Pareto-optimal Nash equilibrium detection using an evolutionary approach

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Abstract. Pareto-optimal Nash equilibrium is a refinement of the Nash equilibrium. An evolutionary method is described in order to detect this equilibrium. Generative relations induces a non-domination concept which is essentially in the detection method. Numerical experiments with games having multiple Nash equilibria are presented. The evolutionary method detects correctly the Pareto-optimal Nash equilibria.

1 Introduction

Equilibrium detection in non-cooperative games is an essential task. Decision making processes can be analysed and predicted using equilibrium detection. The most known equilibrium concept is the Nash equilibrium [9]. Unfortunately this equilibrium has some limitations: if a game has multiple Nash equilibria, a selection problem can appear. Solution to this problem are the refinements of Nash equilibria such as: Aumann equilibrium [1], coalition proof Nash equilibrium [2], modified strong Nash equilibrium [6],[10] (detection of this equilibrium is described in [5]), etc.

The Pareto-optimal Nash equilibrium is one of the most important refinements of the Nash equilibrium that selects the NE that is Pareto non-dominated with respect to the other NE's of the game. The evolutionary method, presented in this paper is the first one, for the best of our knowledge.

In the next part of the paper some basic notions from game theory and a computationally method are described in order to detect the Pareto-optimal Nash equilibrium. The advantage of the proposed method is that the equilibrium can be obtained in one step, and there is no need for two different selection steps.

A finite strategic game is a system $G = ((N, S_i, u_i), i = 1, \ldots, n)$, where:

- N represents a set of players, and n is the number of players;
- for each player $i \in N$, S_i is the set of available actions,

$$S = S_1 \times S_2 \times \cdots \times S_n$$

  is the set of all possible situations of the game and $s \in S$ is a strategy (or strategy profile) of the game;
- for each player $i \in N$, $u_i : S \to R$ represents the payoff function (utility) of the player i.

One of the most important solving concept in non-cooperative game theory is the Nash equilibrium [9]. The Nash equilibrium (NE) of the game G means that no player can increase his/her payoff by unilateral deviation.

Let us denote by (s_i, s_{-i}^*) the strategy profile obtained from s^* by replacing the strategy of player i with s_i :

$$(s_i, s_{-i}^*) = (s_1^*, \ldots, s_i, \ldots, s_n^*).$$

Formally we have the next definition:

Definition 1 *A strategy profile $s^* \in S$ is a Nash equilibrium if the inequality*

$$u_i(s_i, s_{-i}^*) \leq u_i(s^*),$$

holds $\forall i = 1, \ldots, n, \forall s_i \in S_i, s_i \neq s_i^*$.

2 Pareto-optimal Nash equilibrium

To describe the Pareto-optimal Nash equilibrium first we introduce the notion of the Pareto optimality:

Definition 2 *A strategy profile* $s^* \in S$ *is Pareto efficient when it does not exist a strategy* $s \in S$, *such that*

$$u_i(s) \geq u_i(s^*), \, i \in N,$$

with at least one strict inequality.

Pareto-optimal Nash equilibrium [8] is a refinement of the Nash equilibrium. The Pareto-optimal Nash equilibrium is a Nash equilibrium for which there is no other state in which every player is better off.

Formally:

Definition 3 *Let* $s^* \in S$ *be a Nash equilibrium.* s^* *is a Pareto-optimal Nash equilibrium, there exists no* $s \in S$ *such that:*

$$u_i(s) \geq u_i(s^*), \forall i \in N.$$

Let us denote the Pareto-optimal Nash equilibrium by PNE.

Remark 4 *The Pareto-optimal Nash equilibrium is a subset of the Nash equilibrium:*

$$PNE \subseteq NE.$$

3 Generative relation for Pareto-optimal Nash equilibrium

Generative relations are used for equilibrium detection by inducing a domination concept. Two strategy profiles can be dominated, non-dominated, or indifferent with respect to a generative relation. An evolutionary algorithm with a generative relation will approximate a certain equilibrium type.

