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Sets with prescribed lower and upper weighted densities

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Abstract. It is known that we can prescribe the lower and upper asymptotic and logarithmic density of a set of positive integers. The only limitation is the inequality between asymptotic and logarithmic density. We generalize this result.

1 Introduction

Denote by $\mathbb{N}$ the set of all positive integers, let $A \subset \mathbb{N}$ and let $f : \mathbb{N} \to (0, \infty)$ be a weight function. For $A \subset \mathbb{N}$ and $n \in \mathbb{N}$ denote

$$S_f(A, n) = \sum_{\substack{a \le n \\ a \in A}} f(a), \qquad S_f(n) = \sum_{a \le n} f(a)$$

and define

$$\underline{d}_f(A) = \liminf_{n\to\infty} \frac{S_f(A, n)}{S_f(n)} \qquad \text{and} \qquad \overline{d}_f(A) = \limsup_{n\to\infty} \frac{S_f(A, n)}{S_f(n)}$$

the lower and upper f-densities of A, respectively. In the case when $\underline{d}_f(A) = \overline{d}_f(A)$ we say that A possesses f-density $d_f(A)$.

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Notice that the well-known asymptotic density corresponds to $f(n) = 1$ and the logarithmic density corresponds to $f(n) = \frac{1}{n}$. The concept of weighted densities was introduced in [7] and [1]. The continuity of densities given by the weight function n^{α}, $\alpha \geq -1$, was studied in [3]. Inequalities between upper and lower weighted densities for different weight functions were proved in [2].

The independence (within admissible bounds) of the asymptotic and logarithmic densities was proved in [6] and [5] showing that for any given real numbers $0 \leq \alpha \leq \beta \leq \gamma \leq \delta \leq 1$ there exists a set $A \subset \mathbb{N}$ such that

$$\underline{d}_1 = \alpha, \quad \underline{d}_{\frac{1}{n}}(A) = \beta, \quad \overline{d}_{\frac{1}{n}}(A) = \gamma, \quad \overline{d}_1(A) = \delta.$$

We generalize this result. We prove that under some assumptions on the weighted densities an analogous result holds. In [4], generalized asymptotic and logarithmic densities over an arithmetical semigroup were considered.

We call a weight function f *regular* if the corresponding weighted density fulfills the condition that for arbitrary positive integers a, b we have

$$d_f(a\mathbb{N} + b) = \frac{1}{a}$$

(f-density of the terms of arbitrary infinite arithmetical progression with the same difference are equal). Note that from this condition follows that

$$\sum_{n=1}^{\infty} f(n) = \infty, \qquad \lim_{n\to\infty} \frac{f(n)}{S_f(n)} = 0. \tag{1}$$

2 Results

The following lemma will be useful

Lemma 1 *Let* f, g *be regular weight functions. Let* B *be a subset of positive integers such that*

$$\underline{d}_f(B) = 0, \quad \overline{d}_f(B) = 1 \quad \textit{and} \quad d_g(B) = 0.$$

Then for any $0 \leq \alpha \leq \beta \leq \gamma \leq \delta \leq 1$, *rational numbers there exists a set* $E \subset \mathbb{N}$ *such that*

$$d_g(E) = \underline{d}_f(E) = \gamma \quad \textit{and} \quad \overline{d}_f(E) = \delta$$

and a set $H \subset \mathbb{N}$ *with the property*

$$d_g(H) = \overline{d}_f(H) = \beta \quad \textit{and} \quad \underline{d}_f(H) = \alpha.$$

Proof. Write γ and δ as fractions with a common denominator, let $\gamma = \frac{p}{t}$ and $\delta = \frac{q}{t}$. Define

$$E = \bigcup_{i=1}^{p} (t\mathbb{N} + i) \cup \Big(B \cap \big(\bigcup_{i=p+1}^{q} (t\mathbb{N} + i)\big)\Big).$$

As $d_g(B) = 0$, therefore

$$d_g(E) = d_g\big(\bigcup_{i=1}^{p} (t\mathbb{N} + i)\big) = \frac{p}{t} = \gamma.$$

Analogously we get $\underline{d}_f(E) = \frac{p}{t} = \gamma$.

