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Shepard interpolation with stationary points

László Pál
Sapientia University, Faculty of Business and Humanities, Department of Mathematics and Informatics, Miercurea-Ciuc, Romania
email: pallaszlo@sapientia.siculorum.ro

Róbert Oláh-Gál
Babeş-Bolyai University, Faculty of Mathematics and Informatics, Extension of Miercurea-Ciuc, Romania
email: olah_gal@topnet.ro

Zoltán Makó
Sapientia University, Faculty of Business and Humanities, Department of Mathematics and Informatics, Miercurea-Ciuc, Romania
email: makozoltan@sapientia.siculorum.ro

Abstract. In the paper [10] we analyzed how it is possible to approximate spatial points which mark real methane probes. The origin of the discussed problem is the modelling of real geological reserve calculating. In most of the cases the specialists consider that the contact points of borings to the envelope surface are stationary points. In this paper we will study an interesting feature of the Shepard's interpolation method in m dimensional space, where the control points are stationary and the interpolation function is continuous derivable. Using this interpolation in the real three dimensional space, we will show that the envelope surface may be approximated with this interpolation function.

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1 Introduction

In the paper [10] we analyzed how it is possible to approximate spatial points which mark real methane probes. The origin of the discussed problem is the modelling of real geological reserve calculating ([5], [8]).

There are given planar, spatial or m dimensional points and we are looking for a smooth hypersurface which could interpolate so that these points are stationary.

The classical methods are the Lagrange interpolation method and Gauss' least square method. Both of them have applicable features but there are also disadvantages. The Lagrange interpolation method may fluctuate between two points while with the Gauss method we must choose the desired hipersurface in advance. None of them is natural because the points provide the only piece of information that we have. There is no rule which could describe the mathematical instruments that we must use with the interpolation and approximation methods. Therefore in most of the cases people choose polynomials of $1, 2, 3, ..., n$ degrees because they can solve them more easily. Thus the modern graphical computing evolved and it has the most flexible methods such as the Bezier curves and the B-splines.

Our task comes from a practical problem: how can we reconstruct the methane reserve samples provided by the probes? We partially solved this problem and the results ware published in our paper [10]. Therefore we analyzed the raised question and mathematically rephrased the problem. Thus we came to the conclusion that we must search for a hypersurface to which the given points are stationary.

During the the examination of the literature we found out that in the papers [1], [9] approximation-interpolation methods were presented to terrain models. This is called the *calculating of the arithmetic mean weighted by the inverse of distance interpolation function* (4).

We have tested and compared the Shepard's interpolation method ([2], [3], [6], [7], [4], [11]) given by the functions (5), (7) in two and three dimensions: *arithmetic mean weighted by the distance approximation* (1), with the *arithmetic mean weighted by the square of the distance approximation* (2), with *Lagrange interpolation* (3), with *the arithmetic mean weighted by the inverse of distance interpolation* (4). These comparisons we summarized in Figure 1 and Figure 2. These figures show with the enumerated methods the evolution of the discrete approximation-interpolation in the two respectively three dimensional space.

We can observe that the first two methods (1), (2) – first two rows from

Figure 1 and Figure 2 – only approximate the control points and the curve or surface is determined by these points. The third method (3) – third row from Figure 1 and Figure 2 – interpolate the control points but these points are not stationary. The fourth method (4)-fourth row from Figure 1 and Figure 2- also interpolates, the control points are extremely, but the curve and surface are not smooth in the control points. In the second section, we will show that the Shepard's method (5), (7) – fifth row from Figure 1 and Figure 2 – also interpolates, the control points are stationary, and the curve and surface are smooth.

