}{\delta}M_k''(2).$$

We obtain

$$\left[(M_k''\circ M_k')\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)\right]^{p_k} t_k = \left[M_k''\left\{M_k'\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)\right\}\right]^{p_k} t_k = \Big[M_k''(y_k)\Big]^{p_k} t_k$$
$$\leq \max\left\{\sup_k\Big([M_k''(1)]^{p_k}\Big), \sup_k\Big([kM_k''(2)\delta^{-1}]^{p_k}\Big)\right\}[y_k]^{p_k} t_k \to 0, \;\text{ as }\; k\to\infty.$$

Similarly, we can prove the other cases.

(ii) Suppose $x = (x_k) \in c_0\{M'_k, \Lambda, p, q\} \cap c_0\{M''_k, \Lambda, p, q\}$, then there exist $\rho_1,\ \rho_2 > 0$ such that

$$\left\{\left(M'_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_1}\right)\right)^{p_k} t_k\right\} \to 0, \text{ as } k \to \infty$$

and

$$\left\{\left(M''_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_2}\right)\right)^{p_k} t_k\right\} \to 0, \text{ as } k \to \infty.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The remaining proof follows from the inequality

$$\left\{\left[(M'_k + M''_k)\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)\right]^{p_k} t_k\right\} \le D\left\{\left[M'_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_1}\right)\right]^{p_k} t_k + \left[M''_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_2}\right)\right]^{p_k} t_k\right\}.$$

Hence $c_0\{M'_k, \Lambda, p, q\} \cap c_0\{M''_k, \Lambda, p, q\} \subseteq c_0\{M'_k + M''_k, \Lambda, p, q\}$. Similarly we can prove the other cases. □

Theorem 6 (i) *If* $0 < \inf p_k \le p_k < 1$, *then* $l_\infty\{\mathcal{M}, \Lambda, p, q\} \subset l_\infty\{\mathcal{M}, \Lambda, q\}$.
(ii) *If* $1 \le p_k \le \sup p_k < \infty$, *then* $l_\infty\{\mathcal{M}, \Lambda, q\} \subset l_\infty\{\mathcal{M}, \Lambda, p, q\}$.

Proof. (i) Let $x = (x_k) \in l_\infty\{\mathcal{M}, \Lambda, p, q\}$. Since $0 < \inf p_k \le 1$, we have

$$\sup_k\left\{\left[M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_2}\right)\right]\right\} \le \sup_k\left\{\left[M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_2}\right)\right]^{p_k} t_k\right\}$$

and hence $x = (x_k) \in l_\infty\{\mathcal{M}, \Lambda, q\}$.
(ii) Let $p_k \ge 1$ for each k and $\sup_k p_k < \infty$. Let $x = (x_k) \in l_\infty\{\mathcal{M}, \Lambda, q\}$, then for each $\epsilon,\ \ 0 < \epsilon < 1$, there exists a positive integer $n_0 \in \mathbb{N}$ such that

$$\sup_k\left\{M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)\right\} \le \epsilon < 1.$$

This implies that

$$\sup_k\left\{\left[M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)\right]^{p_k} t_k\right\} \le \sup_k\left\{M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)\right\}.$$

Thus $x = (x_k) \in l_\infty\{\mathcal{M}, \Lambda, p, q\}$ and this completes the proof. □

References

[1] T. Bilgin, Some new difference sequences spaces defined by an Orlicz function, *Filomat*, **17** (2003), 1–8.

[2] A. Esi, Some new paranormed sequence spaces defined by Orlicz function, I*nternational Journal of Science, Environment and Technology*, **1** (2012), 49–55.

[3] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, **10** (1971), 379–390.

[4] L. Maligranda, *Orlicz spaces and interpolation*, Seminars in Mathematics **5**, Polish Academy of Science, 1989.

[5] M. Mursaleen, Generalized spaces of difference sequences, *J. Math. Anal. Appl.*, **203** (1996), 738–745.

[6] M. Mursaleen and A. K. Noman, On some new sequence spaces of non absolute type related to the spaces l_p and l_∞ I, *Filomat*, **25** (2011), 33–51.

[7] M. Mursaleen and A. K. Noman, On some new sequence spaces of non absolute type related to the spaces l_p and l_∞ II, *Math. Commun.*, **16** (2011), 383–398.

[8] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, **1034** (1983).

[9] K. Raj, A. K. Sharma and S. K. Sharma, A Sequence space defined by Musielak-Orlicz functions, *Int. J. Pure Appl. Math.*, **67** (2011), 475–484.

[10] K. Raj, S. K. Sharma and A. K. Sharma, Some difference sequence spaces in n-normed spaces defined by Musielak-Orlicz function, *Armenian J. Math.,* **3** (2010), 127–141.

[11] K. Raj and S. K. Sharma, Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz function, *Acta. Univ. Saprientia Math.*, **3** (2011), 97–109.

[12] K. Raj and S. K. Sharma, Some multiplier sequence spaces defined by a Musielak-Orlicz function in n-normed spaces, *New Zealand J. Math.*, **42** (2012), 45–56.

[13] B. C. Tripathy and B. Sarma, Some Some classes of difference paranormed sequence spaces defined by Orlicz function, *Thai J. Math.*, **3** (2005), 209–218.

[14] A. Wilansky, *Summability through Functional Analysis*, North- Holland Math. Stud. **85** (1984).