Clearly $\overline{d}_f(E) \le \delta = \frac{q}{t}$. The case $\overline{d}_f(E) < \frac{q}{t}$ yields a contradiction because

$$\begin{aligned} 1 &= \overline{d}_f(B \cap \mathbb{N}) = \overline{d}_f\big(\bigcup_{i=1}^{q} (B \cap (t\mathbb{N} + i))\big) + \overline{d}_f\big(\bigcup_{i=q+1}^{t} (B \cap (t\mathbb{N} + i))\big) \le \\ &\le \overline{d}_f(E) + \overline{d}_f\big(\bigcup_{i=q+1}^{t} (t\mathbb{N} + i)\big) < \frac{q}{t} + \frac{t-q}{t} = 1. \end{aligned}$$

In analogous way we can prove the existence of the set H with the prescribed properties. For $\alpha = \frac{r}{t}$ and $\beta = \frac{s}{t}$ let

$$H = \bigcup_{i=1}^{s} (t\mathbb{N} + i) \setminus \Big(B \cap \big(\bigcup_{i=r+1}^{s} (t\mathbb{N} + i)\big)\Big).$$

Note, from the construction of the sets E, H follows $H \subset E$. ■

Theorem 1 *Let* f, g *be regular weight functions. Let* B *be a subset of positive integers such that*

$$\underline{d}_f(B) = 0, \quad \overline{d}_f(B) = 1 \quad \textit{and} \quad d_g(B) = 0.$$

Let $0 \le \alpha \le \beta \le \gamma \le \delta \le 1$ *be given real numbers. Then there exists a set* $A \subset \mathbb{N}$ *such that*

$$\underline{d}_f(A) = \alpha, \quad \underline{d}_g(A) = \beta, \quad \overline{d}_g(A) = \gamma, \quad \overline{d}_f(A) = \delta.$$

Proof. On the contrary, we suppose that there exist rational numbers $0 < \alpha < \beta < \gamma < \delta < 1$ and $\varepsilon > 0$ such that at least one of the following inequalities does not hold.

$$|\underline{d}_f(A) - \alpha| < \varepsilon, \quad |\underline{d}_g(A) - \beta| < \varepsilon, \quad |\overline{d}_g(A) - \gamma| < \varepsilon, \quad |\overline{d}_f(A) - \delta| < \varepsilon$$

Using the sets H, E defined in the previous lemma we construct a set A such that

$$H \subset A \subset E.$$

Then clearly

$$\underline{d}_f(A) \geq \underline{d}_f(H), \quad \underline{d}_g(A) \geq \underline{d}_g(H)$$

and

$$\overline{d}_f(A) \leq \overline{d}_f(E), \quad \overline{d}_g(A) \leq \overline{d}_g(E).$$

Define the set A by "intertwinning" the sets E and H

$$A = H \bigcup_{k=1}^{\infty} [n_{2k}, n_{2k+1}] \cap E,$$

where $n_1 = 1$ and for $k = 1, 2, \dots$ let n_k be sufficiently large, such that for some i, j between n_{k-1} and n_k the

$$\left|\frac{S_f(A,i)}{S_f(i)} - \alpha\right| < \varepsilon \quad \text{and} \quad \left|\frac{S_g(A,j)}{S_g(j)} - \beta\right| < \varepsilon \tag{2}$$

inequalities hold. Analogously, sufficiently large n_{2k+1} guarantees the inequalities

$$\left|\frac{S_f(A,m)}{S_f(m)} - \delta\right| < \varepsilon \quad \text{and} \quad \left|\frac{S_g(A,l)}{S_g(l)} - \gamma\right| < \varepsilon \tag{3}$$

for some m, l. From this we can deduce that (2) and (3) hold for infinitely many i, j, m, l what is a contradiction to our assumption. ■

Roughly speaking, the proved theorem says that under some conditions to prove the existence of a set A with prescribed upper and lower weighted densities it is sufficient to consider only one, the "worst" case.

Lemma 2 *If the function* $f : \mathbb{N} \to (0, \infty)$ *satisfies the conditions*

$$\sum_{n=1}^{\infty} f(n) = \infty, \tag{4}$$

$$\lim_{n\to\infty} \frac{f(n)}{S_f(n)} = 0, \tag{5}$$

then for the function g *defined as*

$$g(n) = \frac{f(n)}{\sum_{i=1}^{n} f(i)} \tag{6}$$

we have

$$\sum_{n=1}^{\infty} g(n) = \infty \tag{7}$$

and

$$\lim_{n\to\infty} \frac{g(n)}{S_g(n)} = 0. \tag{8}$$

Proof. We prove only (7), using the fact $\lim_{n\to\infty} g(n) = 0$ together with (7) it follows immediately (8).