In the m dimensional space different $A_i, i = \overline{1, n}$ points are given. We denote by $\mathbf{r}_i$ the positional vector of A_i. For every point A_i we assign a z_i scalar value. Let us define the interpolation functions of the enumerated methods in the following manner:

$$E_1(\mathbf{r}) = \frac{\sum\limits_{i=1}^{n} d_i z_i}{\sum\limits_{i=1}^{n} d_i}; \tag{1}$$

$$E_2(\mathbf{r}) = \frac{\sum\limits_{i=1}^{n} d_i^2 z_i}{\sum\limits_{i=1}^{n} d_i^2}; \tag{2}$$

$$L(\mathbf{r}) = \sum_{i=1}^{n} z_i \frac{\prod\limits_{\substack{k=1\\k\neq i}}^{n} (\mathbf{r} - \mathbf{r}_k)}{\prod\limits_{\substack{k=1\\k\neq i}}^{n} d_{ki}}; \tag{3}$$

$$E_3(\mathbf{r}) = \begin{cases} \frac{\sum_{i=1}^{n} \frac{z_i}{d_i}}{\sum_{i=1}^{n} \frac{1}{d_i}}, & \text{if} \quad \mathbf{r} \neq \mathbf{r}_i, \\ z_i, & \text{if} \quad \mathbf{r} = \mathbf{r}_i, \end{cases} \tag{4}$$

where d_i is the length of the vector $\mathbf{r} - \mathbf{r}_i$ and d_{ki} is the distance between points A_k and A_i.

2 The stationary points of the Shepard's interpolation

2.1 Curve interpolation in the plane

Theorem 1 *Given* (x_i, y_i), $i = \overline{1, n}$ *points in the real plane where* $x_i \neq x_j$ *if* $i \neq j$. *Let us define the* $G : \mathbf{R} \to \mathbf{R}$ *function where*

$$G(x) = \begin{cases} \dfrac{\sum\limits_{i=1}^{n} \frac{y_i}{(x-x_i)^2}}{\sum\limits_{i=1}^{n} \frac{1}{(x-x_i)^2}} & \text{if} \quad x \neq x_i, \\ y_i & \text{if} \quad x = x_i. \end{cases} \tag{5}$$

The G *is continuous derivable and* $G'(x_i) = 0$ *for all* $i = \overline{1, n}$.

The theorem is a particular case of the Theorem 2 in one dimensional space.

2.2 Hypersurface interpolation in the m dimensional space

We can generalize the G function in the m dimensional space.

Theorem 2 *Given* $A_i, i = \overline{1, n}$ *different points in the* m *dimensional space. We denote by* $\mathbf{r}_i$ *the positional vector of* A_i. *For every point* A_i *we assign a* z_i *scalar value. Let us define the function* $F : \mathbf{R}^m \to \mathbf{R}$, *where*

$$F(\mathbf{r}) = \begin{cases} \dfrac{\sum\limits_{i=1}^{n} \frac{z_i}{d_i^2}}{\sum\limits_{i=1}^{n} \frac{1}{d_i^2}} & \text{if} \quad \mathbf{r} \neq \mathbf{r}_i, \\ z_i & \text{if} \quad \mathbf{r} = \mathbf{r}_i, \end{cases} \tag{6}$$

where d_i *is the length of the vector* $\mathbf{r} - \mathbf{r}_i$. *The* F *function is continuous derivable and* $F'(\mathbf{r}_i) = 0$ *for all* $i = \overline{1, n}$.

Proof. From the definition we get:

$$F(\mathbf{r}) = \frac{\sum\limits_{i=1}^{n} \frac{z_i}{d_i^2}}{\sum\limits_{i=1}^{n} \frac{1}{d_i^2}} = \frac{\frac{z_1}{d_1^2} + \sum\limits_{i=2}^{n} \frac{z_i}{d_i^2}}{\frac{1}{d_1^2} + \sum\limits_{i=2}^{n} \frac{1}{d_i^2}} = \frac{\frac{1}{d_1^2}\left(z_1 + \sum\limits_{i=2}^{n} \left(\frac{d_1}{d_i}\right)^2 z_i\right)}{\frac{1}{d_1^2}\left(1 + \sum\limits_{i=2}^{n} \left(\frac{d_1}{d_i}\right)^2\right)}$$

and

$$\lim_{\mathbf{r} \to \mathbf{r}_1} F(\mathbf{r}) = \lim_{d_1 \to 0} F(\mathbf{r}) = z_1.$$

