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Some characterizations of special curves in the Euclidean space E^4

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Abstract. In this work, first, we give some characterizations of helices and ccr curves in the Euclidean 4-space. Thereafter, relations among Frenet-Serret invariants of Bertrand curve of a helix are presented. Moreover, in the same space, some new characterizations of involute of a helix are presented.

1 Introduction

In the local differential geometry, we think of curves as a geometric set of points, or locus. Intuitively, we are thinking of a curve as the path traced out by a particle moving in E^4. So, investigating position vectors of the curves is a classical aim to determine behavior of the particle (curve).

Natural scientists have long held a fascination, sometimes bordering on mystical obsession for helical structures in nature. Helices arise in nanosprings, carbon nanotubes, $\alpha-$helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is

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structure of DNA [3]. This fact was published for the first time by Watson and Crick in 1953 [25]. They constructed a molecular model of DNA in which there were two complementary, antiparallel (side-by-side in opposite directions) strands of the bases guanine, adenine, thymine and cytosine, covalently linked through phosphodiesterase bonds. Each strand forms a helix and two helices are held together through hydrogen bonds, ionic forces, hydrophobic interactions and van der Waals fores forming a double helix, lipid bilayers, bacterial flagella in Salmonella and E. coli, aerial hyphae in actinomycete, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells (helico-spiral structures) [4, 5].

Helix is one of the most fascinating curves in science and nature. Also we can see the helix curve or helical structures in fractal geometry, for instance hyperhelices [23]. In the field of computer aided design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion or the design of highways, etc. [26]. From the view of differential geometry, a helix is a geometric curve with non-vanishing constant curvature κ and non-vanishing constant torsion τ [2]. The helix may be called a *circular helix* or *W-curve* [12, 17].

It is known that straight line ($\kappa(s) = 0$) and circle ($\tau(s) = 0$) are degenerate-helix examples [13]. In fact, circular helix is the simplest three-dimensional spirals. One of the most interesting spiral examples are k-Fibonacci spirals. These curves appear naturally from studying the k-Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ and the related hyperbolic k-Fibonacci function. Fibonacci numbers and the related Golden Mean or Golden section appear very often in theoretical physics and physics of the high energy particles [7, 8]. Three-dimensional k-Fibonacci spirals was studied from a geometric point of view in [9].

Indeed, in Euclidean 3-space E^3, a helix is a special case of the *general helix*. A curve of constant slope or general helix in Euclidean 3-space is defined by the property that the tangent makes a constant angle with a fixed straight line called the axis of the general helix. A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (see [22] for details) says that: *A necessary and sufficient condition that a curve be a general helix is that the ratio $\frac{\kappa}{\tau}$ is constant along the curve, where κ and τ denote the curvature and the torsion, respectively.*

The notation of a generalized helix in E^3 can be generalized to higher dimensions in the same definition is proposed but in E^n, i.e., a generalized helix as a curve $\psi : R \to E^n$ such that its tangent vector forms a constant angle with a given direction U in E^n [20].

Two curves which, at any point, have a common principal normal vector are called Bertrand curves. The notion of Bertrand curves was discovered by J. Bertrand in 1850. Bertrand curves have been investigated in E^n and many characterizations are given in [10]. Thereafter, by theory of relativity, investigators extend some of classical differential geometry topics to Lorentzian manifolds. For instance, one can see, Bertrand curves in E^n_1 [6], in E^3_1 for null curves [1], and in E^4_1 for space-like curves [27]. In the fourth section of this paper, we follow the same procedure as in [27].

In this work, first, we aim to give some new characterizations of helices and ccr curves in terms of recently obtained theorems. Thereafter, we investigate relations among Frenet-Serret invariants of Bertrand curve couples, when one of is helix, in the Euclidean 4-space. Moreover, we observe that Bertrand curve of a helix is also a helix; and cannot be a spherical curve, a general helix and a 3-type slant helix, respectively. We also express some characterizations of involute of a helix. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

2 Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E^4 are briefly presented (A more complete elementary treatment can be found in [11]).

