



## Soft covered ideals in semigroups

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**Abstract.** In this work, soft covered ideals in semigroups are constructed and in this concept, soft covered semigroups, soft covered left (right) ideals, soft covered interior ideals, soft covered (generalized) bi-ideals and soft covered quasi ideals of a semigroup are defined. Various properties of these ideals are introduced and the interrelations of these soft covered ideals and the relations of soft anti covered ideals and soft covered ideals are investigated.

### 1 Introduction

Soft set theory introduced by Molodstov in 1999 [1] is applied uncertainty complicated problems especially in economy, medical, engineering and over classical Mathematics such as groups [2], semirings [3], rings [4], BCK/BCI-algebras [5, 6, 7], d-algebras [8], BL-algebras [9], BCH-algebras [10] and near-rings [11].

The works over soft set theory progressed rapidly and after then, this theory started to extend over fuzzy set, rough set and so on and a lot of mathematical structures were constructed over these sets. Moreover, soft set theory has

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received abroad attention by several authors in game theory, intelligent system and especially decision making method. After decision making method was applied over uncertainty sets, it could solve complicated problems including medical diagnoses, finding the best alternative solution for a company for example; the best worker, the best job etc., logistic industry and so on. A relation has been developed between soft set and decision making method and it has overcome several problems [12, 13, 14, 15].

A lot of ideals were introduced over fuzzy sets. Mandal [16] introduced fuzzy ideals and definition of fuzzy interior ideals of ordered semirings. Dib and Galhum [17] defined fuzzy groupoid, a fuzzy semigroup and also fuzzy ideals and fuzzy bi-ideals of a semigroup. Moreover, Kazanci and Yamak [18] gave a kind of generalized fuzzy bi-ideals of semigroups. Kavikumar and Khamis [19] studied fuzzy ideals and fuzzy quasi ideals in ternary semirings. Then, Changphas and Summagrap [20] worked semigroups in covered ideals. Sezer and others wrote several papers [21, 22] over soft ideals in semigroups. In one of papers, they introduced soft union semigroup, ideals and bi-ideals [21]. After then, they defined soft intersection quasi ideals, generalized bi-ideals of a semigroup and surveyed regular, weakly regular, intra regular, completely regular, quasi regular semigroups with help of these ideals [22].

Fabrici [23, 24] investigated covered left, right ideal of a semigroup. By inspiring covered ideals, anti-covered (AC)-left (right), interior ideals were defined in [25] and Xie and Yan [25] obtained fuzzy anti-covered (AC)-left (right), bi-ideal, interior ideal of a semigroup and studied their basic properties. In [26], soft anti covered (AC)-ideals of semigroups were defined and studied in detailed.

In this manuscript, we first give the basic definitions of soft sets and covered ideals in semigroups. By using these basic definitions, we introduce covered ideals of semigroups by defining soft covered semigroups, soft covered left (right) ideals, soft covered bi-ideals, soft covered interior ideals, soft covered quasi-ideals, soft covered generalized bi-ideals. We discuss their properties and the interrelations with each others. Also, we study the relationship between soft anti covered ideals defined in [26] and soft covered ideals which is the most important point in this article.

## 2 Methodology

From now on,  $U$  refers to an initial universe,  $E$  is a set of parameters,  $P(U)$  is the power set of  $U$  and  $A, B, C \subseteq E$ .

**Definition 1** [1, 13] A soft set  $f_A$  over  $U$  is a set defined by

$$f_A : E \rightarrow P(U) \text{ such that } f_A(x) = \emptyset \text{ if } x \notin A.$$

Here  $f_A$  is also called an approximate function. A soft set over  $U$  can be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}.$$

It is clear to see that a soft set is a parameterized family of subsets of the set  $U$ . Note that the set of all soft sets over  $U$  will be denoted by  $S(U)$ .

**Definition 2** [13] Let  $f_A, f_B \in S(U)$ . Then,  $f_A$  is called a soft subset of  $f_B$  and denoted by  $f_A \subseteq f_B$ , if  $f_A(x) \subseteq f_B(x)$  for all  $x \in E$ .

**Definition 3** [13] Let  $f_A, f_B \in S(U)$ . Then, union of  $f_A$  and  $f_B$ , denoted by  $f_A \widetilde{\cup} f_B$ , is defined as  $f_A \widetilde{\cup} f_B = f_{A \widetilde{\cup} B}$ , where  $f_{A \widetilde{\cup} B}(x) = f_A(x) \cup f_B(x)$ , intersection of  $f_A$  and  $f_B$ , denoted by  $f_A \widetilde{\cap} f_B$ , is defined as  $f_A \widetilde{\cap} f_B = f_{A \widetilde{\cap} B}$ , where  $f_{A \widetilde{\cap} B}(x) = f_A(x) \cap f_B(x)$  for all  $x \in E$ .

**Definition 4** [13] Let  $f_A, f_B \in S(U)$ . Then,  $\widetilde{\wedge}$ -product of  $f_A$  and  $f_B$ , denoted by  $f_A \widetilde{\wedge} f_B$ , is defined as  $f_A \widetilde{\wedge} f_B = f_{A \widetilde{\wedge} B}$ , where  $f_{A \widetilde{\wedge} B}(x, y) = f_A(x) \cap f_B(y)$  for all  $(x, y) \in E \times E$ .

**Definition 5** [27] Let  $f_A$  and  $f_B$  be soft sets over the common universe  $U$  and  $\Psi$  be a function from  $A$  to  $B$ . Then, soft image of  $f_A$  under  $\Psi$ , denoted by  $\Psi(f_A)$ , is a soft set over  $U$  by

$$(\Psi(f_A))(b) = \begin{cases} \bigcap \{f_A(a) \mid a \in A \text{ and } \Psi(a) = b\}, & \text{if } \Psi^{-1}(b) \neq \emptyset, \\ \emptyset, & \text{otherwise} \end{cases}$$

for all  $b \in B$ . And soft pre-image (or soft inverse image) of  $f_B$  under  $\Psi$ , denoted by  $\Psi^{-1}(f_B)$ , is a soft set over  $U$  by  $(\Psi^{-1}(f_B))(a) = f_B(\Psi(a))$  for all  $a \in A$ .

From now on,  $S$  denotes a semigroup.

**Definition 6** [21] Let  $f_S$  and  $g_S$  be soft sets over the common universe  $U$ . Then, soft union product  $f_S * g_S$  is defined by

$$(f_S * g_S)(x) = \begin{cases} \bigcap_{x=yz} \{f_S(y) \cup g_S(z)\}, & \text{if } \exists y, z \in S \text{ such that } x = yz, \\ \emptyset, & \text{otherwise} \end{cases}$$

for all  $x \in S$ .

**Theorem 1** [21] *Let  $f_S, g_S, h_S \in S(\mathcal{U})$ . Then,*

- i)  $(f_S * g_S) * h_S = f_S * (g_S * h_S)$ .
- ii)  $f_S * g_S \neq g_S * f_S$ , generally.
- iii)  $f_S * (g_S \widetilde{\cup} h_S) = (f_S * g_S) \widetilde{\cup} (f_S * h_S)$  and  $(f_S \widetilde{\cup} g_S) * h_S = (f_S * h_S) \widetilde{\cup} (g_S * h_S)$ .
- iv)  $f_S * (g_S \widetilde{\cap} h_S) = (f_S * g_S) \widetilde{\cap} (f_S * h_S)$  and  $(f_S \widetilde{\cap} g_S) * h_S = (f_S * h_S) \widetilde{\cap} (g_S * h_S)$ .
- v) If  $f_S \subseteq g_S$ , then  $f_S * h_S \subseteq g_S * h_S$  and  $h_S * f_S \subseteq h_S * g_S$ .
- vi) If  $t_S, l_S \in S(\mathcal{U})$  such that  $t_S \subseteq f_S$  and  $l_S \subseteq g_S$ , then  $t_S * l_S \subseteq f_S * g_S$ .

From now on, if  $f_S : S \rightarrow P(\mathcal{U})$  is a soft set satisfying  $f_S(x) = \emptyset$  for all  $x \in S$ , then  $f_S$  is denoted by  $\theta$  and if  $f_S : S \rightarrow P(\mathcal{U})$  is a soft set satisfying  $f_S(x) = \mathcal{U}$  for all  $x \in S$ , then  $f_S$  is denoted by  $\mathbb{S}$ .

**Lemma 1** *Let  $f_S$  be any soft set over  $\mathcal{U}$ . Then, we have the followings.*

- i)  $\theta * \theta \supseteq \theta$ .
- ii)  $(\mathbb{S} - f_S) * \theta \supseteq \theta$  and  $\theta * (\mathbb{S} - f_S) \supseteq \theta$ .
- iii)  $(\mathbb{S} - f_S) \widetilde{\cup} \theta = (\mathbb{S} - f_S)$  and  $(\mathbb{S} - f_S) \widetilde{\cap} \theta = \theta$ .

Soft anti covered (AC)-ideals of a semigroup were defined in [26] as following:

**Definition 7** [26]  $f_S$  is called as soft AC-semigroup over  $\mathcal{U}$ , if

$$f_S(xy) \supseteq (\mathbb{S} - f_S)(x) \cap (\mathbb{S} - f_S)(y)$$

for all  $x, y \in S$ .

**Definition 8** [26]  $f_S$  is called a soft AC-left ideal over  $\mathcal{U}$ , if  $f_S(xy) \supseteq (\mathbb{S} - f_S)(y)$ , AC-right ideal of  $S$  over  $\mathcal{U}$ , if  $f_S(xy) \supseteq (\mathbb{S} - f_S)(x)$ , AC-ideal of  $S$  over  $\mathcal{U}$ , if  $f_S(xy) \supseteq (\mathbb{S} - f_S)(y)$  and  $f_S(xy) \supseteq (\mathbb{S} - f_S)(x)$  for all  $x, y \in S$ .

**Definition 9** [26] A soft AC-semigroup  $f_S$  over  $\mathcal{U}$  is called a soft AC-bi-ideal over  $\mathcal{U}$ , if

$$f_S(xyz) \supseteq (\mathbb{S} - f_S)(x) \cap (\mathbb{S} - f_S)(z)$$

**Definition 10** [26]  $f_S$  is called a soft AC-interior ideal over  $\mathcal{U}$ , if

$$f_S(xyz) \supseteq (\mathbb{S} - f_S)(y)$$

and is called soft AC-generalized bi-ideal of  $S$ , if

$$f_S(xyz) \supseteq (\mathbb{S} - f_S)(x) \cap (\mathbb{S} - f_S)(y)$$

for all  $x, y, z \in S$ .

From now on, all covered ideals are denoted by C-ideals for the sake of brevity. C-ideals of a semigroup (and fuzzy C-ideals of a semigroup) are defined in [23, 24] as following:

$L$  is called C-left ideal of  $S$ , if  $L \subseteq S(S - L)$  and  $R$  is called C-right ideal of  $S$ , if  $R \subseteq (S - R)S$ . By *C-two-sided ideal*, it is meant a subset of  $S$ , which is both C-left and C-right ideal of  $S$ .  $X$  is called a *C-interior* of  $S$  if  $X \subseteq (S - X)S(S - X)$ .  $X$  is called C-bi-ideal of  $S$  if  $X \subseteq S(S - X)S$  and  $X$  is called C-quasi-ideal of  $S$  if  $X \subseteq (S - X)S \cup S(S - X)$ .

### 3 Results and discussion

#### 3.1 Soft C-semigroup

In this section, we construct soft covered semigroup and study some properties of it.

