Acta Univ. Sapientiae, Mathematica, **11**, 1 (2019) 87–98

DOI: 10.2478/ausm-2019-0008

Sharp bounds of Fekete-Szegő functional for quasi-subordination class

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Abstract. In the present paper, we introduce a certain subclass $\mathcal{K}_q(\lambda, \gamma, h)$ of analytic functions by means of a quasi-subordination. Sharp bounds of the Fekete-Szegő functional for functions belonging to the class $\mathcal{K}_q(\lambda, \gamma, h)$ are obtained. The results presented in the paper give improved versions for the certain subclasses involving the quasi-subordination and majorization.

1 Introduction and definitions

Let $\mathcal{A}$ denote the family of normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. If $f \in \mathcal{A}$ satisfies $f(z_1) \neq f(z_2)$ for any $z_1 \in \mathbb{U}$ and $z_2 \in \mathbb{U}$ with $z_1 \neq z_2$, then f is said to be univalent in $\mathbb{U}$ and denoted by $f \in \mathcal{S}$.

2010 Mathematics Subject Classification: 30C45
Key words and phrases: univalent functions, subordination, quasi-subordination, Fekete-Szegő coefficients

Let g and f be two analytic functions in $\mathbb{U}$ then function g is said to be subordinate to f if there exists an analytic function w in the unit disk $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$g(z) = f(w(z)) \quad (z \in \mathbb{U}).$$

We denote this subordination by $g \prec f$. In particular, if the f is univalent in $\mathbb{U}$, the above subordination is equivalent to $g(0) = f(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. Further, function g is said to be quasi-subordinate [18] to f in the unit disk $\mathbb{U}$ if there exist the functions w (with constant coefficient zero) and ϕ which are analytic and bounded by one in the unit disk $\mathbb{U}$ such that

$$g(z) = \phi(z) f(w(z))$$

and this is equivalent to

$$\frac{g(z)}{\phi(z)} \prec f(z) \quad (z \in \mathbb{U}).$$

We denote this quasi-subordination by $g \prec_q f$. It is observed that if $\phi(z) = 1$ $(z \in \mathbb{U})$, then the quasi-subordination $\prec_q$ become the usual subordination $\prec$, and for the function $w(z) = z$ $(z \in \mathbb{U})$, the quasi-subordination $\prec_q$ become the majorization '$\ll$'. In this case

$$g(z) = \phi(z) f(w(z)) \Rightarrow g(z) \ll f(z), \quad (z \in \mathbb{U}).$$

Some typical problems in geometric function theory are to study functionals made up of combinations of the coefficients of f. In 1933, Fekete and Szegő [5] obtained a sharp bound of the functional $\lambda a_2^2 - a_3$, with real $\lambda (0 \le \lambda \le 1)$ for a univalent function f. Since then, the problem of finding the sharp bounds for this functional of any compact family of functions $f \in \mathcal{A}$ with any complex λ is known as the classical Fekete-Szegő problem or inequality. Lawrence Zalcman posed a conjecture in 1960 that the coefficients of $\mathcal{S}$ satisfy the sharp inequality

$$|a_n^2 - a_{2n-1}| \le (n-1)^2, \quad n \ge 2.$$

More general versions of Zalcman conjecture have also been considered ([4, 12, 13, 14]) for the functional such as

$$\lambda a_n^2 - a_{2n-1} \text{ and } \lambda a_m a_n - a_{m+n-1}$$

for certain positive value of λ. These functionals can be seen as generalizations of the Fekete-Szegő functional $\lambda a_2^2 - a_3$. Several authors including [1]–[4], [9]–[15], [17, 20] have investigated the Fekete-Szegő and Zalcman functionals for various subclasses of univalent and multivalent functions.

Throughout this paper it is assumed that functions ϕ and h are analytic in $\mathbb{U}$. Also let

$$\phi(z) = A_0 + A_1 z + A_2 z^2 + \cdots \quad (|\phi(z)| \leq 1,\ z \in \mathbb{U}) \tag{2}$$

and

$$h(z) = 1 + B_1 z + B_2 z^2 + \cdots \qquad (B_1 \in \mathbb{R}^+). \tag{3}$$

Motivated by earlier works in ([6], [7], [15], [17], [19]) on quasi-subordination, we introduce here the following subclass of analytic functions:

Definition 1 *For* $0 \leq \lambda \leq 1$ *and* $\gamma \in \mathbb{C}\backslash\{0\}$*, a function* $f \in \mathcal{A}$ *given by* (1) *is said to be in the class* $\mathcal{K}_q(\lambda, \gamma, h)$ *if the following condition are satisfied:*

$$\frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) \prec_q (h(z) - 1), \tag{4}$$

where h *is given by* (3) *and* $z \in \mathbb{U}$*.*

It follows that a function f is in the class $\mathcal{K}_q(\lambda, \gamma, h)$ if and only if there exists an analytic function ϕ with $|\phi(z)| \leq 1$, in $\mathbb{U}$ such that

$$\frac{\frac{1}{\gamma}\left(\frac{zf'(z)+z^2f''(z)}{(1-\lambda)z+\lambda zf'(z)} - 1\right)}{\phi(z)} \prec (h(z) - 1)$$

where h is given by (3) and $z \in \mathbb{U}$.
If we set $\phi(z) \equiv 1$ $(z \in \mathbb{U})$, then the class $\mathcal{K}_q(\lambda, \gamma, h)$ is denoted by $\mathcal{K}(\lambda, \gamma, h)$ satisfying the condition that

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) \prec h(z) \quad (z \in \mathbb{U}).$$

In the present paper, we find sharp bounds on the Fekete-Szegő functional for functions belonging in the class $\mathcal{K}_q(\lambda, \gamma, h)$. Several known and new consequences of these results are also pointed out. In order to derive our main results, we have to recall here the following well-known lemma:

Let Ω be class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + \cdots \tag{5}$$

in the unit disk $\mathbb{U}$ satisfying the condition $|w(z)| < 1$.

Lemma 1 ([8], p.10) *If $w \in \Omega$, then for any complex number ν:*

$$|w_1| \leq 1, |w_2 - \nu w_1^2| \leq 1 + (|\nu| - 1)|w_1^2| \leq \max\{1, |\nu|\}.$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

2 Main results

Theorem 1 *Let $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$. If $f \in \mathcal{A}$ of the form (1) belonging to the class $\mathcal{K}_q(\lambda, \gamma, h)$, then*

$$|a_2| \leq \frac{|\gamma| B_1}{2(2-\lambda)} \tag{6}$$

and for any $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{3(3-\lambda)} \max\left\{1, \left|\frac{B_2}{B_1} - QB_1\right|\right\}, \tag{7}$$

where

$$Q = \gamma\left(\frac{3\nu(3-\lambda)}{4(2-\lambda)^2} - \frac{\lambda}{2-\lambda}\right). \tag{8}$$

The results are sharp.

Proof. Let $f \in \mathcal{K}_q(\lambda, \gamma, h)$. In view of Definition 1, there exist then Schwarz functions w and an analytic function ϕ such that

$$\frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) = \phi(z)(h(w(z)) - 1) \quad (z \in \mathbb{U}). \tag{9}$$

Series expansions for f and its successive derivatives from (1) gives us

$$\frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) = \frac{1}{\gamma}\left[2(2-\lambda)a_2 z + \left(3(3-\lambda)a_3 - 4\lambda(2-\lambda)a_2^2\right)z^2 + \ldots\right]. \tag{10}$$

Similarly from (2), (3) and (5), we obtain

$$h(w(z)) - 1 = B_1 w_1 z + (B_1 w_2 + B_2 w_1^2) z^2 + \cdots$$

and

$$\phi(z)\big(h(w(z)) - 1\big) = A_0 B_1 w_1 z + [A_1 B_1 w_1 + A_0 (B_1 w_2 + B_2 w_1^2)] z^2 + \cdots . \quad (11)$$

Equating (10) and (11) in view of (9) and comparing the coefficients of z and z^2, we get

$$a_2 = \frac{\gamma A_0 B_1 w_1}{2(2-\lambda)} \quad (12)$$

and

$$a_3 = \frac{\gamma B_1}{3(3-\lambda)} \left[A_1 w_1 + A_0 \{ w_2 + \left(\frac{\gamma \lambda A_0 B_1}{2-\lambda} + \frac{B_2}{B_1} \right) w_1^2 \} \right]. \quad (13)$$

Thus, for any $\nu \in \mathbb{C}$, we have

$$a_3 - \nu a_2^2 = \frac{\gamma B_1}{3(3-\lambda)} \left[A_1 w_1 + \left(w_2 + \frac{B_2}{B_1} w_1^2 \right) A_0 - \left(\frac{3(3-\lambda)\gamma}{4(2-\lambda)^2} \nu - \frac{\gamma \lambda}{2-\lambda} \right) B_1 A_0^2 w_1^2 \right]$$

$$= \frac{\gamma B_1}{3(3-\lambda)} \left[A_1 w_1 + \left(w_2 + \frac{B_2}{B_1} w_1^2 \right) A_0 - Q B_1 A_0^2 w_1^2 \right], \quad (14)$$

where Q is given by (8).