Let $\alpha : I \subset \mathbb{R} \to E^4$ be an arbitrary curve in the Euclidean space E^4. Recall that the curve α is said to be of unit speed (or parameterized by arclength function s) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle .,. \rangle$ is the standard scalar (inner) product of E^4 given by

$$\langle \xi, \zeta \rangle = \xi_1\zeta_1 + \xi_2\zeta_2 + \xi_3\zeta_3 + \xi_4\zeta_4,$$

for each $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$, $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in E^4$. In particular, the norm of a vector $\xi \in E^4$ is given by

$$\|\xi\| = \sqrt{\langle \xi, \xi \rangle}.$$

Let $\{T(s), N(s), B(s), E(s)\}$ be the moving frame along the unit speed curve α. Then the Frenet-Serret formulas are given by [10, 21]

$$\begin{bmatrix} T' \\ N' \\ B' \\ E' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \\ E \end{bmatrix}. \tag{1}$$

Here T, N, B and E are called, respectively, the tangent, the normal, the binormal and the trinormal vector fields of the curve and the functions $\kappa(s), \tau(s)$ and $\sigma(s)$ are called, respectively, the first, the second and the third curvature of a curve in E^4. Also, the functions $H_1 = \frac{\kappa}{\tau}$ and $H_2 = \frac{H_1'}{\sigma}$ are called *Harmonic Curvatures* of the curves in E^4, where $\kappa \neq 0, \tau \neq 0$ and $\sigma \neq 0$. Let $\alpha : I \subset R \to E^4$ be a regular curve. If tangent vector field T of α forms a constant angle with unit vector U, this curve is called an inclined curve or a general helix in E^4. Recall that, A curve $\psi = \psi(s)$ is called a 3-type slant helix if the trinormal lines of α make a constant angle with a fixed direction in E^4 [24]. Recall that if a regular curve has constant Frenet curvatures ratios, (i.e., $\frac{\tau}{\kappa}$ and $\frac{\sigma}{\tau}$ are constants), then it is called a *ccr-curve* [16]. It is worth noting that: the W-curve, in Euclidean 4-space E^4, is a special case of a ccr-curve.

Let $\alpha(s)$ and $\alpha^*(s)$ be regular curves in E^4. $\alpha(s)$ and $\alpha^*(s)$ are called Bertrand Curves if for each s_0, the principal normal vector to α at $s = s_0$ is the same as the principal normal vector to $\alpha^*(s)$ at $s = s_0$. We say that $\alpha^*(s)$ is a Bertrand mate for $\alpha(s)$ if $\alpha(s)$ and $\alpha^*(s)$ are Bertrand Curves.

In [14] Magden defined in the same space, a vector product and gave a method to establish the Frenet-Serret frame for an arbitrary curve by the following definition and theorem:

Definition 1 Let $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ and $c = (c_1, c_2, c_3, c_4)$ be vectors in E^4. The vector product in E^4 is defined by the determinant

$$a \wedge b \wedge c = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}, \tag{2}$$

where e_1, e_2, e_3 and e_4 are mutually orthogonal vectors (coordinate direction vectors) satisfying equations

$$e_1 \wedge e_2 \wedge e_3 = e_4, \; e_2 \wedge e_3 \wedge e_4 = e_1, \; e_3 \wedge e_4 \wedge e_1 = e_2, \; e_4 \wedge e_1 \wedge e_2 = e_3.$$

Theorem 1 *Let $\alpha = \alpha(t)$ be an arbitrary regular curve in the Euclidean space E^4 with above Frenet-Serret equations. The Frenet apparatus of α can be written as follows:*

$$T = \frac{\alpha'}{\|\alpha'\|},$$

$$N = \frac{\|\alpha'\|^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha'}{\left\| \|\alpha'\|^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha' \right\|},$$

$$B = \mu\, E \wedge T \wedge N,$$

$$E = \mu \frac{T \wedge N \wedge \alpha'''}{\|T \wedge N \wedge \alpha'''\|},$$

$$\kappa = \frac{\left\| \|\alpha'\|^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha' \right\|}{\|\alpha'\|^4},$$

$$\tau = \frac{\|T \wedge N \wedge \alpha'''\| \, \|\alpha'\|}{\left\| \|\alpha'\|^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha' \right\|},$$

and

$$\sigma = \frac{\left\langle \alpha^{(IV)}, E \right\rangle}{\|T \wedge N \wedge \alpha'''\| \, \|\alpha'\|},$$

where μ *is taken* -1 *or* $+1$ *to make* $+1$ *the determinant of* $[T, N, B, E]$ *matrix.*