**Definition 11** Let  $S$  be semigroup and  $f_S$  be soft set over  $\mathcal{U}$ .  $f_S$  is called a soft covered semigroup, if

$$f_S(xy) \subseteq (\mathbb{S} - f_S)(x) \cup (\mathbb{S} - f_S)(y)$$

for all  $x, y \in S$ .

From now on, soft covered semigroup is denoted by soft C-semigroup for the sake of brevity.

**Example 1** Consider the semigroup  $S = \{a, b, c, d\}$  constructed by the following table:

.	$a$	$b$	$c$	$d$
$a$	$a$	$a$	$a$	$a$
$b$	$a$	$a$	$a$	$a$
$c$	$a$	$a$	$b$	$a$
$d$	$a$	$a$	$b$	$b$

Let  $\mathbf{U} = \mathbf{D}_3 = \{< x, y >: x^3 = y^2 = e, xy = yx^2\} = \{e, x, x^2, y, yx, yx^2\}$  be the universal set and  $f_S$  be a soft set over  $\mathbf{U}$  such that  $f_S(a) = \emptyset$ ,  $f_S(b) = \{y\}$ ,  $f_S(c) = \{e, yx^2\}$ ,  $f_S(d) = \{e, x^2, yx, yx^2\}$  and so  $(S - f_S)(a) = \{e, x, x^2, y, yx, yx^2\}$ ,  $(S - f_S)(b) = \{e, x, x^2, yx, yx^2\}$ ,  $(S - f_S)(c) = \{x, x^2, y, yx\}$ ,  $(S - f_S)(d) = \{x, y\}$ . One can show that  $f_S$  is a soft  $C$ -semigroup. However, if  $f_S(b) = \{x, x^2, yx\}$ , we can easily show that  $f_S$  is not a soft  $C$ -semigroup.

It is easy to see that if  $f_S(x) = \emptyset$  for  $x \in S$ , then  $f_S$  is a soft  $C$ -semigroup over  $\mathbf{U}$ . We denote such kind of  $C$ -semigroup by  $\theta$ .

**Definition 12** [21] Let  $X$  be a subset of  $S$ . We denote by  $\mathcal{S}_X^c$  the soft anti characteristic function of  $X$  and define as

$$\mathcal{S}_X^c(x) = \begin{cases} \emptyset, & x \in X \\ \mathbf{U}, & x \in S - X \end{cases}$$

It is clear that soft anti characteristic function is a soft set over  $\mathbf{U}$  clearly,

$$\mathcal{S}_X^c : S \rightarrow P(\mathbf{U}).$$

**Theorem 2** If  $X$  is a  $C$ -semigroup of  $S$ , then  $\mathcal{S}_X^c$  is a soft  $C$ -semigroup of  $S$ .

**Proof.** Let  $X$  be a  $C$ -semigroup and  $x = pq \in X$ . Since  $X \subseteq (S - X)(S - X)$ , then it follows that  $x = pq \in (S - X)(S - X)$ , and so  $p, q \in (S - X)$ . In this statement,  $\mathcal{S}_X^c(pq) \subseteq (S - \mathcal{S}_X^c)(p) \cup (S - \mathcal{S}_X^c)(q)$ , that is,  $\mathcal{S}_X^c$  is a soft  $C$ -semigroup of  $S$ . In fact,

$$\begin{aligned} \emptyset &= \mathcal{S}_X^c(pq) \\ &\subseteq (S - \mathcal{S}_X^c)(p) \cup (S - \mathcal{S}_X^c)(q) \\ &= (\mathbf{U} - \mathbf{U}) \cup (\mathbf{U} - \mathbf{U}) \\ &= \emptyset. \end{aligned}$$

□

**Theorem 3** Let  $f_S$  be a soft set over  $\mathbf{U}$ . Then,  $f_S$  is a soft  $AC$ -semigroup over  $\mathbf{U}$  of  $S$  if and only if  $f_S^c$  is a soft  $C$ -semigroup over  $\mathbf{U}$  of  $S$ .

**Proof.** Let  $f_S$  be a soft  $AC$ -semigroup over  $\mathbf{U}$  of  $S$ . In this statement,

$$\begin{aligned} f_S^c(xy) &= (\mathbf{U} - f_S)(xy) \\ &\subseteq \mathbf{U} - ((S - f_S)(x) \cap (S - f_S)(y)) \\ &= (\mathbf{U} - (S - f_S)(x)) \cup (\mathbf{U} - (S - f_S)(y)) \\ &= (\mathbf{U} - (\mathbf{U} - f_S(x))) \cup (\mathbf{U} - (\mathbf{U} - f_S(y))) \\ &= (\mathbf{U} - f_S^c(x)) \cup (\mathbf{U} - f_S^c(y)) \\ &= (S - f_S^c)(x) \cup (S - f_S^c)(y) \end{aligned}$$

for all  $x, y \in S$ . Conversely, let  $f_S^c$  be a soft C-semigroup over  $\mathcal{U}$  of  $S$ . Then,

$$\begin{aligned}
 f_S(xy) &= (\mathcal{U} - f_S^c)(xy) \\
 &\supseteq \mathcal{U} - ((S - f_S^c)(x) \cup (S - f_S^c)(y)) \\
 &\supseteq (\mathcal{U} - (S - f_S^c)(x)) \cap (\mathcal{U} - (S - f_S^c)(y)) \\
 &= (\mathcal{U} - (\mathcal{U} - f_S^c(x))) \cap (\mathcal{U} - (\mathcal{U} - f_S^c(y))) \\
 &= (\mathcal{U} - f_S(x)) \cap (\mathcal{U} - f_S(y)) \\
 &= (S - f_S)(x) \cap (S - f_S)(y)
 \end{aligned}$$

for all  $x, y \in S$ . This completes the proof.  $\square$

**Theorem 4** Let  $f_S$  be a soft set over  $\mathcal{U}$ . Then,  $f_S$  is a soft C-semigroup over  $\mathcal{U}$  if and only if we have:

$$f_S \widetilde{\subseteq} (S - f_S) * (S - f_S).$$

**Proof.** To prove this, we assume that  $f_S$  is a soft C-semigroup over  $\mathcal{U}$ . If  $f_S = \emptyset$ , then it is trivial since

$$f_S(x) \subseteq ((S - f_S) * (S - f_S))(x)$$

thus,  $f_S \subseteq (S - f_S) * (S - f_S)$ . Otherwise, there exist elements  $m, n \in S$  such that  $x = mn$ . Then, since  $f_S$  is a soft C-semigroup over  $\mathcal{U}$ , we have:

$$\begin{aligned}
 ((S - f_S) * (S - f_S))(x) &= \bigcap_{x=mn} (S - f_S)(m) \cup (S - f_S)(n) \\
 &\supseteq \bigcap_{x=mn} f_S(mn) \\
 &= f_S(x)
 \end{aligned}$$

thus,  $f_S \widetilde{\subseteq} (S - f_S) * (S - f_S)$ .

Conversely, suppose that  $f_S \widetilde{\subseteq} (S - f_S) * (S - f_S)$ . Let  $m, n \in S$  and  $x = mn$ . Hence, we have:

$$\begin{aligned}
 f_S(mn) &= f_S(x) \\
 &\subseteq ((S - f_S) * (S - f_S))(x) \\
 &= \bigcap_{x=mn} (S - f_S)(m) \cup (S - f_S)(n) \\
 &\subseteq (S - f_S)(m) \cup (S - f_S)(n).
 \end{aligned}$$

Then, this means that  $f_S$  is a soft C-semigroup over  $\mathcal{U}$ . This completes the proof.  $\square$

**Proposition 1** Let  $f_S$  and  $f_T$  be soft C-semigroups over  $\mathcal{U}$ . Then,  $f_S \widetilde{\wedge} f_T$  is a soft C-semigroup over  $\mathcal{U}$ .

**Proof.** Let  $(x_1, y_1), (x_2, y_2)$  be any two elements of  $S \times T$ . Then,

$$\begin{aligned}
 f_{S \tilde{\wedge} T}((x_1, y_1), (x_2, y_2)) &= f_{S \wedge T}(x_1 x_2, y_1 y_2) \\
 &= f_S(x_1 x_2) \cap f_T(y_1 y_2) \\
 &\subseteq [(\mathbb{S} - f_S)(x_1) \cup (\mathbb{S} - f_S)(x_2)] \cap [(\mathbb{S} - f_T)(y_1) \\
 &\quad \cup (\mathbb{S} - f_T)(y_2)] \\
 &= [\mathbb{U} - (f_S(x_1) \cap f_S(x_2))] \cap [\mathbb{U} - (f_T(y_1) \cap f_T(y_2))] \\
 &= \mathbb{U} - [(f_S(x_1) \cap f_S(x_2)) \cup (f_T(y_1) \cap f_T(y_2))] \\
 &\subseteq \mathbb{U} - [(f_S(x_1) \cap f_S(x_2)) \cap (f_T(y_1) \cap f_T(y_2))] \\
 &= \mathbb{U} - [(f_S(x_1) \cap f_T(y_1)) \cap (f_S(x_2) \cap f_T(y_2))] \\
 &= (\mathbb{S} - f_{S \tilde{\wedge} T})(x_1, y_1) \cup (\mathbb{S} - f_{S \tilde{\wedge} T})(x_2, y_2).
 \end{aligned}$$

This implies that  $f_S \tilde{\wedge} f_T$  is a soft C-semigroup over  $\mathbb{U}$ . □

**Theorem 5** Let  $f_S$  and  $g_S$  be two soft C-semigroups over  $\mathbb{U}$ , then  $f_S \tilde{\cap} g_S$  is also so.

**Proof.** Let  $f_S$  and  $g_S$  be two soft C-semigroups over  $\mathbb{U}$  for  $x, y \in S$ . Then,

$$\begin{aligned}
 (f_S \tilde{\cap} g_S)(xy) &= f_S(xy) \cap g_S(xy) \\
 &\subseteq [(\mathbb{S} - f_S)(x) \cup (\mathbb{S} - f_S)(y)] \cap [(\mathbb{S} - g_S)(x) \cup (\mathbb{S} - g_S)(y)] \\
 &= [\mathbb{U} - (f_S(x) \cap f_S(y))] \cap [\mathbb{U} - (g_S(x) \cap g_S(y))] \\
 &= \mathbb{U} - [(f_S(x) \cap f_S(y)) \cup (g_S(x) \cap g_S(y))] \\
 &\subseteq \mathbb{U} - [(f_S(x) \cap f_S(y)) \cap (g_S(x) \cap g_S(y))] \\
 &= \mathbb{U} - [(f_S(x) \cap g_S(x)) \cap ((f_S(y) \cap g_S(y)))] \\
 &= \mathbb{U} - [(f_S \tilde{\cap} g_S)(x) \cap (f_S \tilde{\cap} g_S)(y)] \\
 &= (\mathbb{S} - (f_S \tilde{\cap} g_S))(x) \cup (\mathbb{S} - (f_S \tilde{\cap} g_S))(y).
 \end{aligned}$$

This completes the proof. □

The union of two C-semigroups needs not to be a soft C-semigroup as shown in following example.

**Example 2** Let  $f_S$  and  $g_S$  be two soft sets over  $\mathbb{U} = D_3$  of semigroup  $S = \{a, b, c, d\}$ . We accept that  $f_S$  is the same soft set in Example 1 and define the soft set,  $g_S$  as following;  $g_S(a) = \emptyset$ ,  $g_S(b) = \{e, x\}$ ,  $g_S(c) = \{x^2, yx, yx^2\}$  and  $g_S(d) = \{y, yx, yx^2\}$ . Since  $(f_S \tilde{\cup} g_S)(dd) = (f_S \tilde{\cup} g_S)(b) \not\subseteq (\mathbb{S} - (f_S \tilde{\cup} g_S))(d) \cup (\mathbb{S} - (f_S \tilde{\cup} g_S))(d)$ ,  $f_S \tilde{\cup} g_S$  is not a soft C-semigroup over  $\mathbb{U}$ .