Since $\phi(z) = A_0 + A_1 z + A_2 z^2 + \cdots$ is analytic and bounded by one in $\mathbb{U}$, therefore we have (see [16], p 172)

$$|A_0| \le 1 \ \text{ and } \ A_1 = (1 - A_0^2) y \qquad (y \le 1). \quad (15)$$

From (14) and (15), we obtain

$$a_3 - \nu a_2^2 = \frac{\gamma B_1}{3(3-\lambda)} \left[y w_1 + \left(w_2 + \frac{B_2}{B_1} w_1^2 \right) A_0 - \left(B_1 Q w_1^2 + y w_1 \right) A_0^2 \right]. \quad (16)$$

If A_0=0 in (16), we at once get

$$|a_3 - \nu a_2^2| \le \frac{|\gamma| B_1}{3(3-\lambda)}. \quad (17)$$

But if $A_0 \neq 0$, let us then suppose that

$$G(A_0) = y w_1 + \left(w_2 + \frac{B_2}{B_1} w_1^2 \right) A_0 - \left(B_1 Q w_1^2 + y w_1 \right) A_0^2$$

which is a quadratic polynomial in A_0 and hence analytic in $|A_0| \leq 1$ and maximum value of $|G(A_0)|$ is attained at $A_0 = e^{\iota\theta}$ $(0 \leq \theta < 2\pi)$, we find that

$$\begin{aligned} \max|G(A_0)| &= \max_{0\leq\theta<2\pi} |G(e^{\iota\theta})| = |G(1)| \\ &= \left| w_2 - \left(QB_1 - \frac{B_2}{B_1} \right) w_1^2 \right|. \end{aligned}$$

Therefore, it follows from (16) that

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma|B_1}{3(3-\lambda)} \left| w_2 - \left(QB_1 - \frac{B_2}{B_1} \right) w_1^2 \right|, \tag{18}$$

which on using Lemma 1, shows that

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma|B_1}{3(3-\lambda)} \max \left\{ 1, \left| \frac{B_2}{B_1} - QB_1 \right| \right\},$$

and this last above inequality together with (17) establish the results. The result are sharps for the function f given by

$$1 + \frac{1}{\gamma} \left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1 \right) = h(z),$$

$$1 + \frac{1}{\gamma} \left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1 \right) = h(z^2)$$

and

$$1 + \frac{1}{\gamma} \left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1 \right) = z(h(z) - 1).$$

This completes the proof of Theorem 1. □

For $\lambda = 0$ the Theorem 1 reduces to following corollary:

Corollary 1 *If* $f \in \mathcal{A}$ *of the form* (1) *satisfies*

$$\frac{1}{\gamma}(f'(z) + zf''(z) - 1) \prec_q (h(z) - 1) \qquad (z \in \mathbb{U},\ \gamma \in \mathbb{C}\backslash\{0\}),$$

then

$$|a_2| \leq \frac{|\gamma|B_1}{4},$$

and for some $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma|B_1}{9} \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{9\nu|\gamma|B_1}{16} \right| \right\}.$$

The results are sharp.

Remark 1 *In Corollary 1, if we set* $\phi \equiv 1$*, then above result match with the result given in* [3].

Remark 2 *For* $\phi \equiv 1$*,* $\gamma = \lambda = 1$*, Theorem 1 reduces to an improved result of given in* [15].

The next theorem gives the result based on majorization.

Theorem 2 *Let* $0 \le \lambda \le 1$ *and* $\gamma \in \mathbb{C}\backslash\{0\}$*. If* $f \in \mathcal{A}$ *of the form* (1) *satisfies*

$$\frac{1}{\gamma}\left(\frac{zf'(z)+z^2f''(z)}{(1-\lambda)z+\lambda zf'(z)}-1\right) \ll (h(z)-1) \quad (z \in \mathbb{U}), \tag{19}$$

then

$$|a_2| \le \frac{|\gamma|B_1}{2(2-\lambda)}$$

and for any $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \le \frac{|\gamma|B_1}{3(3-\lambda)} \max\left\{1, \left|\frac{B_2}{B_1} - QB_1\right|\right\},$$

where Q is given by (8)*. The results are sharp.*

Proof. Assume that (19) holds. From the definition of majorization, there exist an analytic function ϕ such that

$$\frac{1}{\gamma}\left(\frac{zf'(z)+z^2f''(z)}{(1-\lambda)z+\lambda zf'(z)}-1\right) = \phi(z)(h(z)-1) \quad (z \in \mathbb{U}).$$