First generative relation has been introduced in [7] for detecting the Nash equilibrium.

To obtain the Pareto-optimal Nash equilibrium a new generative relation is introduced next.

Consider two strategy profiles s^* and s from S. Denote by $pn(s^*, s)$ the number of strategies, for which some players can benefit deviating.

We may express $pn(s^*, s)$ as:

$$
\begin{aligned}
pn(s^*, s) \quad = \quad &\mathrm{card}\{i \in N, \, u_i(s) > u_i(s^*), s \neq s^*\} \\
&+ \mathrm{card}\{i \in N, u_i(s_i, s^*_{-i}) > u_i(s^*)), s^*_i \neq s_i\},
\end{aligned}
$$

where $\mathrm{card}\{R\}$ denotes the cardinality of the set R.

Definition 5 *Let $s^*, s \in S$. We say the strategy s^* is better than the strategy s with respect to Pareto-optimal Nash equilibrium, and we write $s^* \prec_{PN} s$, if the inequality*

$$pn(s^*, s) < pn(s, s^*)$$

holds.

Definition 6 *The strategy profile $s^* \in S$ is a Pareto-optimal Nash non-dominated strategy, if there is no strategy $s \in S$, $s \neq s^*$ such that s dominates s^* with respect to $\prec_{PN}$ i.e.*

$$s \prec_{PN} s^*.$$

Denote by PNS the set of all non-dominated strategies with respect to the relation $\prec_{PN}$.

We may consider relation $\prec_{PN}$ as a candidate for generative relation of the Pareto-optimal Nash equilibrium. This means the set of the non-dominated strategies with respect to the relation $\prec_{PN}$ equals to the set of Pareto-optimal Nash equilibria.

It can be considered that the set of all Pareto-optimal Nash equilibrium strategies as representing the Pareto-optimal Nash equilibrium (PNE) of the game.

4　Evolutionary equilibria detection

Differential Evolution [11] is an evolutionary algorithm designed for continuous function optimization. It is a simple but very efficient algorithm, these two advantages have made DE one of the most popular optimization technique for real value single objective optimization. It has also been extended to multi-objective optimization. For the trial vector generation we use the strategy *rand/1/bin* proposed in [11]. This version of DE has only four parameters: n – population size, stopping criterion – number of generations, $F \in [0, 1]$ – mutation factor, and $Cr \in [0, 1]$ – crossover rate. High values for F assure the exploration of the search space while high Cr values assure the space exploitation. The DE procedure is given by Algorithm 1.
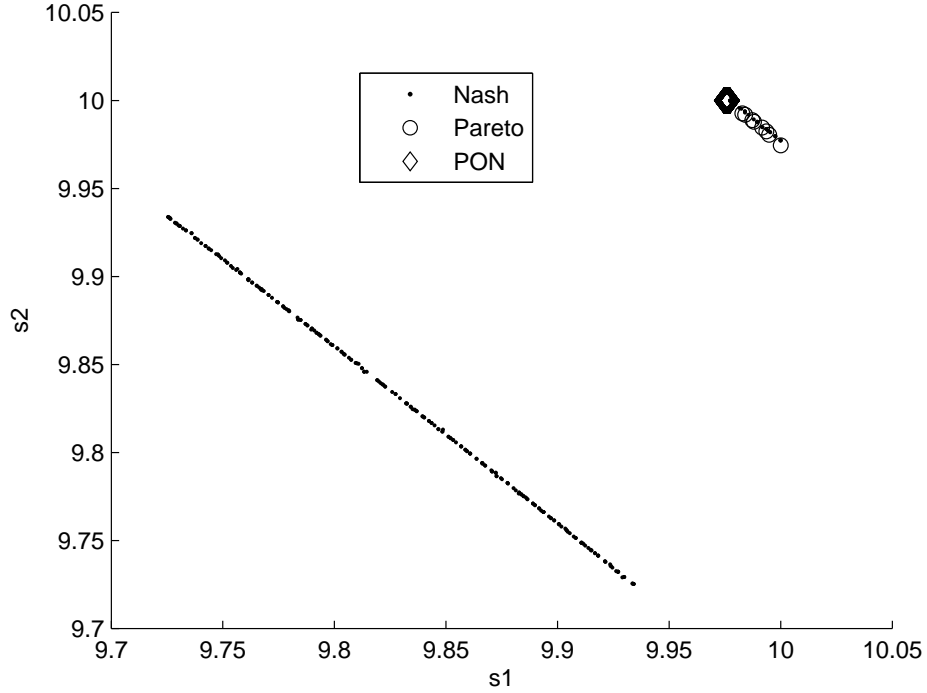
Figure 1: Detected Pareto front, Nash equilibrium, and Pareto-optimal Nash equilibrium strategies for Game 1