**Proposition 2** Let  $f_S$  and  $f_T$  be soft C-semigroups over  $\mathbb{U}$  and  $\Psi$  is a semi-group isomorphism from  $S$  to  $T$ . If  $f_S$  is a soft C-semigroup over  $\mathbb{U}$ , then  $\Psi(f_S)$  is a soft C-semigroup.



**Proof.** Let  $m_1, m_2 \in T$ . Since  $\Psi$  is surjective, then there exist  $k_1, k_2 \in S$  such that  $\Psi(k_1) = m_1$  and  $\Psi(k_2) = m_2$ . Then,

$$\begin{aligned}
 (\Psi(f_S))(m_1 m_2) &= \bigcap \{f_S(k) : k \in S, \Psi(k) = m_1 m_2\} \\
 &= \bigcap \{f_S(k) : k \in S, k = \Psi^{-1}(m_1 m_2)\} \\
 &= \bigcap \{f_S(k) : k \in S, k = \Psi^{-1}(\Psi(k_1 k_2)) = k_1 k_2\} \\
 &= \bigcap \{f_S(k_1 k_2) : k_i \in S, \Psi(k_i) = m_i, i = 1, 2\} \\
 &\subseteq \bigcap \{(\mathbb{S} - f_S)(k_1) \cup (\mathbb{S} - f_S)(k_2) : m_i \in S, \Psi(k_i) = m_i, i = 1, 2\} \\
 &= (\bigcap \{(\mathbb{S} - f_S)(k_1) : k_1 \in S, \Psi(k_1) = m_1\}) \\
 &\quad \cup (\bigcap \{(\mathbb{S} - f_S)(k_2) : k_2 \in S, \Psi(k_2) = m_2\}) \\
 &\subseteq (\bigcup \{(\mathbb{S} - f_S)(k_1) : k_1 \in S, \Psi(k_1) = m_1\}) \\
 &\quad \cup (\bigcup \{(\mathbb{S} - f_S)(k_2) : k_2 \in S, \Psi(k_2) = m_2\}) \\
 &= \{\mathbb{U} - \bigcap (f_S(k_1)) : k_1 \in S, \Psi(k_1) = m_1\} \\
 &\quad \cup \{\mathbb{U} - \bigcap (f_S(k_2)) : k_2 \in S, \Psi(k_2) = m_2\} \\
 &= (\mathbb{S} - \Psi(f_S))(m_1) \cup (\mathbb{S} - \Psi(f_S))(m_2).
 \end{aligned}$$

Consequently,  $\Psi(f_S)$  is a soft C-semigroup over  $\mathbb{U}$ . □

**Proposition 3** Let  $f_S$  and  $f_T$  be soft C-semigroups over  $\mathbb{U}$  and  $\Psi$  be a semigroup homomorphism from  $S$  to  $T$ . If  $f_T$  is a soft C-semigroup over  $\mathbb{U}$ , then so  $\Psi^{-1}(f_T)$  is.

**Proof.** Suppose  $k_1, k_2 \in S$ . Then,

$$\begin{aligned}
 (\Psi^{-1}(f_T))(k_1 k_2) &= f_T(\Psi(k_1 k_2)) \\
 &= f_T(\Psi(k_1) \Psi(k_2)) \\
 &\subseteq (\mathbb{S} - f_T)(\Psi(k_1)) \cup (\mathbb{S} - f_T)(\Psi(k_2)) \\
 &= \mathbb{U} - (f_T(\Psi(k_1)) \cap f_T(\Psi(k_2))) \\
 &= \mathbb{U} - (\Psi^{-1}(f_T(k_1)) \cap \Psi^{-1}(f_T(k_2))) \\
 &= (\mathbb{S} - \Psi^{-1}(f_T))(k_1) \cup (\mathbb{S} - \Psi^{-1}(f_T))(k_2).
 \end{aligned}$$

Hence,  $\Psi^{-1}(f_T)$  is a soft C-semigroup over  $\mathbb{U}$ . □

### 3.2 Soft C-left (right), C-ideals of semigroups

In this subsection, soft covered left (right) ideal are introduced and also we survey some properties of these ideals.

**Definition 13** Let  $S$  be semigroup and  $f_S$  be soft set over  $\mathbb{U}$ .  $f_S$  is called a soft covered left ideal over  $\mathbb{U}$  if  $f_S(xy) \subseteq (\mathbb{S} - f_S)(y)$ ; covered right ideal of  $S$  over  $\mathbb{U}$  if  $f_S(xy) \subseteq (\mathbb{S} - f_S)(x)$ ; covered ideal of  $S$  over  $\mathbb{U}$  if  $f_S(xy) \subseteq (\mathbb{S} - f_S)(y)$  and  $f_S(xy) \subseteq (\mathbb{S} - f_S)(x)$  for all  $x, y \in S$ .

From now on, soft covered left ideal is denoted by soft C-left ideal, soft covered right ideal by soft C-right ideal and soft covered ideal by soft C-ideal for the sake of brevity.

**Example 3** Let  $S = \{a, b, c, d\}$  in Example 1. Define the soft set  $f_S$  over  $U = D_4$  as following,  $f_S(a) = \emptyset$ ,  $f_S(b) = \{x, y\}$ ,  $f_S(c) = \{e, x^2, yx^2\}$ ,  $f_S(d) = \{x^2, yx, yx^2\}$  and so  $(S - f_S)(a) = \{e, x, x^2, y, yx, yx^2\}$ ,  $(S - f_S)(b) = \{e, x^2, yx, yx^2\}$ ,  $(S - f_S)(c) = \{x, y, yx\}$ ,  $(S - f_S)(d) = \{e, x, y\}$ . Then,  $f_S$  forms a soft C-left ideal of  $S$  over  $U$  but now suppose  $f_S(b) = \{x, x^2, yx, yx^2\}$ , then since  $f_S(b) = f_S(dd) \not\subseteq (S - f_S)(d)$ ,  $f_S$  is not a soft C-left ideal.

It is easy to see that if  $f_S(x) = \emptyset$  for  $x \in S$ , then  $f_S$  is a soft C-left (right) ideal over  $U$ . We denote such kind of C-left (right) by  $\emptyset$ .

**Theorem 6** If  $X$  is a C-left ideal of  $S$ , then  $\mathcal{S}_X^c$  is a soft C-left ideal of  $S$ .

**Proof.** Assume that  $X$  is a C-left ideal and  $x = mn \in X$ . Then, since  $X \subseteq S(S - X)$ ,  $x = mn \in S(S - X)$ , implying that  $m \in S$  and  $n \in S - X$ . Hence,  $\mathcal{S}_X^c(mn) \subseteq (S - \mathcal{S}_X^c)(n)$ . In fact,

$$\emptyset = \mathcal{S}_X^c(mn) \subseteq (S - \mathcal{S}_X^c)(n) = U - U = \emptyset.$$

Thus,  $\mathcal{S}_X^c$  is a soft C-left ideal of  $S$ . □

**Theorem 7** Let  $f_S$  be a soft set over  $U$ . Then,  $f_S$  is a soft AC-left (right, ideal) over  $U$  of  $S$  if and only if  $f_S^c$  is a soft C-left (right, ideal) ideal over  $U$  of  $S$ .

**Proof.** We give the proof for soft AC-left ideals. Let  $f_S$  be a soft AC-left ideal over  $U$  of  $S$ . In this statement,

$$\begin{aligned} f_S^c(xy) &= (U - f_S)(xy) \\ &\subseteq U - (S - f_S)(y) \\ &\subseteq U - (U - f_S(y)) \\ &= (U - f_S^c(y)) \\ &= (S - f_S^c)(y) \end{aligned}$$

for all  $x, y \in S$ . Conversely, let  $f_S^c$  be a soft C-left ideal over  $U$  of  $S$ . Then,

$$\begin{aligned} f_S(xy) &= (U - f_S^c)(xy) \\ &\supseteq U - ((S - f_S^c)(y)) \\ &\supseteq U - (U - f_S^c(y)) \\ &= (U - f_S(y)) \\ &= (S - f_S)(y) \end{aligned}$$

for all  $x, y \in S$ . This completes the proof.  $\square$

**Theorem 8** *Let  $f_S$  be soft set over  $U$ . Then,  $f_S$  is a soft C-left ideal over  $U$  if and only if*

$$f_S \widetilde{\subseteq} \theta * (S - f_S)$$

**Proof.** Assume that  $f_S$  is a soft C-left ideal over  $U$ . In this statement, if  $f_S = \emptyset$ , it is clear that  $f_S \widetilde{\subseteq} (\theta * (S - f_S))$ . According to assume, since  $f_S$  is a soft C-left ideal for  $m, n \in S$  over  $U$ , we have:

$$\begin{aligned} (\theta * (S - f_S))(mn) &= \bigcap_{s=mn} \theta(m) \cup (S - f_S)(n) \\ &\supseteq \bigcap_{s=mn} \emptyset \cup (f_S)(mn) \\ &= \bigcap_{s=mn} (f_S)(mn) \\ &= f_S(s) \\ &= f_S(mn) \end{aligned}$$

hence,  $f_S \widetilde{\subseteq} \theta * (S - f_S)$ . Conversely, suppose that  $f_S \widetilde{\subseteq} (\theta * (S - f_S))$ . Let  $m, n \in S$  and  $s = mn$ . Then, we have:

$$\begin{aligned} (f_S)(mn) &= (f_S)(s) \\ &\subseteq (\theta * (S - f_S))(mn) \\ &= \bigcap_{mn=xy} \theta(x) \cup (S - f_S)(y) \\ &\subseteq \emptyset \cup (S - f_S)(n) \\ &= (S - f_S)(n). \end{aligned}$$

$\square$

**Theorem 9** *Let  $f_S$  be soft set over  $U$ . Then,  $f_S$  is a soft C-right ideal over  $U$  if and only if*

$$f_S \widetilde{\subseteq} (S - f_S) * \theta.$$

**Theorem 10** *Let  $f_S$  be soft set over  $U$ . Then,  $f_S$  is a soft C-ideal over  $U$  if and only if  $f_S \widetilde{\subseteq} \theta * (S - f_S)$  and  $f_S \widetilde{\subseteq} (S - f_S) * \theta$ .*

**Theorem 11** *Let  $f_S$  and  $g_S$  be two soft subsets of  $S$  such that  $g_S \widetilde{\subseteq} f_S$ . If  $f_S$  is a soft C-left ideal, then  $g_S$  is also a soft C-left ideal.*

**Proof.** First assume that  $f_S$  is soft C-left ideal and  $g_S \widetilde{\subseteq} f_S$ . Let  $p, q \in S$  and

$s = pq$ . Then,

$$\begin{aligned}
 g_S(pq) \subseteq f_S(pq) &\subseteq (\theta * (\mathbb{S} - f_S))(pq) \\
 &= \bigcap_{s=pq} \theta(p) \cup (\mathbb{S} - f_S)(q) \\
 &= \bigcap_{s=pq} \emptyset \cup (\mathbb{S} - f_S)(q) \\
 &= \bigcap_{s=pq} (\mathbb{S} - f_S)(q) \\
 &\subseteq \bigcap_{s=pq} (\mathbb{S} - g_S)(q) \\
 &= \bigcap_{s=pq} \theta(p) \cup (\mathbb{S} - g_S)(q) \\
 &= (\theta * (\mathbb{S} - g_S))(pq).
 \end{aligned}$$

Consequently,  $g_S(pq) \subseteq (\theta * (\mathbb{S} - g_S))(pq)$  meaning that  $g_S$  is a soft C-left ideal.  $\square$

**Definition 14** *Soft union left and soft C-left ideal  $f_S$  of  $S$  is called a completely soft C-left ideal of  $S$ .*

**Theorem 12** *Let  $f_S$  be a soft subset of  $S$ . Then, the followings are equivalent.*

- i)  $f_S$  is a completely soft C-left ideal of  $S$ .
- ii)  $(\forall x, y \in S) f_S(xy) \subseteq f_S(y) \cap (\mathbb{S} - f_S)(y)$ .
- iii)  $(\forall s \in S^2) f_S(s) \subseteq \bigcap_{s=xy} f_S(y) \cap \bigcap_{s=xy} (\mathbb{S} - f_S)(y)$ .