Consequently F is continuous. Furthermore if x is one of the coordinates of $\mathbf{r}$ then

$$\begin{aligned}
\frac{\partial F(\mathbf{r})}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\sum_{j=1}^{n}\frac{z_j}{d_j^2}}{\sum_{i=1}^{n}\frac{1}{d_i^2}}\right) \\
&= \frac{-\sum_{i=1}^{n}\frac{1}{d_i^2}\cdot\sum_{j=1}^{n}\frac{2z_j\cdot(x-x_j)}{d_j^4}+\sum_{j=1}^{n}\frac{z_j}{d_j^2}\cdot\sum_{i=1}^{n}\frac{2\cdot(x-x_i)}{d_i^4}}{\left(\sum_{i=1}^{n}\frac{1}{d_i^2}\right)^2} \\
&= \frac{2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(x-x_j)(z_i-z_j)}{d_j^4 d_i^2}}{\left(\sum_{i=1}^{n}\frac{1}{d_i^2}\right)^2}
\end{aligned}$$

If $\mathbf{r}\to\mathbf{r}_1$ then $x\to x_1$, $d_1\to 0$ and

$$\begin{aligned}
\lim_{\mathbf{r}\to\mathbf{r}_1}\frac{\partial F(\mathbf{r})}{\partial x} &= 2\cdot\lim_{\mathbf{r}\to\mathbf{r}_1}\frac{\sum_{i=1}^{n}\left[\frac{(x-x_j)}{d_j^4}\left(\sum_{i=1}^{n}\frac{z_i-z_j}{d_i^2}\right)\right]}{\left(\sum_{i=1}^{n}\frac{1}{d_i^2}\right)^2} \\
&= 2\cdot\lim_{\mathbf{r}\to\mathbf{r}_1}\frac{\frac{(x-x_1)}{d_1^4}\cdot\sum_{i=2}^{n}\frac{z_i-z_1}{d_i^2}+\sum_{j=2}^{n}\left[\frac{(x-x_j)}{d_j^4}\left(\sum_{i=1}^{n}\frac{z_i-z_j}{d_i^2}\right)\right]}{\left(\sum_{i=1}^{n}\frac{1}{d_i^2}\right)^2} \\
&= 2\cdot\lim_{\mathbf{r}\to\mathbf{r}_1}\frac{\frac{(x-x_1)}{d_1^4}\cdot\sum_{i=2}^{n}\frac{z_i-z_1}{d_i^2}}{\left(\frac{1}{d_1}\right)^4\cdot\left(\sum_{i=1}^{n}\left(\frac{d_1}{d_i}\right)^2\right)^2} \\
&\quad + 2\cdot\lim_{\mathbf{r}\to\mathbf{r}_1}\frac{\sum_{j=2}^{n}\left[\frac{(x-x_j)}{d_j^4}\left(\frac{z_1-z_j}{d_1^2}+\sum_{i=2}^{n}\frac{z_i-z_j}{d_i^2}\right)\right]}{\left(\frac{1}{d_1}\right)^4\cdot\left(\sum_{i=1}^{n}\left(\frac{d_1}{d_i}\right)^2\right)^2} \\
&= 0.
\end{aligned}$$

Consequently F is derivable and

$$F'(\mathbf{r}_i) = 0, \text{ for all } i = \overline{1, n}.$$

■

2.3 Surface interpolation in the space

There is a special case. If $m = 2$ we will get the H function in the real space:

$$H(x, y) = \begin{cases} \dfrac{\sum\limits_{i=1}^{n} \frac{z_i}{d_i^2}}{\sum\limits_{i=1}^{n} \frac{1}{d_i^2}} & \text{if} \quad x \neq x_i \text{ or } y \neq y_i, \\ z_i & \text{if} \quad x = x_i \text{ and } y = y_i, \end{cases} \tag{7}$$

where $d_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$ is the euclidean distance and (x_i, y_i, z_i), $i = \overline{1, n}$ are the points we want to interpolate.

3 Conclusions

The F function it is like the (4) but here the weights are the inverse of the distance's square. This function has the properties of Lagrange's interpolation method and those of the arithmetic mean weighted by the inverse of distance method, because it interpolates the control points. The F function also has an important property. It is continuous derivable and the control points are stationary. We illustrate these features in the fifth row of the Figure 1 and Figure 2.

Consequently, with the Shepard's interpolation function we can derive smooth curves, surfaces and it allows the making of beautiful and aesthetic drawings in computer graphics.