Algorithm 1: DE/rand/1/bin

1: Evaluate fitness
2: **for** $i = 0 \rightarrow max - iterations$ **do**
3:     Create difference offspring by mutation and recombination
4:     Evaluate fitness
5:     **if** the offspring is better than the parent **then**
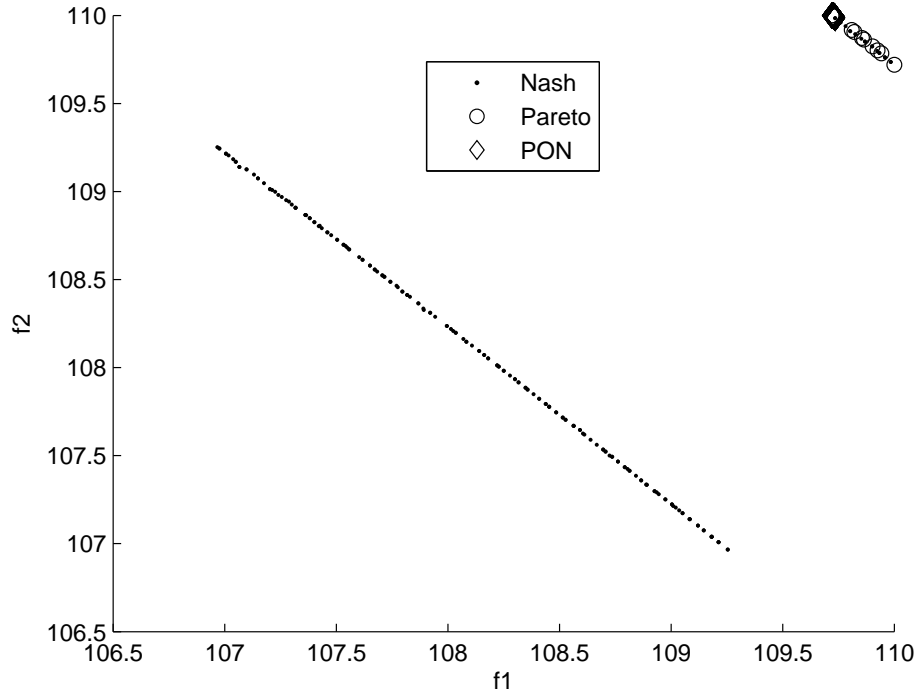6:   Replace the parent by offspring in the next generation
7:     **end if**
8: **end for**

Figure 2: Detected Pareto front, Nash equilibrium, and Pareto-optimal Nash equilibrium payoff for Game 1

5   Numerical experiments

Parameter settings used in numerical experiments are the following: population size $n = 100$, number of generation $= 1000$, $Cr = 0.7$, $F = 0.25$.