**Proof.**

$i \Rightarrow ii$  Suppose that  $f_S$  is a completely soft C-left ideal of  $S$ . Then, since  $f_S$  is a soft C-left ideal  $f_S(xy) \subseteq (\mathbb{S} - f_S)(y)$ . Since  $f_S$  is a soft union left ideal of  $S$ ,  $f_S(xy) \subseteq f_S(y)$ . This means that  $f_S(xy) \subseteq f_S(y) \cap (\mathbb{S} - f_S)(y)$ .

$ii \Rightarrow iii$  Accept that  $(\forall x, y \in S) f_S(xy) \subseteq f_S(y) \cap (\mathbb{S} - f_S)(y)$ . Then, by taking into account the sets,  $f_S(xy) \subseteq f_S(y)$  and  $f_S(xy) \subseteq (\mathbb{S} - f_S)(y)$ . Let  $s = xy$  for any  $x, y \in S$ . This implies that  $f_S(s) = f_S(xy) \subseteq \bigcap_{s=xy} f_S(y) \cap \bigcap_{s=xy} (\mathbb{S} - f_S)(y)$ .

$iii \Rightarrow i$  Assume that  $\forall s = xy \in S^2$ ,  $f_S(s) \subseteq \bigcap_{s=xy} f_S(y) \cap \bigcap_{s=xy} (\mathbb{S} - f_S)(y)$ . Thus,  $f_S(xy) \subseteq \bigcap_{s=xy} f_S(y)$  and  $f_S(xy) \subseteq \bigcap_{s=xy} (\mathbb{S} - f_S)(y)$ . Then,

$$\begin{aligned}
 f_S(s) = f_S(xy) &\subseteq \bigcap_{s=xy} (\mathbb{S} - f_S)(y) \\
 &= \bigcap_{s=xy} \emptyset \cup (\mathbb{S} - f_S)(y) \\
 &= \bigcap_{s=xy} \theta(x) \cup (\mathbb{S} - f_S)(y) \\
 &= (\theta * (\mathbb{S} - f_S))(s)
 \end{aligned}$$

and also,

$$\begin{aligned}
 f_S(xy) &\subseteq \bigcap_{s=xy} f_S(y) \\
 &\subseteq f_S(y).
 \end{aligned}$$

Clearly,  $f_S$  is a completely soft C-left ideal of  $S$ .  $\square$

**Theorem 13** *If  $f_S$  and  $g_S$  are two soft C-left ideals of  $S$  over  $U$ , then  $f_S \widetilde{\cap} g_S$  is a soft C-left ideal over  $U$ .*

**Proof.** Assume that  $f_S$  and  $g_S$  are two soft C-left ideals. Then,

$$\begin{aligned} (f_S \widetilde{\cap} g_S)(mn) &= f_S(mn) \cap g_S(mn) \\ &\subseteq (\mathbb{S} - f_S)(n) \cap (\mathbb{S} - g_S)(n) \\ &= U - (f_S(n) \cup g_S(n)) \\ &\subseteq U - (f_S(n) \cap g_S(n)) \\ &= (\mathbb{S} - (f_S \widetilde{\cap} g_S))(n) \end{aligned}$$

for all  $m, n \in S$ . This means that  $f_S \widetilde{\cap} g_S$  is a soft C-left ideal over  $U$ .

Now, we show that if  $f_S$  and  $g_S$  are two soft C-left ideals of  $S$  over  $U$ , then  $f_S \widetilde{\cup} g_S$  is not a soft C-left ideal of  $S$  with the following example.  $\square$

**Example 4** *Let consider the semigroup  $S = \{a, b, c, d\}$  over  $U = D_4$  in Example 1 and introduce the soft set  $f_S$  over  $U$  as following  $f_S(a) = \emptyset$ ,  $f_S(b) = \{e, x\}$ ,  $f_S(c) = \{x^2, yx, yx^2\}$ ,  $f_S(d) = \{y, x^2, yx^2\}$  so  $(\mathbb{S} - f_S)(a) = \{e, x, x^2, y, yx, yx^2\}$ ,  $(\mathbb{S} - f_S)(b) = \{y, x^2, yx, yx^2\}$ ,  $(\mathbb{S} - f_S)(c) = \{e, x, y\}$ ,  $(\mathbb{S} - f_S)(d) = \{e, x, yx\}$ . Also let  $g_S(a) = \emptyset$ ,  $g_S(b) = \{e\}$ ,  $g_S(c) = \{y, x^2, yx, yx^2\}$ ,  $g_S(d) = \{x, x^2, yx, yx^2\}$  so  $(\mathbb{S} - g_S)(a) = \{e, x, x^2, y, yx, yx^2\}$ ,  $(\mathbb{S} - g_S)(b) = \{x, y, x^2, yx^2\}$ ,  $(\mathbb{S} - g_S)(c) = \{e, x\}$ ,  $(\mathbb{S} - g_S)(d) = \{e, y\}$ . This shows that  $(f_S \widetilde{\cup} g_S)(dd) = (f_S \widetilde{\cup} g_S)(b) \not\subseteq (\mathbb{S} - (f_S \widetilde{\cup} g_S))(d)$ .*

**Proposition 4**  *$f_S$  is a soft C-ideal over  $U$  of  $S$  if and only if*

$$f_S(mn) \subseteq (\mathbb{S} - f_S)(m) \cap (\mathbb{S} - f_S)(n)$$

for all  $m, n \in S$ .

**Proof.** Let  $f_S$  be a soft C-ideal of  $S$  over  $U$  and  $m, n \in S$ . Then,  $f_S(mn) \subseteq (\mathbb{S} - f_S)(m)$  and  $f_S(mn) \subseteq (\mathbb{S} - f_S)(n)$ . Then,  $f_S(mn) \subseteq (\mathbb{S} - f_S)(m) \cap (\mathbb{S} - f_S)(n)$ .

Conversely, suppose that  $f_S(mn) \subseteq (\mathbb{S} - f_S)(m) \cap (\mathbb{S} - f_S)(n)$  for all  $m, n \in S$ . It follows that

$$f_S(mn) \subseteq (\mathbb{S} - f_S)(m) \cap (\mathbb{S} - f_S)(n) \subseteq (\mathbb{S} - f_S)(m)$$

and

$$f_S(mn) \subseteq (\mathbb{S} - f_S)(m) \cap (\mathbb{S} - f_S)(n) \subseteq (\mathbb{S} - f_S)(n)$$

so  $f_S$  is a soft C-ideal of  $S$  over  $U$ .  $\square$

**Theorem 14** *Let  $f_S$  be soft set over  $\mathcal{U}$ . If  $f_S$  is a soft C-left (right) ideal over  $\mathcal{U}$ ,  $f_S$  is a soft C-semigroup over  $\mathcal{U}$ .*

**Proof.** We give the proof for soft C-left ideals. Similarly, it can be indicated for soft C-right ideals. Let  $f_S$  be a soft C-left ideal of  $S$  over  $\mathcal{U}$ . Then,  $f_S(pq) \subseteq (S - f_S)(q)$  for all  $p, q \in S$ . Thus,  $f_S(pq) \subseteq (S - f_S)(q) \subseteq (S - f_S)(p) \cup (S - f_S)(q)$ , therefore  $f_S$  is a soft C-semigroup.  $\square$

**Proposition 5** *Let  $f_S$  be soft set over  $\mathcal{U}$ . Then,  $(S - f_S) \widetilde{\cap} (\theta * (S - f_S))$  is a soft C-left ideal over  $\mathcal{U}$  and  $(S - f_S) \widetilde{\cap} ((S - f_S) * \theta)$  is a soft C-right ideal of  $S$  over  $\mathcal{U}$ .*

**Proof.** Assume that  $f_S$  is a soft C-left ideal of  $S$ . Then,

$$\begin{aligned} \theta * [(S - f_S) \widetilde{\cap} (\theta * (S - f_S))] &= [(\theta * (S - f_S)) \widetilde{\cap} ((\theta * (\theta * (S - f_S))))] \\ &= (\theta * (S - f_S)) \widetilde{\cap} ((\theta * \theta) * (S - f_S)) \\ &\supseteq (\theta * (S - f_S)) \widetilde{\cap} ((\theta * (S - f_S))) \\ &= (\theta * (S - f_S)) \\ &\supseteq (S - f_S) \widetilde{\cap} (\theta * (S - f_S)). \end{aligned}$$

Clearly,  $(S - f_S) \widetilde{\cap} (\theta * (S - f_S))$  is a soft C-left ideal of  $S$  over  $\mathcal{U}$ . Also,

$$\begin{aligned} [((S - f_S) \widetilde{\cap} ((S - f_S) * \theta))] * \theta &= [((S - f_S) * \theta) \widetilde{\cap} (((S - f_S) * \theta) * \theta)] \\ &= ((S - f_S) * \theta) \widetilde{\cap} (((S - f_S) * \theta * \theta)) \\ &\supseteq ((S - f_S) * \theta) \widetilde{\cap} ((S - f_S) * \theta) \\ &= ((S - f_S) * \theta) \\ &\supseteq (S - f_S) \widetilde{\cap} ((S - f_S) * \theta). \end{aligned}$$

Hence,  $(S - f_S) \widetilde{\cap} ((S - f_S) * \theta)$  is a soft C-right ideal of  $S$  over  $\mathcal{U}$ .  $\square$

**Theorem 15** *Let  $f_S$  and  $g_S$  be a soft C-right ideal and soft C-left ideal of  $S$  over  $\mathcal{U}$ , respectively. Then,*

$$f_S \widetilde{\cup} g_S \widetilde{\subseteq} (S - f_S) * (S - g_S).$$

**Proof.** We know that  $f_S$  is a soft C-R-right ideal of  $S$  over  $\mathcal{U}$  and  $g_S$  is a soft C-left ideal of  $S$  over  $\mathcal{U}$  and also  $\theta \widetilde{\subseteq} (S - f_S)$ ,  $\theta \widetilde{\subseteq} (S - g_S)$ . Thus,

$$g_S \widetilde{\subseteq} \theta * (S - g_S) \widetilde{\subseteq} (S - f_S) * (S - g_S)$$

and

$$f_S \widetilde{\subseteq} (S - f_S) * \theta \widetilde{\subseteq} (S - f_S) * (S - g_S)$$

from here

$$f_S \widetilde{\cup} g_S \subseteq (\mathbb{S} - f_S) * (\mathbb{S} - g_S).$$

Now, we survey that again, let  $f_S$  and  $g_S$  be a soft C-right ideal a soft C-left ideal of  $S$  over  $U$ , respectively. In this statement,

$$f_S \widetilde{\cap} g_S \not\subseteq (\mathbb{S} - f_S) * (\mathbb{S} - g_S).$$

with the following example. □

**Example 5** Think the semigroup  $S = \{a, b, c, d\}$  over  $U = D_4$  in Example 1 and let the soft set  $f_S$  and  $g_S$  be  $f_S(b) = \{x, yx\}$ ,  $(\mathbb{S} - f_S)(c) = \{e, x, yx\}$ ,  $(\mathbb{S} - f_S)(d) = \{x, y, yx\}$  and  $g_S(b) = \{yx, yx^2\}$ ,  $(\mathbb{S} - g_S)(c) = \{x, yx, yx^2\}$ ,  $(\mathbb{S} - g_S)(d) = \{yx, yx^2\}$ . Then,  $f_S$  is a soft C-right ideal and  $g_S$  is a soft C-left ideal.

$$\begin{aligned} ((\mathbb{S} - f_S) * (\mathbb{S} - g_S))(b) &= \{(\mathbb{S} - f_S)(d) \cup (\mathbb{S} - g_S)(d)\} \cap \{(\mathbb{S} - f_S)(c) \cup (\mathbb{S} - g_S)(c)\} \\ &\cap \{(\mathbb{S} - f_S)(d) \cup (\mathbb{S} - g_S)(c)\} \\ &= \{x, yx, yx^2\} \\ &\not\subseteq (f_S \widetilde{\cap} g_S)(b) \\ &= \{yx\}. \end{aligned}$$

**Proposition 6** Let  $f_S$  and  $f_T$  be soft C-left (right) ideals of  $S$  over  $U$ . Again,  $f_S \widetilde{\wedge} f_T$  is a soft C-left (right) ideal of  $S \times T$  over  $U$ .