Following similar steps as in the proof of Theorem 1, and by setting $w(z) \equiv z$, so that $w_1 = 1, w_n = 0, n \ge 2$, we obtain

$$a_2 = \frac{\gamma A_0 B_1}{2(2-\lambda)}$$

and also we obtain that

$$a_3 - \nu a_2^2 = \frac{\gamma B_1}{3(3-\lambda)}\left[A_1 + \frac{B_2}{B_1}A_0 - QB_1A_0^2\right].$$

On putting the value of A_1 from (15), we obtain

$$a_3 - \nu a_2^2 = \frac{\gamma B_1}{3(3-\lambda)}\left[y + \frac{B_2}{B_1}A_0 - (QB_1 + y)A_0^2\right]. \tag{20}$$

If A_0=0 in (20), we at once get

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{3(3-\lambda)}, \tag{21}$$

But if $A_0 \neq 0$, let us then suppose that

$$T(A_0) = y + \frac{B_2}{B_1} A_0 - \Big(QB_1 + y\Big) A_0^2,$$

which is a quadratic polynomial in A_0, hence analytic in $|A_0| \leq 1$ and maximum value of $|T(A_0)|$ is attained at $A_0 = e^{\iota\theta}$ $(0 \leq \theta < 2\pi)$, we find that

$$\max |T(A_0)| = \max_{0 \leq \theta < 2\pi} |T(e^{\iota\theta})| = |T(1)|.$$

Hence, from (20), we obtain

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{3(3-\lambda)} \left| QB_1 - \frac{B_2}{B_1} \right|.$$

Thus, the assertion of Theorem 2 follows from this last above inequality together with (21). The results are sharp for the function given by

$$1 + \frac{1}{\gamma} \left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1 \right) = h(z),$$

which completes the proof of Theorem 2. □

Theorem 3 *Let* $0 \leq \lambda \leq 1$ *and* $\gamma \in \mathbb{C} \backslash \{0\}$. *If* $f \in \mathcal{A}$ *of the form* (1) *belonging to the class* $\mathcal{K}(\lambda, \gamma, h)$, *then*

$$|a_2| \leq \frac{|\gamma| B_1}{2(2-\lambda)}$$

and for any $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{3(3-\lambda)} \max \left\{ 1, \left| \frac{B_2}{B_1} - QB_1 \right| \right\},$$

where Q *is given by* (8), *the results are sharp.*

Proof. The proof is similar to Theorem 1, Let $f \in \mathcal{K}(\lambda, \gamma, h)$.
If $\phi(z) = 1$, then $A_0 = 1, A_n = 0$ $(n \in \mathbb{N})$. Therefore, in view of (12) and (14) and by application of Lemma 1, we obtain the desired assertion. The results are sharp for the function f given by

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) = h(z),$$

or

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) = h(z^2).$$

Thus, the proof of Theorem 3 is completed. □

Now, we determine the bounds on the functional $|a_3 - \nu a_2^2|$ for real ν.

Theorem 4 *Let $0 \leq \lambda \leq 1$. If $f \in \mathcal{A}$ of the form (1) belonging to the class $\mathcal{K}_q(\lambda, \gamma, h)$, then for real ν and γ, we have*

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|\gamma| B_1}{3(3-\lambda)}\left[B_1\gamma\left(\frac{\lambda}{2-\lambda} - \frac{3(3-\lambda)}{4(2-\lambda)^2}\nu\right) + \frac{B_2}{B_1}\right] & (\nu \leq \sigma_1), \\ \frac{|\gamma| B_1}{3(3-\lambda)} & (\sigma_1 \leq \nu \leq \sigma_1 + 2\rho), \\ -\frac{|\gamma| B_1}{3(3-\lambda)}\left[B_1\gamma\left(\frac{\lambda}{2-\lambda} - \frac{3(3-\lambda)}{4(2-\lambda)^2}\nu\right) + \frac{B_2}{B_1}\right] & (\nu \geq \sigma_1 + 2\rho), \end{cases} \tag{22}$$

where

$$\sigma_1 = \frac{4\lambda(2-\lambda)}{3(3-\lambda)} - \frac{4(2-\lambda)^2}{3\gamma(3-\lambda)}\left(\frac{1}{B_1} - \frac{B_2}{B_1^2}\right) \tag{23}$$

and

$$\rho = \frac{4(2-\lambda)^2}{3\gamma(3-\lambda)B_1}. \tag{24}$$

Each of the estimates in (22) are sharp.