Furthermore it is an important geological requirement and an empirical fact that methane and petrol have the shape of a mushroom. They cannot have a polyhedron like, plicate surface. In most of the cases when people make borings, first they find the maximum points of the methane. Therefore if we want to appreciate the volume of the methane we need a surface which crosses the maximum point and it has the form of a mushroom. On Figure 2 is visible that the Shepard's function has approximately this form in the maximum points, but the other functions do not have this feature. Naturally we have tested this function with higher powers of distance but we didn't get better interpolations.

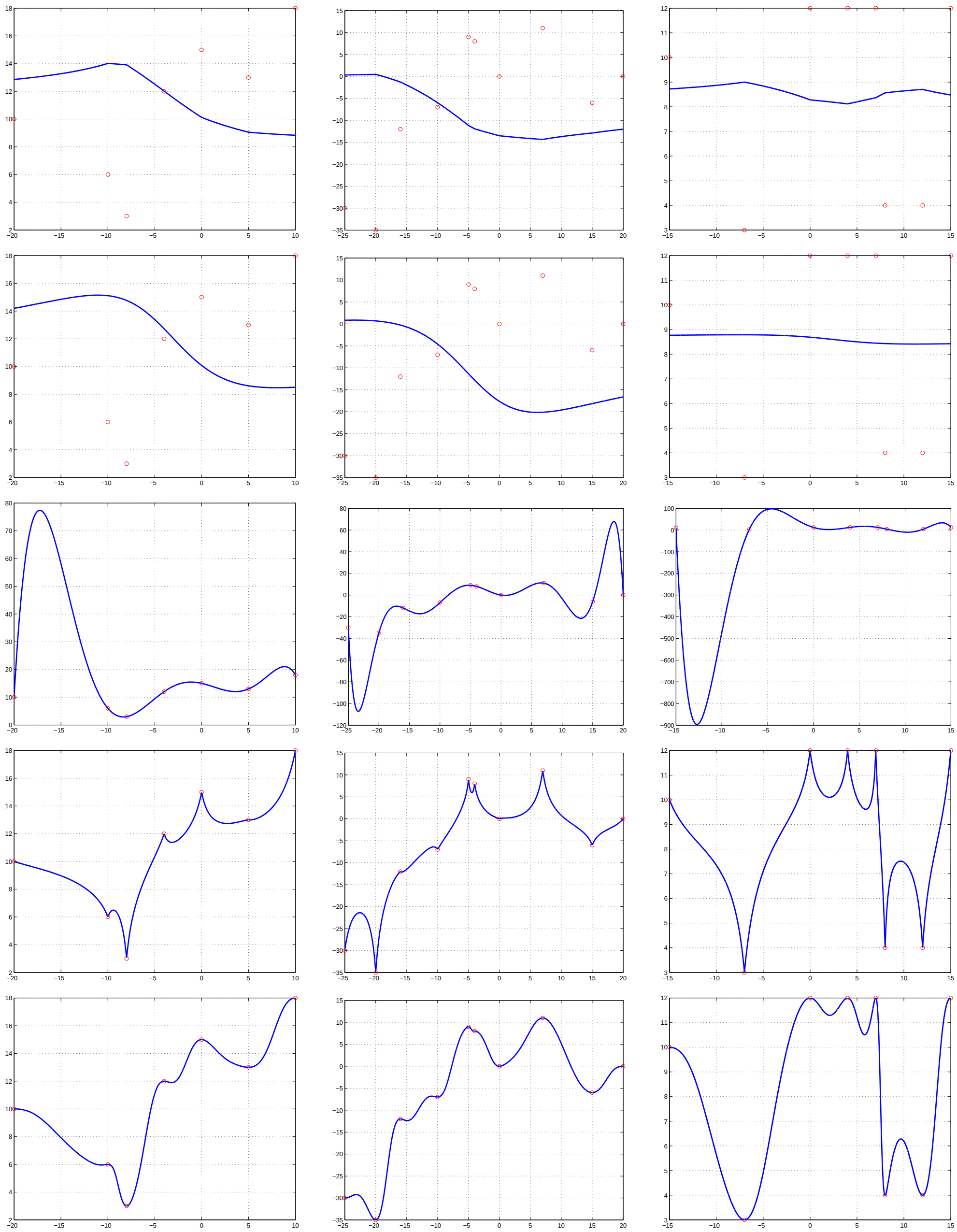

Figure 1: Approximation-interpolation in two dimension

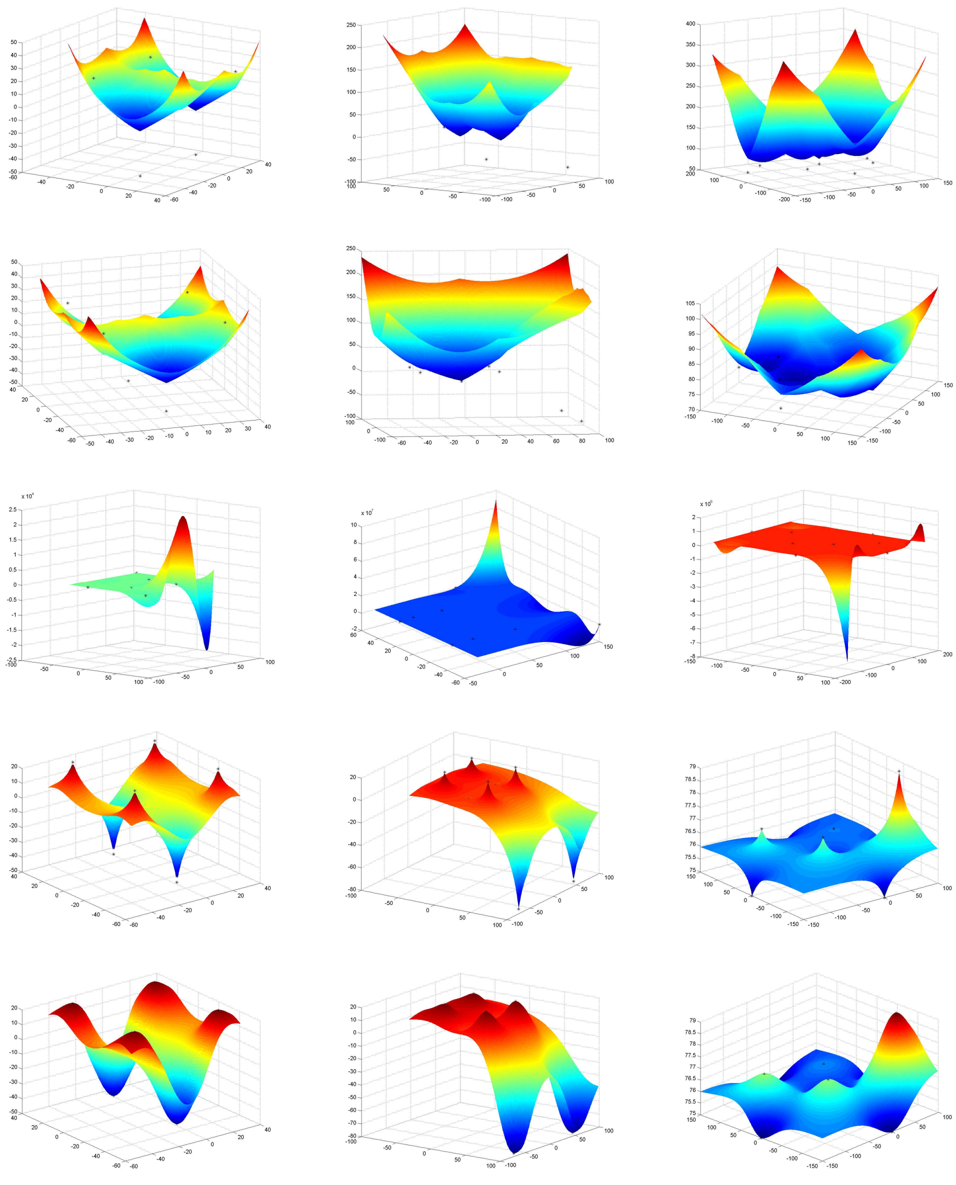

Figure 2: Approximation-interpolation in three dimension

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