5.1   Game 1

Let us consider the two person game G_1 [4], having the following payoff functions:

$$u_1(s_1, s_2) = s_1(10 - \sin(s_1^2 + s_2^2)),$$

$$u_2(s_1, s_2) = s_2(10 - \sin(s_1^2 + s_2^2)),$$

$$s_1, s_2 \in [0, 10].$$
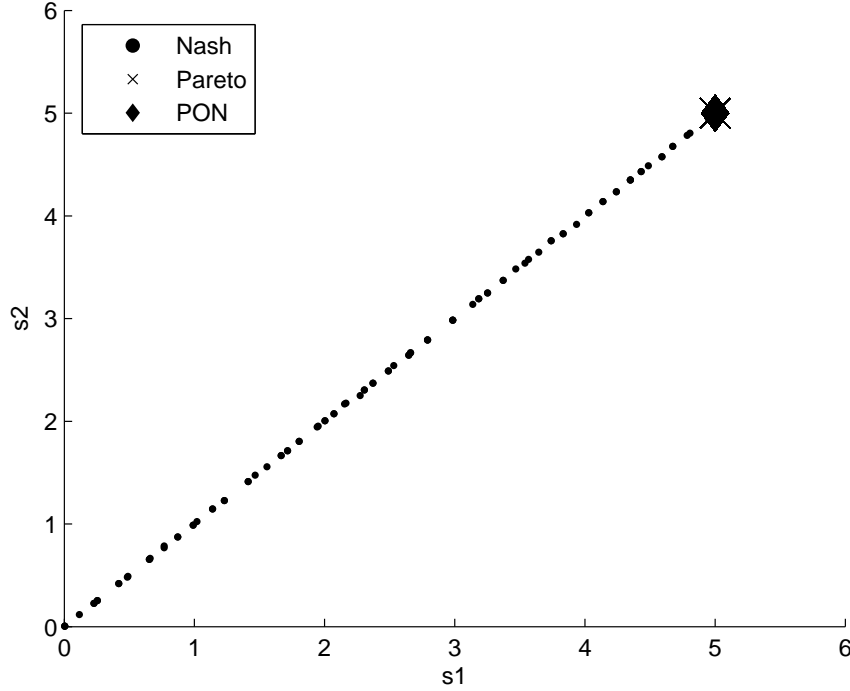
Figure 3: Detected Pareto front, Nash equilibrium, and Pareto-optimal Nash equilibrium strategies for the two-player version of Game 2

Obtained strategies are depicted in Figure 3, obtained payoffs are depicted in Figure 2. The Pareto-optimal Nash equilibrium reduces the set of Nash equilibria.

5.2 Game 2

Let us consider the Bryant game [3], where the payoff function is the following:

$$u_i(s) = c_i - s_i,$$

where $c_i = \alpha \min\{s_1, s_2, \ldots, s_n\}$, $\alpha = 2$, and $s_i \in [0, 5], i = 1, \ldots, n$.

Experiments with the two-person version of the game are presented in Figure 3 (strategies) and 4 (corresponding payoff). The set of the Nash equilibrium is the whole interval $s^* = (s_1^*, s_2^*) \in [0, 5]$, but the Pareto-optimal Nash equilibrium is only the point $(5, 5)$.
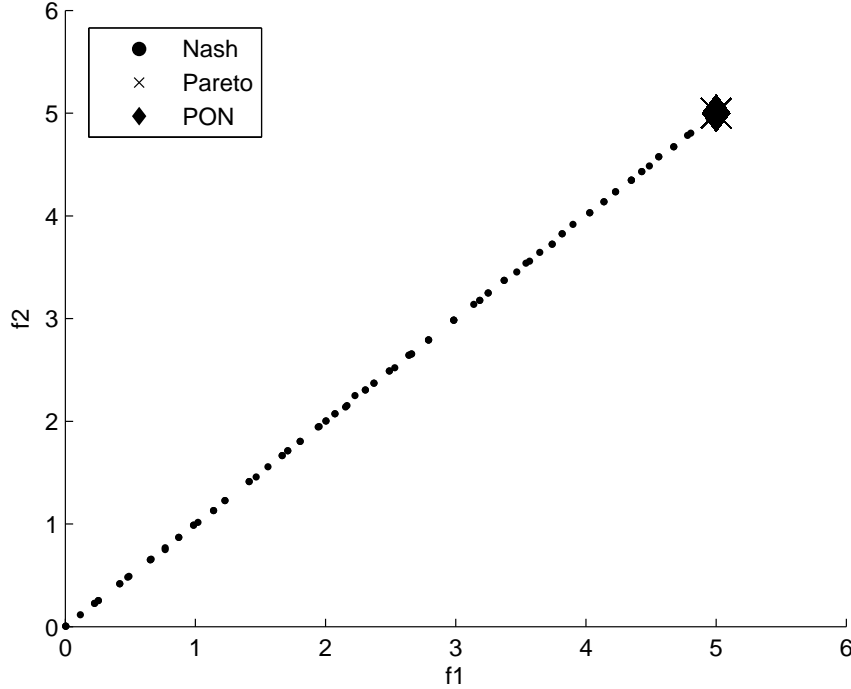
Figure 4: Detected Pareto front, Nash equilibrium, and Pareto-optimal Nash equilibrium payoff for the two-player version of Game 2