Proof. For real values of ν and γ the above bounds can be obtained from (7), respectively, under the following cases:

$$B_1 Q - \frac{B_2}{B_1} \leq -1, \; -1 \leq B_1 Q - \frac{B_2}{B_1} \leq 1 \text{ and } B_1 Q - \frac{B_2}{B_1} \geq 1,$$

where Q is given by (8). We also note the following:

(i) When $\nu < \sigma_1$ or $\nu > \sigma_1 + 2\rho$, then the equality holds if and only if $\phi(z) \equiv 1$ and $w(z) = z$ or one of its rotations.

(ii) When $\sigma_1 < \nu < \sigma_1 + 2\rho$, then the equality holds if and only if $\phi(z) \equiv 1$ and $w(z) = z^2$ or one of its rotations.

(iii) Equality holds for $\nu = \sigma_1$ if and only if $\phi(z) \equiv 1$ and $w(z) = \frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \leq \epsilon \leq 1)$, or one of its rotations, while for $\nu = \sigma_1 + 2\rho$, the equality holds if and only if $\phi(z) \equiv 1$ and $w(z) = -\frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \leq \epsilon \leq 1)$, or one of its rotations. □

The bounds of the functional $a_3 - \nu a_2^2$ for real values of ν and γ for the middle range of the parameter ν can be improved further as follows:

Theorem 5 *Let $0 \leq \lambda \leq 1$. If $f \in \mathcal{A}$ of the form (1) belonging to the class $\mathcal{K}_q(\lambda, \gamma, h)$, then for real ν and γ, we have*

$$|a_3 - \nu a_2^2| + (\nu - \sigma_1)|a_2|^2 \leq \frac{|\gamma|B_1}{3(3-\lambda)} \quad (\sigma_1 \leq \nu \leq \sigma_1 + \rho) \tag{25}$$

and

$$|a_3 - \nu a_2^2| + (\sigma_1 + 2\rho - \nu)|a_2|^2 \leq \frac{|\gamma|B_1}{3(3-\lambda)} \quad (\sigma_1 + \rho \leq \nu \leq \sigma_1 + 2\rho), \tag{26}$$

where σ_1 and ρ are given by (23) and (24), respectively.

Proof. Let $f \in \mathcal{K}_q(\lambda, \gamma, h)$. For real ν satisfying $\sigma_1 + \rho \leq \nu \leq \sigma_1 + 2\rho$ and using (12) and (18) we get

$$|a_3 - \nu a_2^2| + (\nu - \sigma_1)|a_2|^2$$
$$\leq \frac{|\gamma|B_1}{3(3-\lambda)}\left[|w_2| - \frac{3|\gamma|B_1(3-\lambda)}{4(2-\lambda)^2}(\nu - \sigma_1 - \rho)|w_1|^2 + \frac{3|\gamma|B_1(3-\lambda)}{4(2-\lambda)^2}(\nu - \sigma_1)|w_1|^2\right].$$

Therefore, by virtue of Lemma 1, we get

$$|a_3 - \nu a_2^2| + (\nu - \sigma_1)|a_2|^2 \leq \frac{|\gamma|B_1}{3(3-\lambda)}[1 - |w_1|^2 + |w_1|^2],$$

which yields the assertion (25).

If $\sigma_1 + \rho \leq \nu \leq \sigma_1 + 2\rho$, then again from (12), (18) and the application of Lemma 1, we have

$$\begin{aligned}|a_3 - \nu a_2^2| + (\sigma_1 + 2\rho - \nu)|a_2|^2 &\leq \frac{|\gamma|B_1}{3(3-\lambda)}\left[|w_2| + \frac{3|\gamma|B_1(3-\lambda)}{4(2-\lambda)^2}(\nu - \sigma_1 - \rho)|w_1|^2\right.\\ &\left.\quad + \frac{3|\gamma|B_1(3-\lambda)}{4(2-\lambda)^2}(\sigma_1 + 2\rho - \nu)|w_1|^2\right]\\ &\leq \frac{|\gamma|B_1}{3(3-\lambda)}[1 - |w_1|^2 + |w_1|^2],\end{aligned}$$

which estimates (26). □

Conflicts of interest

The authors declare that there are no conflict of interest regarding the publication of this paper.

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Received: June 7, 2018