For arbitrary positive integers $m > n$ we have

$$\sum_{k=n}^{m} g(k) = \sum_{k=n}^{m} \frac{f(k)}{\sum_{i=1}^{k} f(i)} \geq \frac{\sum_{k=n}^{m} f(k)}{\sum_{i=1}^{m} f(i)}.$$

The proof is completed by showing that for given n and sufficiently large m

$$\sum_{k=n}^{m} g(k) \geq \frac{1}{2}.$$

From (4) we see that $\sum\limits_{i=1}^{\infty} f(i) = \infty$, therefore for arbitrary $n \in \mathbb{N}$

$$\lim_{m\to\infty} \sum_{k=n}^{m} g(k) \geq \lim_{m\to\infty} \frac{\sum\limits_{k=n}^{m} f(k)}{\sum\limits_{i=1}^{m} f(i)} = 1$$

and the lemma follows. ■

Theorem 2 *Let the functions* $f, g : \mathbb{N} \to (0, \infty)$ *satisfy the assumptions (4)–(6). Then there exists a set* $B \subset \mathbb{N}$ *such that*

$$\underline{d}_f(B) = 0, \quad \overline{d}_f(B) = 1 \quad \textit{and} \quad d_g(B) = 0.$$

Proof. Consider

$$B = \bigcup_{k=1}^{\infty} [n_{2k}, n_{2k+1}].$$

Let $n_1 = 1$. Assume $n_1, n_2, \ldots, n_{2k-1}$ are given. We are looking for n_{2k} such that

$$\frac{f(n)}{f(1) + f(2) + \cdots + f(n)} < \frac{1}{k+1} \quad \text{for arbitrary } n \geq n_{2k}, \tag{9}$$

$$\frac{f(1)+f(2)+\cdots+f(n_{2k-1})}{f(1)+f(2)+\cdots+f(n_{2k})} < \frac{1}{k}, \tag{10}$$

$$\frac{g(1)+g(2)+\cdots+g(n_{2k-1})}{g(1)+g(2)+\cdots+g(n_{2k})} < \frac{1}{k}, \tag{11}$$

$$g(1)+g(2)+\cdots+g(n_{2k}) > k^2. \tag{12}$$

Moreover, let n_{2k+1} satisfy the inequalities

$$\frac{k-1}{k+1} < \frac{f(n_{2k})+f(n_{2k}+1)+\cdots+f(n_{2k+1})}{f(1)+f(2)+\cdots+f(n_{2k+1})} < \frac{k}{k+1}. \tag{13}$$

Inequalities (10)–(13) follow from the prescribed conditions on the functions f and g.

By (10) we have $\underline{d}_f(B) = 0$, by (11) we have $\underline{d}_g(B) = 0$ and taking into account (13) we get $\overline{d}_f(B) = 0$.

We proceed to show that $\overline{d}_g(B) = 0$. In virtue of [2], Lemma 2.1 it is sufficient to consider only the values $\frac{S_g(B,n_k)}{S_g(n_k)}$. We have

$$\frac{S_g(B,n_{2k+1})}{S_g(n_{2k+1})} \leq$$

$$\frac{g(1)+g(2)+\cdots+g(n_{2k-1})+g(n_{2k})+g(n_{2k}+1)+\cdots+g(n_{2k+1})}{g(1)+g(2)+\cdots+g(n_{2k+1})} <$$

$$\frac{1}{k} + \frac{g(n_{2k})+g(n_{2k}+1)+\cdots+g(n_{2k+1})}{g(1)+g(2)+\cdots+g(n_{2k})} \leq \frac{1}{k} + \frac{\frac{f(n_{2k})+f(n_{2k}+1)+\cdots+f(n_{2k+1})}{f(1)+f(2)+\cdots+f(n_{2k-1})}}{g(1)+g(2)+\cdots+g(n_{2k})}.$$

Using (13) we can show the inequality

$$\frac{f(n_{2k})+f(n_{2k}+1)+\cdots+f(n_{2k+1})}{f(1)+f(2)+\cdots+f(n_{2k-1})} < k.$$

Using this together with (12) we have

$$\frac{S_g(B,n_{2k+1})}{S_g(n_{2k+1})} < \frac{1}{k} + \frac{k}{k^2} = \frac{2}{k}$$

and hence $\overline{d}_g(B) = 0$ and $d_g(B) = 0$ follows. ■

It is not hard to show that if a monotone function f satisfies (4)–(5), then it is regular (see, e.g. [2], Example 2.1). If the function g defined by (6) is monotonely decreasing, then it is regular, too. We have

Corollary 1 *Let the monotone function* f *and monotone decreasing function* g *satisfy the assumptions (4)–(6). Let* $0 \le \alpha \le \beta \le \gamma \le \delta \le 1$ *be given real numbers. Then there exists a set* $A \subset \mathbb{N}$ *such that*

$$\underline{d}_f(A) = \alpha, \quad \underline{d}_g(A) = \beta, \quad \overline{d}_g(A) = \gamma, \quad \overline{d}_f(A) = \delta.$$

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