**Proof.** We accept that  $f_S$  and  $f_T$  are soft C-left ideals of  $S$  over  $U$  and  $(x_1, y_1), (x_2, y_2) \in S \times T$ ,

$$\begin{aligned} f_{S \widetilde{\wedge} T}((x_1, y_1), (x_2, y_2)) &= f_{S \widetilde{\wedge} T}(x_1 x_2, y_1 y_2) \\ &= f_S(x_1 x_2) \cap f_T(y_1 y_2) \\ &\subseteq (\mathbb{S} - f_S)(x_2) \cap (\mathbb{S} - f_T)(y_2) \\ &= U - (f_S(x_2) \cup f_T(y_2)) \\ &\subseteq U - (f_S(x_2) \cap f_T(y_2)) \\ &= (\mathbb{S} - f_{S \widetilde{\wedge} T})(x_2, y_2). \end{aligned}$$

Therefore,  $f_S \widetilde{\wedge} f_T$  is a soft C-left ideal over  $U$ . □

We give following propositions without proof. The proofs are similar to those in section 2.

**Proposition 7** Let  $f_S$  and  $h_S$  be soft sets over  $U$  and  $\Psi$  is a semigroup isomorphism from  $S$  to  $T$ . If  $f_S$  is a soft C-left (right) ideal of  $S$  over  $U$ , then so is  $\Psi(f_S)$  of  $T$  over  $U$ .

**Proposition 8** Let  $f_S$  and  $h_S$  be soft sets over  $\mathcal{U}$  and  $\Psi$  is a semigroup homomorphism from  $S$  to  $T$ . If  $f_T$  is a soft  $C$ -left (right) ideal of  $T$  over  $\mathcal{U}$ , then so is  $\Psi^{-1}(f_T)$  of  $S$  over  $\mathcal{U}$ .

### 3.3 Soft $C$ -bi-ideals of semigroups

In this subsection, we define soft covered bi-ideals and provide their basic properties by using soft set operations and soft intersection products and also support them with examples.

**Definition 15** A soft  $C$ -semigroup  $f_S$  over  $\mathcal{U}$  is called a soft covered bi-ideal over  $\mathcal{U}$  if

$$f_S(xyz) \subseteq (\mathbb{S} - f_S)(x) \cup (\mathbb{S} - f_S)(z).$$

For the sake of brevity, soft covered bi-ideal is denoted by soft  $C$ -bi-ideal.

**Example 6** Define operation over  $S = \mathbb{Z}_4 = \{0, 1, 2, 3\}$  by the following table:

.	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

Now let  $\mathcal{U} = D_2 = \{e, x, y, yx\}$  be universal set and  $f_S$  be a soft set over  $\mathcal{U}$  defined by  $f_S(0) = \emptyset$ ,  $f_S(1) = \{x\}$ ,  $f_S(2) = \{x, y\}$ ,  $f_S(3) = \{yx\}$  and so  $(\mathbb{S} - f_S)(0) = \mathcal{U}$ ,  $(\mathbb{S} - f_S)(1) = \{e, y, yx\}$ ,  $(\mathbb{S} - f_S)(2) = \{e, yx\}$ ,  $(\mathbb{S} - f_S)(3) = \{e, x, y\}$ . Then,  $f_S$  is a soft  $C$ -bi-ideal, but if  $f_S(1) = \{e, yx\}$ , then  $f_S$  is not a soft  $C$ -bi-ideal.

It is easy to see that if  $f_S(x) = \emptyset$  for  $x \in S$ , then  $f_S$  is a soft  $C$ -bi-ideal over  $\mathcal{U}$ . We denote such kind of  $C$ -bi-ideal by  $\emptyset$ .

**Theorem 16** If  $X$  is a  $C$ -bi-ideal of  $S$ , then  $\mathcal{S}_X$  is a soft  $C$ -bi-ideal of  $S$ .

**Proof.** We accept that  $X$  is a  $C$ -bi-ideal of  $S$ . Let  $x = mnp \in X$ , then it is clear that  $x = mnp \in (S - X)S(S - X)$ , implying that  $m, p \in S - X$  and  $n \in S$ . In this statement,  $\mathcal{S}_X(mnp) \subseteq (\mathbb{S} - \mathcal{S}_X)(m) \cup (\mathbb{S} - \mathcal{S}_X)(p)$ . In fact,

$$\begin{aligned} \emptyset &= \mathcal{S}_X^c(mnp) \\ &\subseteq (\mathbb{S} - \mathcal{S}_X^c)(m) \cup (\mathbb{S} - \mathcal{S}_X^c)(p) \\ &= (\mathcal{U} - \mathcal{U}) \cup (\mathcal{U} - \mathcal{U}) \\ &= \emptyset. \end{aligned}$$

□



**Theorem 17** Let  $f_S$  be a soft set over  $U$ . Then,  $f_S$  is a soft AC-bi-ideal over  $U$  of  $S$  if and only if  $f_S^c$  is a soft C-bi-ideal over  $U$  of  $S$ .

**Proof.** Let  $f_S$  be a soft AC-bi-ideal over  $U$  of  $S$ . In this statement,

$$\begin{aligned}
 f_S^c(xyz) &= (U - f_S)(xyz) \\
 &\subseteq U - ((S - f_S)(x) \cap (S - f_S)(z)) \\
 &= U - ((U - f_S(x)) \cap (U - f_S(z))) \\
 &= (U - (U - f_S(x))) \cup (U - (U - f_S(z))) \\
 &= (U - f_S^c(x)) \cup (U - f_S^c(z)) \\
 &= (S - f_S^c)(x) \cup (S - f_S^c)(z)
 \end{aligned}$$

for all  $x, y, z \in S$ . Conversely, let  $f_S^c$  be a soft C-bi-ideal over  $U$  of  $S$ . Then,

$$\begin{aligned}
 f_S(xyz) &= (U - f_S^c)(xyz) \\
 &\supseteq U - ((S - f_S^c)(x) \cup (S - f_S^c)(z)) \\
 &= (U - (U - f_S^c(x))) \cap (U - (U - f_S^c(z))) \\
 &= (U - f_S(x)) \cap (U - f_S(z)) \\
 &= (S - f_S)(x) \cap (S - f_S)(z)
 \end{aligned}$$

for all  $x, y, z \in S$ . This completes the proof.  $\square$

**Theorem 18** A soft subset  $f_S$  of  $S$  is a soft C-bi-ideal if and only if

$$f_S \widetilde{\subseteq} (S - f_S) * \theta * (S - f_S)$$

**Proof.** Let  $f_S$  be a soft C-bi-ideal of  $S$ . Assume that  $f_S = \emptyset$ , then it is clear  $f_S \widetilde{\subseteq} ((S - f_S) * \theta) * (S - f_S)$ . Otherwise, let  $a, b, t, p, z \in S$ . Since  $f_S$  is a soft C-bi-ideal of  $S$  over  $U$ , then,

$$\begin{aligned}
 ((S - f_S) * \theta * (S - f_S))(s) &= \bigcap_{s=ab} [((S - f_S) * \theta)(a) \cup (S - f_S)(b)] \\
 &= \bigcap_{s=ab} [\bigcap_{a=tp} (S - f_S)(t) \cup \theta(p) \cup (S - f_S)(b)] \\
 &= \bigcap_{s=ab} [\bigcap_{a=tp} (S - f_S)(t) \cup \emptyset \cup (S - f_S)(b)] \\
 &= \bigcap_{s=ab} \bigcap_{a=tp} (S - f_S)(t) \cup (S - f_S)(b) \\
 &\supseteq \bigcap_{s=ab} \bigcap_{a=tp} f_S(tpb) \\
 &= \bigcap_{s=ab} f_S(ab) \\
 &= f_S(s)
 \end{aligned}$$

hence,  $f_S \widetilde{\subseteq} ((S - f_S) * \theta * (S - f_S))$ .

Conversely, let us assume that  $f_S \widetilde{\subseteq} ((S - f_S) * \theta * (S - f_S))$ . Let  $s = abz \in S$ ,

$$\begin{aligned}
 f_S(abz) &= f_S(s) \\
 &\subseteq ((S - f_S) * \theta * (S - f_S))(s) \\
 &= \bigcap_{s=abz=mn} ((S - f_S) * \theta)(m) \cup (S - f_S)(n) \\
 &\subseteq ((S - f_S) * \theta)(ab) \cup (S - f_S)(z) \\
 &= \bigcap_{ab=tp} (S - f_S)(t) \cup \theta(p) \cup (S - f_S)(z) \\
 &\subseteq (S - f_S)(a) \cup \emptyset \cup (S - f_S)(z) \\
 &= (S - f_S)(a) \cup (S - f_S)(z).
 \end{aligned}$$

Consequently,  $f_S$  is a soft C-bi-ideal of  $S$  over  $U$ . This completes the proof.  $\square$

**Theorem 19** *The intersection of two soft C-bi-ideals over  $U$  is a soft C-bi-ideal over  $U$ .*

**Proof.** Let  $f_S$  and  $g_S$  be soft C-bi-ideals over  $U$ . Then,

$$\begin{aligned}
 (f_S \widetilde{\cap} g_S)(mnp) &= f_S(mnp) \cap g_S(mnp) \\
 &\subseteq [(S - f_S)(m) \cup (S - f_S)(p)] \cap [(S - g_S)(m) \cup (S - g_S)(p)] \\
 &= [U - (f_S(m) \cap f_S(p))] \cap [U - (g_S(m) \cap g_S(p))] \\
 &= U - [(f_S(m) \cap f_S(p)) \cup (g_S(m) \cap g_S(p))] \\
 &\subseteq U - [(f_S(m) \cap f_S(p) \cap (g_S(m) \cap g_S(p)))] \\
 &= U - ((f_S(m) \cap g_S(m)) \cap ((f_S(p) \cap g_S(p))) \\
 &= U - ((f_S \widetilde{\cap} g_S)(m) \cap (f_S \widetilde{\cap} g_S)(p)) \\
 &= (U - (f_S \widetilde{\cap} g_S)(m)) \cup (U - (f_S \widetilde{\cap} g_S)(p)) \\
 &= (S - (f_S \widetilde{\cap} g_S))(m) \cup (S - (f_S \widetilde{\cap} g_S))(p)
 \end{aligned}$$

for  $m, n, p \in S$ . This completes the proof. Now, we show that if  $f_S$  and  $g_S$  are two C-bi-ideals of  $S$  over  $U$ , then  $f_S \widetilde{\cup} g_S$  is not a soft C-bi-ideal of  $S$  with the following example.  $\square$