Figure 5 presents the same game with three players. The results are the same, the algorithm approximates well the Pareto-optimal Nash equilibrium. Numerical experiments were also conducted for 4, 5 players, and the algorithm finds correctly the Pareto-optimal Nash equilibrium of the game.

6   Conclusions

Nash equilibrium is not always the best solution in all non-cooperative games. In games having several Nash equilibria a selection problem can appear, which Nash equilibrium is best from the detected equilibria.

Pareto-optimal Nash equilibrium is a refinement of the Nash equilibrium. An evolutionary method based on generative relations is considered in order to detect this equilibrium. For the best of our knowledge this is the first evolutionary method for detecting the Pareto optimal Nash equilibrium.
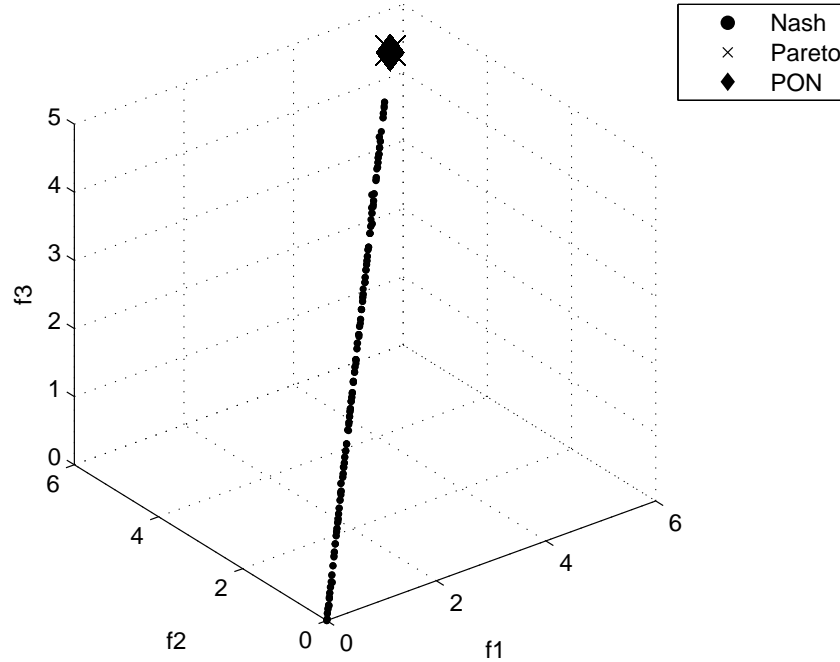
Figure 5: Detected Pareto front, Nash equilibrium, and Pareto-optimal Nash equilibrium payoff for the three-player version of Game 2

Numerical experiments show the potential of the proposed method. The method can deal with continuous and also discrete games. Further work will address games with multiple players.

Acknowledgements

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