3 Some new results of helices and ccr curves

In this section we state some related theorems and some important results about helices and ccr curves:

Theorem 2 *Let* $\alpha = \alpha(s)$ *be a regular curve in* E^4 *parameterized by arclength with curvatures* κ, τ *and* σ. *Then* $\alpha = \alpha(s)$ *lies on the hypersphere of center* m *and radius* $r \in \Re^+$ *in* E^4 *if and only if*

$$\rho^2 + \left(\frac{1}{\tau} \frac{d\rho}{ds} \right)^2 + \frac{1}{\sigma^2} \left[\rho\tau + \frac{d}{ds} \left(\frac{1}{\tau} \frac{d\rho}{ds} \right) \right]^2 = r^2, \tag{3}$$

where $\rho = \frac{1}{\kappa}$ *[16].*

Theorem 3 *Let* $\alpha = \alpha(s)$ *be a regular curve in* E^4 *parameterized by arclength with curvatures* κ, τ *and* σ. *Then* α *is a generalized helix if and only if*

$$H_2' + \sigma H_1 = 0, \tag{4}$$

where $H_1 = \frac{\kappa}{\tau}$ *and* $H_2 = \frac{1}{\sigma} H_1'$ *are the Harmonic Curvatures of* α *[15].*

Theorem 4 *Let* $\alpha = \alpha(s)$ *be a regular curve in* E^4 *parameterized by arclength with curvatures* κ, τ *and* σ. *Then* α *is a type 3-slant helix (its second binormal vector* E *makes a constant angle with a fixed diretion* U*) if and only if*

$$\tilde{H}_2' + \sigma \tilde{H}_1 = 0, \tag{5}$$

where $\tilde{H}_1 = \frac{\sigma}{\tau}$ *and* $\tilde{H}_2 = \frac{1}{\kappa} \tilde{H}_1'$ *are the Anti-Harmonic Curvatures of* α *[18].*

With the aid of the above theorems, one can easily obtain the following important results:

Theorem 5 *Let* $\alpha = \alpha(s)$ *be a helix in* E^4 *with non-zero curvatures.*
1. α *can not be a generalized helix*
2. α *can not be a 3-type slant helix*
3. *If* α *lies on the hypersphere* S^3, *then, the sphere's radius is equal to* $\frac{\sqrt{\tau^2+\sigma^2}}{\kappa\,\sigma}$.

Theorem 6 *Let* $\alpha = \alpha(s)$ *be a ccr curve in* E^4 *with non-zero curvatures* $\kappa(s)$, $\tau(s) = a\,\kappa(s)$ *and* $\sigma(s) = b\,\kappa(s)$. *Then*
1. α *can not be a generalized helix*
2. α *can not be a 3-type slant helix*
3. *If* α *lies on the hypersphere* S^3, *then, if and only if, the following equation is satisfied:*

$$f^2 + \frac{f'^2}{4a^2} + \frac{f}{4a^2b^2}(2a^2 + f'')^2 = r^2, \tag{6}$$

where the function $f = f(s) = \rho^2(s) = \frac{1}{\kappa^2(s)}$.

4 Bertrand curve of a helix

In this section we investigate relations among Frenet-Serret invariants of Bertrand curve of a helix in the space E^4.

Theorem 7 *Let* $\delta = \delta(s)$ *be a helix in* E^4. *Moreover,* ξ *be Bertrand mate of* δ. *Frenet-Serret apparatus of* ξ, $\{T_\xi, N_\xi, B_\xi, E_\xi, \kappa_\xi, \tau_\xi, \sigma_\xi\}$, *can be formed by Frenet apparatus of* δ $\{T, N, B, E, \kappa, \tau, \sigma\}$.