**Example 7** *Consider the semigroup  $S = Z_4 = \{0, 1, 2, 3\}$  and define the soft set  $f_S$  and  $g_S$  over  $U = D_2$  in Example 6 as following, respectively.  $f_S(0) = \emptyset$ ,  $f_S(1) = \{x\}$ ,  $f_S(2) = \{e\}$ ,  $f_S(3) = \{y, yx\}$  so  $(S - f_S)(0) = U$ ,  $(S - f_S)(1) = \{e, y, yx\}$ ,  $(S - f_S)(2) = \{x, y, yx\}$ ,  $(S - f_S)(3) = \{e, x\}$  and  $g_S(0) = \emptyset$ ,  $g_S(1) = \{e\}$ ,  $g_S(2) = \{e\}$ ,  $g_S(3) = \{x, yx\}$  so  $(S - g_S)(0) = U$ ,  $(S - g_S)(1) = \{x, y, yx\}$ ,  $(S - g_S)(2) = \{x, y, yx\}$ ,  $(S - g_S)(3) = \{e, y\}$ . Then,  $f_S$  and  $g_S$  are two C-bi-ideals of  $S$  over  $U$ , since  $(f_S \widetilde{\cup} g_S)(333) = (f_S \widetilde{\cup} g_S)(1) \not\subseteq (S - (f_S \widetilde{\cup} g_S)(3)) \cup (S - (f_S \widetilde{\cup} g_S)(3))$ ,  $(f_S \widetilde{\cup} g_S)$  is not a soft C-bi-ideal.*

**Theorem 20** *Every soft C-left (right) ideal of semigroup  $S$  over  $\mathcal{U}$  is a soft C-bi-ideal of  $S$  over  $\mathcal{U}$ .*

**Proof.** Assume that  $f_S$  is a soft C-Left ideal of  $S$  over  $\mathcal{U}$  for  $m, p, q \in S$ . Then,

$$f_S(mpq) = f_S((mp)q) \subseteq (\mathbb{S} - f_S)(q) \subseteq (\mathbb{S} - f_S)(m) \cup (\mathbb{S} - f_S)(q).$$

Hence,  $f_S$  is a soft C-bi-ideal of  $S$ .  $\square$

**Proposition 9** *Let  $f_S$  and  $f_T$  be soft C-bi-ideals of  $S$  over  $\mathcal{U}$ . Then,  $f_S \widetilde{\wedge} f_T$  is a soft C-bi-ideal of  $S \times T$  over  $\mathcal{U}$ .*

**Proof.** We know that  $f_S$  and  $f_T$  are soft C-bi-ideals and there exists  $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in S \times T$ ,

$$\begin{aligned} f_{S \widetilde{\wedge} T}((x_1, y_1), (x_2, y_2), (x_3, y_3)) &= f_{S \wedge T}(x_1 x_2 x_3, y_1 y_2 y_3) \\ &= f_S(x_1 x_2 x_3) \cap f_T(y_1 y_2 y_3) \\ &\subseteq [(\mathbb{S} - f_S)(x_1) \cup (\mathbb{S} - f_S)(x_3)] \cap [(\mathbb{S} - f_T)(y_1) \\ &\quad \cup (\mathbb{S} - f_T)(y_3)] \\ &= [\mathcal{U} - (f_S(x_1) \cap f_S(x_3))] \cap [\mathcal{U} - (f_T(y_1) \cap f_T(y_3))] \\ &= \mathcal{U} - [(f_S(x_1) \cap f_S(x_3)) \cup (f_T(y_1) \cap f_T(y_3))] \\ &\subseteq \mathcal{U} - [(f_S(x_1) \cap f_S(x_3)) \cap (f_T(y_1) \cap f_T(y_3))] \\ &= \mathcal{U} - [(f_S(x_1) \cap f_T(x_3)) \cap (f_S(y_1) \cap f_T(y_3))] \\ &= (\mathbb{S} - f_{S \widetilde{\wedge} T})(x_1, y_1) \cup (\mathbb{S} - f_{S \widetilde{\wedge} T})(x_3, y_3). \end{aligned}$$

This shows that  $f_S \widetilde{\wedge} f_T$  is a soft C-bi-ideal over  $\mathcal{U}$ .  $\square$

**Proposition 10** *Let  $f_S$  and  $h_S$  be soft sets over  $\mathcal{U}$  and  $\Psi$  is a semigroup isomorphism from  $S$  to  $T$ . If  $f_S$  is a soft C-bi-ideal of  $S$  over  $\mathcal{U}$ , then so is  $\Psi(f_S)$  of  $T$  over  $\mathcal{U}$ .*

**Proposition 11** *Let  $f_S$  and  $h_S$  be soft sets over  $\mathcal{U}$  and  $\Psi$  is a semigroup homomorphism from  $S$  to  $T$ . If  $f_T$  is a soft C-bi-ideal of  $T$  over  $\mathcal{U}$ , then so is  $\Psi^{-1}(f_T)$  of  $S$  over  $\mathcal{U}$ .*

### 3.4 Soft C-interior ideal of semigroups

In this section, we introduce soft covered interior ideals of semigroups, obtain their basic properties with respect to soft operations and soft intersection product.

**Definition 16** Let  $f_S$  be soft set over  $\mathcal{U}$  and  $x, y, z \in S$ . If

$$f_S(xyz) \subseteq (\mathbb{S} - f_S)(y)$$

$f_S$  called a soft covered interior ideal over  $\mathcal{U}$ .

For the sake of brevity, soft covered interior ideal is denoted by soft C-interior ideal over  $\mathcal{U}$ .

**Example 8** Let think the semigroup  $\mathbb{Z}_4$  and the soft set  $f_S$  over  $\mathcal{U} = D_2 = \{e, x, y, yx\}$  in Example 6. Then, one can easily show that  $f_S$  is a soft C-interior ideal of  $S$  over  $\mathcal{U}$ . But we accept that  $f_S(1) = \{e, y\}$ , then  $f_S(333) = f_S(1) \not\subseteq (\mathbb{S} - f_S)(3)$  which implies  $f_S$  is not a soft C-interior ideal.

It is easy to see that if  $f_S(x) = \emptyset$  for  $x \in S$ , then  $f_S$  is a soft C-interior-ideal over  $\mathcal{U}$ . We denote such kind of C-interior-ideal by  $\emptyset$ .

**Theorem 21** If  $X$  is a C-interior ideal of  $S$ , then  $\mathcal{S}_X^c$  is a soft C-interior ideal of  $S$ .

**Proof.** Let  $X$  be C-interior ideal and  $x = mnp \in X$ . Since  $X \subseteq S(S - X)S$ , then  $x = mnp \in S(S - X)S$ , implying that  $m, p \in S$  and  $n \in S - X$ . In this statement,  $\mathcal{S}_X^c(mnp) \subseteq (\mathbb{S} - \mathcal{S}_X^c)(n)$ . In fact,

$$\begin{aligned} \emptyset &= \mathcal{S}_X^c(mnp) \\ &\subseteq (\mathbb{S} - \mathcal{S}_X^c)(n) \\ &= \mathcal{U} - \mathcal{U} \\ &= \emptyset. \end{aligned}$$

□

**Theorem 22** Let  $f_S$  be a soft set over  $\mathcal{U}$ . Then,  $f_S$  is a soft AC-interior ideal over  $\mathcal{U}$  of  $S$  if and only if  $f_S^c$  is a soft C-interior ideal over  $\mathcal{U}$  of  $S$ .

**Proof.** Let  $f_S$  is a soft AC-interior ideal over  $\mathcal{U}$  of  $S$ . In this statement,

$$\begin{aligned} f_S^c(xyz) &= (\mathcal{U} - f_S)(xyz) \\ &\subseteq \mathcal{U} - (\mathbb{S} - f_S)(y) \\ &= \mathcal{U} - (\mathcal{U} - f_S(y)) \\ &= \mathcal{U} - f_S^c(y) \\ &= (\mathbb{S} - f_S^c)(y) \end{aligned}$$

for all  $x, y, z \in S$ . Conversely, let  $f_S^c$  be a soft C-interior ideal over  $\mathcal{U}$  of  $S$ . Then,

$$\begin{aligned} f_S(xyz) &= (\mathcal{U} - f_S^c)(xyz) \\ &\supseteq \mathcal{U} - ((S - f_S^c)(y)) \\ &= \mathcal{U} - (\mathcal{U} - f_S^c(y)) \\ &= \mathcal{U} - f_S(y) \\ &= (S - f_S)(y) \end{aligned}$$

for all  $x, y, z \in S$ . This completes the proof.  $\square$

**Theorem 23** Let  $f_S$  be soft set over  $\mathcal{U}$ . Then,  $f_S$  is a soft C-interior ideal over  $\mathcal{U}$  if and only if

$$f_S \widetilde{\subseteq} \theta * (S - f_S) * \theta$$

**Proof.** Let  $f_S$  be a soft C-interior ideal over  $\mathcal{U}$  and  $x \in S$ . If  $f_S = \emptyset$ , it is clear that  $f_S(x) \widetilde{\subseteq} (\theta * (S - f_S) * \theta)(x)$ , thus  $f_S \widetilde{\subseteq} \theta * (S - f_S) * \theta$ . If there exist elements  $y, z, u, v$  of  $S$  such that  $x = yz$  and  $y = mp$ , we can write:

$$f_S(x) = f_S(yz) = f_S(mpz) \subseteq (S - f_S)(p).$$

Then,

$$\begin{aligned} (\theta * (S - f_S) * \theta)(x) &= ((\theta * (S - f_S)) * \theta)(x) \\ &= \bigcap_{x=yz} (\theta * (S - f_S))(y) \cup \theta(z) \\ &= \bigcap_{x=yz} [\bigcap_{y=mp} (\theta(m) \cup (S - f_S)(p))] \cup \theta(z) \\ &= \bigcap_{x=yz} \bigcap_{y=mp} (\emptyset \cup (S - f_S)(p)) \cup \emptyset \\ &\supseteq \bigcap_{x=yz} \bigcap_{y=mp} \emptyset \cup f_S(mpz) \cup \emptyset \\ &= f_S(x). \end{aligned}$$

Thus,  $f_S \widetilde{\subseteq} \theta * (S - f_S) * \theta$ .

Conversely, accept that  $f_S \widetilde{\subseteq} \theta * (S - f_S) * \theta$  for  $x, a, y, m, n, p, q \in S$ . Then,

$$\begin{aligned} f_S(xay) &\subseteq (\theta * (S - f_S) * \theta)(xay) \\ &= \bigcap_{xay=mn} \{(\theta * (S - f_S))(m) * \theta(n)\} \\ &\subseteq (\theta * (S - f_S))(xa) \cup \theta(y) \\ &= (\theta * (S - f_S))(xa) \cup \emptyset \\ &= \bigcap_{xa=pq} \theta(p) \cup (S - f_S)(q) \\ &\subseteq \theta(x) \cup (S - f_S)(a) \\ &= (S - f_S)(a) \end{aligned}$$

hence,  $f_S$  is a soft C-interior ideal. This completes the proof.  $\square$

**Theorem 24** *Every soft union right C-Left ideal of  $S$  is a soft C-interior ideal of  $S$ .*

**Proof.** We accept that  $f_S$  is a soft union right C-Left ideal. Then,

$$\begin{aligned}
 (\theta * (S - f_S) * \theta)(mnp) &= [(\theta * (S - f_S)) * \theta](mnp) \\
 &= \bigcap_{mnp=uv} [(\theta * (S - f_S))(u) \cup \theta(v)] \\
 &= \bigcap_{mnp=uv} (\theta * (S - f_S))(u) \\
 &\supseteq \bigcap_{mnp=uv} f_S(u) \\
 &\supseteq \bigcap_{mnp=uv} f_S(uv) \\
 &= f_S(mnp)
 \end{aligned}$$

$f_S$  is a C-interior ideal of  $S$ . □

**Theorem 25** *Let  $f_S$  and  $g_S$  be two soft C-interior ideals of  $S$  over  $U$ . Then,  $f_S \widetilde{\cap} g_S$  is a soft C-interior ideal over  $U$ .*

**Proof.** Let  $m, n, p \in S$ . Then,

$$\begin{aligned}
 (f_S \widetilde{\cap} g_S)(mnp) &= f_S(mnp) \cap g_S(mnp) \\
 &\subseteq (S - f_S)(n) \cap (S - g_S)(n) \\
 &\subseteq (S - f_S)(n) \cup (S - g_S)(n) \\
 &= (S - (f_S \widetilde{\cap} g_S))(n)
 \end{aligned}$$

thus,  $f_S \widetilde{\cap} g_S$  is a soft C-interior ideal over  $U$ . Now, we show that if  $f_S$  and  $g_S$  are two soft C-interior ideals of  $S$  over  $U$ ,  $f_S \widetilde{\cup} g_S$  is not a soft C-interior ideal with the following example. □

**Example 9** *Let  $\mathbb{Z}_4$  be the semigroup over  $U = D_2 = \{e, x, y, yx\}$  in Example 6. Let the soft set  $f_S$  and  $g_S$  over  $U$  be as following:  $f_S(0) = \emptyset$ ,  $f_S(1) = \{e, x\}$ ,  $f_S(2) = \{x\}$ ,  $f_S(3) = \{y, yx\}$  and  $g_S(0) = \emptyset$ ,  $g_S(1) = \{e\}$ ,  $g_S(2) = \{y\}$ ,  $g_S(3) = \{x, y, yx\}$ . We see that  $(f_S \widetilde{\cup} g_S)(333) = (f_S \widetilde{\cup} g_S)(1) \not\subseteq S - (f_S \widetilde{\cup} g_S)(3)$ .*

**Definition 17** *A soft set  $f_S$  over  $U$  is defined soft semi prime, if for all  $a \in S$ ,*

$$f_S(a) \subseteq f_S(a^2)$$

**Proposition 12** *Let  $f_S$  be soft semi prime C-interior ideal of a semigroup  $S$ . Then,  $f_S(a^n) \subseteq (S - f_S)(a^{n+1})$  for all positive integers  $n$ .*

**Proof.** Let  $n$  be any positive integer. Then,

$$f_S(a^n) \subseteq f_S(a^{2n}) \subseteq f_S(a^{4n}) = f_S(a^{3n-2}a^{n+1}a) \subseteq (\mathbb{S} - f_S)(a^{n+1}). \quad \square$$

**Proposition 13** *Let  $f_S$  and  $h_S$  be soft sets over  $\mathbb{U}$  and  $\Psi$  is a semigroup isomorphism from  $S$  to  $T$ . If  $f_S$  is a soft  $C$ -interior ideal of  $S$  over  $\mathbb{U}$ , then so is  $\Psi(f_S)$  of  $T$  over  $\mathbb{U}$ .*

**Proposition 14** *Let  $f_S$  and  $h_S$  be soft sets over  $\mathbb{U}$  and  $\Psi$  is a semigroup homomorphism from  $S$  to  $T$ . If  $f_S$  is a soft  $C$ -interior ideal of  $T$  over  $\mathbb{U}$ , then so is  $\Psi^{-1}(f_T)$  of  $S$  over  $\mathbb{U}$ .*

### 3.5 Soft $C$ - quasi ideals of semigroups

In this subsection, we introduce soft covered quasi-ideals of semigroups, define their basic properties with respect to soft set operations, soft intersection product and certain kinds of soft  $C$ -ideals.

**Definition 18** *A soft set  $f_S$  over  $\mathbb{U}$  is called a soft covered quasi-ideal of  $S$  over  $\mathbb{U}$  if*

$$f_S \widetilde{\subseteq} ((\mathbb{S} - f_S) * \theta) \widetilde{\cup} (\theta * (\mathbb{S} - f_S)).$$

For the sake of brevity, soft covered quasi-ideal of  $S$  is denoted by soft  $C$ -quasi-ideal.

**Proposition 15** *Every soft  $C$ -quasi-ideal of  $S$  is a soft  $C$ -semigroup of  $S$ .*

**Proof.** We accept that  $f_S$  is a soft  $C$ -quasi-ideal of  $S$ . Then, since  $\theta \subseteq (\mathbb{S} - f_S)$ ,

$$\theta * (\mathbb{S} - f_S) \widetilde{\subseteq} (\mathbb{S} - f_S) * (\mathbb{S} - f_S)$$

and

$$(\mathbb{S} - f_S) * \theta \widetilde{\subseteq} (\mathbb{S} - f_S) * (\mathbb{S} - f_S)$$

from here

$$f_S \widetilde{\subseteq} (\theta * (\mathbb{S} - f_S)) \widetilde{\cup} ((\mathbb{S} - f_S) * \theta) \widetilde{\subseteq} (\mathbb{S} - f_S) * (\mathbb{S} - f_S)$$

since  $f_S$  is a soft  $C$ -quasi-ideal of  $S$ . Hence,  $f_S$  is a soft  $C$ -semigroup of  $S$ .  $\square$

**Theorem 26** *If  $X$  is a  $C$ -quasi-ideal of  $S$ , then  $S_X^c$  is a soft  $C$ -quasi-ideal of  $S$ .*

**Proof.** Let  $X$  be a  $C$ -quasi-ideal and  $x = mn \in X$ . Since  $X \subseteq ((S - X)S) \cup (S(S - X))$ , then it is clear that  $x = mn \in ((S - X)S) \cup (S(S - X))$ , implying that  $mn \in (S - X)S$  or  $mn \in S(S - X)$  and so  $m \in (S - X)$  and  $n \in S$  or  $m \in S$  and  $n \in (S - X)$ . Hence since  $\mathcal{S}_X^c(x) = \mathcal{S}_X^c(mn) = \emptyset$ , in any case

$$\mathcal{S}_X^c \widetilde{\subseteq} ((S - \mathcal{S}_X^c) * \theta) \widetilde{\cup} (\theta * (S - \mathcal{S}_X^c)).$$

In fact, if we consider  $[(S - \mathcal{S}_X^c) * \theta] \widetilde{\cup} (\theta * (S - \mathcal{S}_X^c))(x)$  and if  $m \in (S - X)$ , then

$$\begin{aligned} ((S - \mathcal{S}_X^c) * \theta)(x) &= \bigcap_{x=mn} (S - \mathcal{S}_X^c)(m) \cup \theta(n) \\ &= \emptyset \end{aligned}$$

and

$$\begin{aligned} (\theta * (S - \mathcal{S}_X^c))(x) &= \bigcap_{x=mn} \theta(m) \cup (S - \mathcal{S}_X^c)(n) \\ &= \emptyset \end{aligned}$$

and so  $[(S - \mathcal{S}_X^c) * \theta] \widetilde{\cup} (\theta * (S - \mathcal{S}_X^c))(x) = \emptyset \cup \emptyset = \emptyset$ . Hence,

$$\mathcal{S}_X^c \widetilde{\subseteq} ((S - \mathcal{S}_X^c) * \theta) \widetilde{\cup} (\theta * (S - \mathcal{S}_X^c)).$$

Also if  $m \in X$ , then

$$\begin{aligned} ((S - \mathcal{S}_X^c) * \theta)(x) &= \bigcap_{x=mn} (S - \mathcal{S}_X^c)(m) \cup \theta(n) \\ &= \mathcal{U} \end{aligned}$$

and

$$\begin{aligned} (\theta * (S - \mathcal{S}_X^c))(x) &= \bigcap_{x=mn} \theta(m) \cup (S - \mathcal{S}_X^c)(n) \\ &= \mathcal{U} \end{aligned}$$

and so  $[(S - \mathcal{S}_X^c) * \theta] \widetilde{\cup} (\theta * (S - \mathcal{S}_X^c))(x) = \mathcal{U} \cup \mathcal{U} = \mathcal{U}$ . Hence,

$$\mathcal{S}_X^c \widetilde{\subseteq} ((S - \mathcal{S}_X^c) * \theta) \widetilde{\cup} (\theta * (S - \mathcal{S}_X^c)).$$

□

**Proposition 16** *Every soft  $C$ -left (right) ideal of  $S$  is a soft  $C$ -quasi-ideal of  $S$ .*

**Proof.** We accept that  $f_S$  is a soft  $C$ -Left ideal of  $S$  over  $\mathcal{U}$  which is defined  $f_S \subseteq (\theta * (S - f_S))$ . In this statement, we have:

$$f_S \widetilde{\subseteq} (\theta * (S - f_S)) \widetilde{\subseteq} ((S - f_S) * \theta) \widetilde{\cup} (\theta * (S - f_S))$$

hence,  $f_S$  is a soft  $C$ -quasi-ideal. □

The converse of the above proposition does not hold in general as shown in the table.



**Example 10** Think  $S = \mathbb{Z}_4 = \{0, 1, 2, 3\}$  defined by the following table:

.	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	1	1

Let  $f_S$  be a soft set over  $\mathcal{U} = D_2 = \{e, x, y, yx\}$  and is defined as following,  $f_S(0) = \emptyset$ ,  $f_S(1) = \{e\}$ ,  $f_S(2) = \{e\}$ ,  $f_S(3) = \{y, yx\}$  and so  $(S - f_S)(0) = \mathcal{U}$ ,  $(S - f_S)(1) = \{x, y, yx\}$ ,  $(S - f_S)(2) = \{x, y, yx\}$ ,  $(S - f_S)(3) = \{e, x\}$ . Then, one can show that  $f_S(1) \subseteq ((S - f_S) * \theta)(1) \cup (\theta * (S - f_S))(1) = (S - f_S)(3) \cup ((S - f_S)(3) \cap (S - f_S)(2)) = (S - f_S)(3) = \{e, x\}$  is a soft C-quasi-ideal but since  $f_S(3.2) = f_S(1) \not\subseteq (S - f_S)(2)$ ,  $f_S$  is not a soft C-Left ideal.

**Proposition 17** Every soft C-quasi-ideal is a soft C-bi-ideal of  $S$ .

**Proof.** Let  $f_S$  be a soft C-quasi-ideal of  $S$ . Then,

$$\begin{aligned}
 f_S &\subseteq ((S - f_S) * \theta) \widetilde{\cup} (\theta * (S - f_S)) \\
 &\subseteq ((S - f_S) * (\theta * \theta)) \widetilde{\cup} ((\theta * \theta) * (S - f_S)) \\
 &\subseteq ((S - f_S) * \theta * (S - f_S)) \widetilde{\cup} ((S - f_S) * \theta) * (S - f_S) \\
 &= ((S - f_S) * \theta * (S - f_S)) \widetilde{\cup} ((S - f_S) * \theta * (S - f_S)) \\
 &= (S - f_S) * \theta * (S - f_S)
 \end{aligned}$$

Hence,  $f_S$  is a soft C-bi-ideal of  $S$  together with Proposition 15. The converse of this proposition does not hold in general as shown in the following example.  $\square$

**Example 11** Let  $S = \{a, b, c, d\}$  be the semigroup in Example 6 and  $\mathcal{U} = D_2 = \{e, x, y, yx\}$ . Let  $f_S$  be a soft set over  $\mathcal{U}$  defined by  $f_S(0) = \emptyset$ ,  $f_S(1) = \{x\}$ ,  $f_S(2) = \{x, y\}$ ,  $f_S(3) = \{yx\}$  and so  $(S - f_S)(0) = \mathcal{U}$ ,  $(S - f_S)(1) = \{e, y, yx\}$ ,  $(S - f_S)(2) = \{e, yx\}$ ,  $(S - f_S)(3) = \{e, x, y\}$ . Then,  $f_S$  is a soft C-bi-ideals. However,  $f_S$  is not a soft C-quasi-ideal since  $f_S(1) = \{x\} \not\subseteq ((S - f_S) * \theta)(1) \cup (\theta * (S - f_S))(1) = (S - f_S)(2) \cap (S - f_S)(3) = \{e\}$ .