Proof. Let us consider a helix (W-curve, i.e.) $\delta = \delta(s)$. We may express

$$\xi = \delta + \lambda\,N. \tag{7}$$

We know that $\lambda = c =$ constant (cf. [11]). By this way, we can write that

$$\frac{d\xi}{ds_\xi}\frac{ds_\xi}{ds} = T_\xi\frac{ds_\xi}{ds} = (1-\lambda\,\kappa)T + \lambda\,\tau\,B.$$

So, one can have

$$T_\xi = \frac{(1-\lambda\kappa)T + \lambda\tau B}{\sqrt{(1-\lambda\kappa)^2 + (\lambda\tau)^2}}, \tag{8}$$

and

$$\frac{ds_\xi}{ds} = \left\|\xi'\right\| = \sqrt{(1-\lambda\kappa)^2 + (\lambda\tau)^2}. \tag{9}$$

In order to determine relations, we differentiate:

$$\begin{aligned} \xi'' &= \left[\kappa - \lambda(\kappa^2+\tau^2)\right] N + (\lambda\,\tau\,\sigma)E, \\ \xi''' &= \kappa\left[\lambda(\kappa^2+\tau^2) - \kappa\right] T + \tau[\kappa - \lambda(\kappa^2+\tau^2+\sigma^2)]B, \\ \xi^{(IV)} &= l_1\,N + l_2\,E \end{aligned} \tag{10}$$

where

$$l_1 = \kappa^3(\lambda\,\kappa - 1) + \lambda\,\tau^2(2\kappa^2+\tau^2+\sigma^2),$$

and

$$l_2 = \tau\,\sigma[\kappa - \lambda(\kappa^2+\tau^2+\sigma^2)].$$

Using the above equations, we can form

$$\left\|\xi'\right\|^2\,\xi'' - \left\langle \xi', \xi''\right\rangle\,\xi' = K^2\left[[\kappa - \lambda(\kappa^2+\tau^2)]N + (\lambda\,\tau\,\sigma)E\right],$$

where

$$K = \sqrt{(1-\lambda\kappa)^2 + (\lambda\tau)^2}.$$

Therefore, we obtain the principal normal and the first curvature, respectively,

$$N_\xi = \frac{1}{L}\left[[\kappa - \lambda(\kappa^2+\tau^2)]N + (\lambda\,\tau\,\sigma)E\right], \tag{11}$$

and

$$\kappa_\xi = \frac{L}{K^2}, \tag{12}$$

where

$$L = \sqrt{[\kappa - \lambda(\kappa^2+\tau^2)]^2 + (\lambda\,\tau\,\sigma)^2}.$$

Now, we can compute the vector form $T_\xi \wedge N_\xi \wedge \xi'''$ as the following:

$$\begin{aligned} T_\xi \wedge N_\xi \wedge \xi''' &= \frac{1}{KL}\begin{vmatrix} T & N & B & E \\ 1-\lambda\kappa & 0 & \lambda\tau & 0 \\ 0 & \kappa - \lambda(\kappa^2+\tau^2) & 0 & \lambda\tau\sigma \\ l_1 & 0 & l_2 & 0 \end{vmatrix} \\ &= -\frac{M}{KL}\left[\lambda\,\tau\,\sigma\,N - [\kappa - \lambda(\kappa^2+\tau^2)]E\right], \end{aligned}$$

where

$$M = \tau\left[\lambda(\kappa^2+\tau^2+\sigma^2) - \kappa(1+\lambda^2\sigma^2)\right].$$

Since, we have

$$E_\xi = -\frac{1}{L}\left[\lambda\,\tau\,\sigma\,N - [\kappa - \lambda(\kappa^2+\tau^2)]E\right]. \tag{13}$$

By this way, we have the third curvature as follows:

$$\tau_\xi = \frac{M}{K^2 L}. \tag{14}$$

Besides, considering last equation of Theorem 1, one can calculate

$$\sigma_\xi = \frac{\kappa\,\sigma}{L}. \tag{15}$$

Now, to determine the third vector field of Frenet frame, we write

$$E_\xi \wedge T_\xi \wedge N_\xi = -\frac{1}{KL^2}\begin{vmatrix} T & N & B & E \\ 0 & \lambda\tau\sigma & 0 & \lambda(\kappa^2+\tau^2)-\kappa \\ 1-\lambda\kappa & 0 & \lambda\tau & 0 \\ 0 & \kappa-\lambda(\kappa^2+\tau^2) & 0 & \lambda\tau\sigma \end{vmatrix}.$$

So we obtain:

$$B_\xi = -\frac{1}{K}\left[\lambda\,\tau\,T + (1-\lambda\,\kappa)B\right]. \tag{16}$$

It is worth to note that $\mu = 1$. ■

Considering obtained equations, we get:

Theorem 8 *Let* $\delta = \delta(s)$ *be a helix in* E^4*. Moreover, let* ξ *be a Bertrand mate of* δ*. Then*

1. ξ *is also a helix.*

2. ξ *can not be a generalized helix.*

3. ξ *can not be a 3-type slant helix.*

4. *If* ξ *lies on the hypersphere* S^3*, then, the sphere's radius is equal to* $\frac{\sqrt{\tau_\xi^2+\sigma_\xi^2}}{\kappa_\xi\,\sigma_\xi} = \frac{\sqrt{\tau^2+(1-\lambda\,\kappa)^2\,\sigma^2}}{\kappa\,\sigma}$.

5 Involute-evolute curve of a helix

In this section, first we correct the computations in the paper [19] and then we obtain new results:

Theorem 9 *Let* $\xi = \xi(s)$ *be involute of* δ. *Let* δ *be a helix in* E^4. *The Frenet apparatus of* ξ, $\{T_\xi, N_\xi, B_\xi, E_\xi, \kappa_\xi, \tau_\xi, \sigma_\xi\}$, *can be formed by Frenet apparatus of* δ $\{T, N, B, E, \kappa, \tau, \sigma\}$ *and take the following form.*

$$T_\xi = N, \quad N_\xi = \frac{-\kappa T + \tau B}{\sqrt{\kappa^2 + \tau^2}}, \quad B_\xi = -E, \quad E_\xi = \frac{\tau T + \kappa B}{\sqrt{\kappa^2 + \tau^2}}, \tag{17}$$

and

$$\kappa_\xi = \frac{\sqrt{\kappa^2 + \tau^2}}{\kappa\,|c - s|}, \ \tau_\xi = \frac{\tau\,\sigma}{\kappa\,\sqrt{\kappa^2 + \tau^2}\,|c - s|}, \ \sigma_\xi = -\frac{\sigma}{\sqrt{\kappa^2 + \tau^2}\,|c - s|}, \tag{18}$$

where

$$\frac{ds_\xi}{ds} = \kappa\,|c - s|. \tag{19}$$

Proof. The proof of the above theorem is similar as the proof of the previous theorem. ■

Theorem 10 *Let* ξ *and* δ *be unit speed regular curves in* E^4. ξ *be involute of* δ. *Then*

1. ξ *cannot be a helix.*
2. ξ *is a ccr-curve.*
3. ξ *cannot be a generalized helix.*
4. ξ *cannot be a 3-type slant helix.*
5. ξ *cannot be lies on the hypersphere* S^3.

Proof. The proof of points 1, 2, 3 and 4 are obviously. In the following we will proof the point 5:

Integrating the equation (19), we have

$$|c - s| = \sqrt{\frac{2s_\xi}{\kappa}},$$

which leads to

$$\kappa_\xi = \frac{A_1}{\sqrt{s_\xi}}, \quad \tau_\xi = \frac{A_2}{\sqrt{s_\xi}}, \quad \sigma_\xi = \frac{A_3}{\sqrt{s_\xi}}, \tag{20}$$

where

$$A_1 = \sqrt{\frac{\kappa^2 + \tau^2}{2\kappa}}, \ A_2 = -\frac{\tau\,\sigma}{2\kappa(\kappa^2 + \tau^2)}, \ A_3 = -\frac{\sigma\sqrt{\kappa}}{\sqrt{2(\kappa^2 + \tau^2)}}.$$

Then if the evolute ξ lies in the hypersphere the equation (6) must be satisfied. Substituting $f = \frac{s_\xi}{A_1^2}$, $\kappa_\xi = \frac{A_1}{\sqrt{s_\xi}}$, $B_1 = \frac{\tau_\xi}{\kappa_\xi}$ and $B_2 = \frac{\sigma_\xi}{\kappa_\xi}$ in the equation (6), we have

$$\frac{s_\xi (B_1^2 + B_2^2)}{A_1^2 B_2^2} + \frac{1}{4A_1^2 B_1^2} = r^2,$$

which is contradiction because the radius r of the sphere must be constant and the coefficient of s_ξ can not be equal zero. The proof is completed. ■

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