**Proposition 18** Let  $f_S$  and  $g_S$  be any soft C-Right ideal and soft C-Left ideal of  $S$  over  $\mathcal{U}$ , respectively. Then,  $f_S \widetilde{\cap} g_S$  is a soft C-quasi-ideal.

**Proof.** Let  $f_S$  be any soft C-Right ideal of  $S$  and  $g_S$  be any soft C-Left ideal of  $S$ . Then,

$$\begin{aligned} (((S - (f_S \tilde{\cap} g_S)) * \theta) \tilde{\cup} (\theta * (S - (f_S \tilde{\cap} g_S)))) &\supseteq (((S - (f_S \tilde{\cap} g_S)) * \theta) \\ &\tilde{\cap} (\theta * (S - (f_S \tilde{\cap} g_S)))) \\ &\supseteq ((S - f_S) * \theta) \tilde{\cap} (\theta * (S - g_S)) \\ &\supseteq f_S \tilde{\cap} g_S. \end{aligned}$$

Thus,  $f_S \tilde{\cap} g_S$  is a soft C-quasi-ideal of  $S$  over  $\mathcal{U}$ .  $\square$

**Proposition 19** *Let  $f_S$  and  $g_S$  be any soft C-quasi-ideals of  $S$  over  $\mathcal{U}$ . Then,  $f_S \tilde{\cap} g_S$  is a soft C-quasi-ideal.*

**Proof.** Let  $f_S$  and  $g_S$  be any soft C-quasi-ideals of  $S$ . Then,

$$\begin{aligned} (((S - (f_S \tilde{\cap} g_S)) * \theta) \tilde{\cup} (\theta * (S - (f_S \tilde{\cap} g_S)))) &\supseteq ((S - f_S) * \theta) \tilde{\cup} (\theta * (S - f_S)) \\ &\supseteq f_S \end{aligned}$$

and

$$\begin{aligned} (((S - (f_S \tilde{\cap} g_S)) * \theta) \tilde{\cup} (\theta * (S - (f_S \tilde{\cap} g_S)))) &\supseteq ((S - g_S) * \theta) \tilde{\cup} (\theta * (S - g_S)) \\ &\supseteq g_S. \end{aligned}$$

Thus,  $((S - (f_S \tilde{\cap} g_S)) * \theta) \tilde{\cup} (\theta * (S - (f_S \tilde{\cap} g_S))) \supseteq (f_S \tilde{\cap} g_S)$ . Then,  $f_S \tilde{\cap} g_S$  is a soft C-quasi-ideal.  $\square$

**Proposition 20** *Let  $f_S$  be any soft C-quasi-ideal of a commutative semigroup  $S$  and  $a \in A$ . Then,*

$$f_S(a^{n+1}) \subseteq (S - f_S)(a^n)$$

*for every positive integer  $n$ .*

**Proof.** For any positive integer  $n$ , we have:

$$\begin{aligned} ((S - f_S) * \theta)(a^{n+1}) &= \bigcap_{a^{n+1}=xy} (S - f_S)(x) \cup \theta(y) \\ &\subseteq (S - f_S)(a^n) \cup \emptyset \\ &= (S - f_S)(a^n) \end{aligned}$$

from here  $((S - f_S) * \theta)(a^{n+1}) \subseteq (S - f_S)(a^n)$ .

Moreover, since  $f_S$  is a soft C-quasi-ideal of  $S$ , we have:

$$\begin{aligned} f_S(a^{n+1}) &\subseteq [((S - f_S) * \theta) \tilde{\cup} (\theta * (S - f_S))](a^{n+1}) \\ &= ((S - f_S) * \theta)(a^{n+1}) \cup (\theta * (S - f_S))(a^{n+1}) \\ &\subseteq (S - f_S)(a^n) \cup (S - f_S)(a^n) \\ &= (S - f_S)(a^n). \end{aligned}$$

This completes the proof.  $\square$

### 3.6 Soft C-generalized Bi-ideals of semigroups

In this subsection, we study soft covered generalized bi-ideals of semigroups, introduce their basic prosperities as regards soft set operations, soft intersection product and certain kinds of soft C-ideals.

**Definition 19** *A soft set over  $\mathcal{U}$  is called a soft covered generalized bi-ideal of  $S$ , if*

$$f_S(xyz) \subseteq (\mathbb{S} - f_S)(x) \cup (\mathbb{S} - f_S)(z)$$

for all  $x, y, z \in S$ .

For the sake of brevity, soft covered generalized bi-ideal is denoted by soft C-generalized bi-ideal of  $S$ . Clearly, every soft C-bi-ideal of  $S$  is a soft C-generalized bi-ideal of  $S$  but converse of this statement is not true. This is indicated by following example.

**Example 12** *Let think the semigroup  $S = \{0, 1, 2, 3\}$  in Example 6 and define the soft set  $f_S$  over  $\mathcal{U} = D_4$  such that  $f_S(0) = \emptyset$ ,  $f_S(1) = \{x, x^2, y\}$ ,  $f_S(2) = \{x, yx\}$ ,  $f_S(3) = \{e, yx, yx^2\}$  and so  $(\mathbb{S} - f_S)(0) = \{e, x, x^2, y, yx, yx^2\}$ ,  $(\mathbb{S} - f_S)(1) = \{e, yx, yx^2\}$ ,  $(\mathbb{S} - f_S)(2) = \{e, y, x^2, yx^2\}$ ,  $(\mathbb{S} - f_S)(3) = \{x, y, x^2\}$ .*

*Then, one can easily show that  $f_S$  is a soft C-generalized bi-ideal of  $S$  over  $\mathcal{U}$ . However, since  $f_S(33) = f_S(2) \not\subseteq (\mathbb{S} - f_S)(3) \cup (\mathbb{S} - f_S)(3)$ ,  $f_S$  is not a soft C-bi-ideal of  $S$ .*

**Theorem 27** *If  $X$  is a C-generalized bi-ideal of  $S$ , then  $S_X^c$  is a soft C-generalized bi-ideal of  $S$ .*

**Theorem 28** *Let  $f_S$  be a soft set over  $\mathcal{U}$ . Then,  $f_S$  is a soft C-generalized bi-ideal over  $\mathcal{U}$  of  $S$  if and only if  $f_S^c$  is a soft C-generalized bi-ideal over  $\mathcal{U}$  of  $S$ .*

**Theorem 29** *Let  $f_S$  be a soft set over  $\mathcal{U}$ . Then,  $f_S$  is a soft C-generalized bi-ideal of  $S$  over  $\mathcal{U}$  if and only if*

$$f_S \widetilde{\subseteq} (\mathbb{S} - f_S) * \theta * (\mathbb{S} - f_S).$$

**Theorem 30** *Every soft C-left (right) ideal of a semigroup  $S$  over  $\mathcal{U}$  is a soft C-generalized bi-ideal of  $S$  over  $\mathcal{U}$ .*

**Proof.** Let  $f_S$  be a soft C-left (right) ideal of  $S$  over  $\mathcal{U}$  and  $x, y, z \in S$ . Then,

$$f_S(xyz) = f_S((xy)z) \subseteq (\mathbb{S} - f_S)(z) \subseteq (\mathbb{S} - f_S)(x) \cup (\mathbb{S} - f_S)(z)$$

Thus,  $f_S$  is a soft C-generalized bi-ideal of  $S$ . □

**Proposition 21** Let  $f_S$  and  $f_T$  be soft  $C$ -generalized bi-ideals of  $S$  over  $\mathcal{U}$ . Then,  $f_S \tilde{\wedge} f_T$  is a soft  $C$ -generalized bi-ideal of  $S \times T$  over  $\mathcal{U}$ .

**Proposition 22** Let  $f_S$  and  $h_S$  be soft sets over  $\mathcal{U}$  and  $\Psi$  is a semigroup isomorphism from  $S$  to  $T$ . If  $f_S$  is a soft  $C$ -generalized bi-ideal of  $S$  over  $\mathcal{U}$ , then so is  $\Psi(f_S)$  of  $T$  over  $\mathcal{U}$ .

**Proposition 23** Let  $f_S$  and  $h_S$  be soft sets over  $\mathcal{U}$  and  $\Psi$  is a semigroup homomorphism from  $S$  to  $T$ . If  $f_T$  is a soft  $C$ -generalized bi-ideal of  $T$  over  $\mathcal{U}$ , then so is  $\Psi^{-1}(f_T)$  of  $S$  over  $\mathcal{U}$ .

## 4 Conclusion

In this manuscript, we have introduced soft  $C$ -semigroups,  $C$ -left (right) ideals,  $C$ -bi-ideals,  $C$ -interior ideals,  $C$ -quasi-ideals and  $C$ -generalized bi-ideals. Moreover, we survey the relation between soft  $AC$ -ideals and soft  $C$ -ideals. Addition to, we obtain the interrelations of various soft  $C$ -ideals as in the following figure.

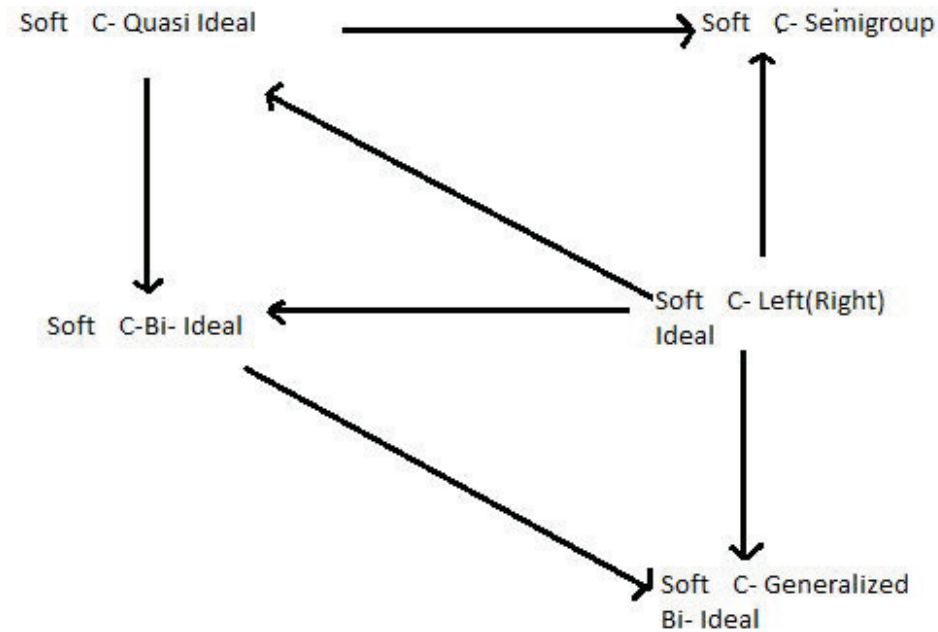


Figure 1

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