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Para-Kenmotsu manifolds admitting semi-symmetric structures

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Abstract. The object of the present paper is to study para-Kenmotsu manifolds satisfying different conditions of semi-symmetric type.

1 Introduction

Recently, A. A. Shaikh and H. Kundu [8] studied the equivalency of various geometric structures. They have proved that the conditions

i) $R \cdot R = 0$, $R \cdot \tilde{C} = 0$ and $R \cdot P = 0$ are equivalent and we call such a class G_1;

ii) $C \cdot R = 0$, $C \cdot \tilde{C} = 0$ and $C \cdot P = 0$ are equivalent and we call such a class G_2;

iii) $\tilde{C} \cdot R = 0$, $\tilde{C} \cdot \tilde{C} = 0$ and $\tilde{C} \cdot P = 0$ are equivalent and we call such a class G_3;

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iv) $H \cdot R = 0$, $H \cdot \tilde{C} = 0$ and $H \cdot P = 0$ are equivalent and we call such a class G_4;
v) $R \cdot C = 0$ and $R \cdot H = 0$ are equivalent and we call such a class G_5;
vi) $C \cdot H = 0$ and $C \cdot C = 0$ are equivalent and we call such a class G_6;
vii) $\tilde{C} \cdot H = 0$ and $\tilde{C} \cdot C = 0$ are equivalent and we call such a class G_7;
viii) $H \cdot H = 0$ and $H \cdot C = 0$ are equivalent and we call such a class G_8,
where R, C, $\tilde{C}$, H and P are the Riemannian, conformal, concircular, conharmonic and projective curvature tensors, respectively.

In an n-dimensional Riemannian manifold (M^n, g) $(n > 3)$, the conformal curvature tensor C [2], conharmonic curvature tensor H [3], concircular curvature tensor $\tilde{C}$ [13] and projective curvature tensor P [7] are defined respectively by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1}$$

$$\begin{aligned} H(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{2}$$

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{3}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{4}$$

where Q, S and r are the Ricci operator, the Ricci curvature tensor and the scalar curvature of M^n. The Ricci operator Q and the $(0,2)$-tensor S^2 are defined as

$$S(X,Y) = g(QX,Y) \text{ and } S^2(X,Y) = S(QX,Y) = g(Q^2X,Y).$$

The present paper is structured as follows. In Section 2, we briefly recall some known results for para-Kenmotsu manifolds. In Section 3, we study para-Kenmotsu manifolds belonging to the class G_i $(i = 1, 2, ..., 8)$ and we prove that a para-Kenmotsu manifold belonging to the class G_1 is Einstein, whereas such a manifold belonging to the class G_5 is η-Einstein.

2 Para-Kenmotsu manifolds

The notion of almost paracontact structure was introduced by I. Sato. According to his definition [9], an *almost paracontact structure* (Φ, ξ, η) on an odd-dimensional manifold M^n consists of a $(1,1)$-tensor field Φ, called the structure endomorphism, a vector field ξ, called the characteristic vector field and a 1-form η, called the contact form, which satisfy the following conditions:

$$\Phi^2 = I - \eta \otimes \xi, \tag{5}$$

$$\eta(\xi) = 1, \tag{6}$$

$$\Phi\xi = 0, \ \eta \circ \Phi = 0, \ \text{rank } \Phi = n - 1. \tag{7}$$

Moreover, if g is a pseudo-Riemannian metric satisfying

$$g(\Phi X, \Phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{8}$$

for any vector fields X and Y on M^n, then the manifold M^n [9] is said to admit an *almost paracontact Riemannian structure* (Φ, ξ, η, g). Remark that from the above conditions we get

$$g(X, \xi) = \eta(X), \tag{9}$$

for any vector field X on M^n. Examples of almost paracontact metric structures are given in [4, 1].

An analogue of the Kenmotsu manifold [5] in paracontact geometry will be further considered.

Definition 1 [6] *The almost paracontact metric structure* (Φ, ξ, η, g) *is called para-Kenmotsu if the Levi-Civita connection* ∇ *of* g *satisfies*

$$(\nabla_X \Phi)Y = g(\Phi X, Y)\xi - \eta(Y)\Phi X,$$

for any vector fields X *and* Y *on* M^n*.*

The para-Kenmotsu structure was also considered by J. Welyczko in [12] for 3-dimensional normal almost paracontact metric structures. A similar notion called P-Kenmotsu structure appears in the paper of B. B. Sinha and K. L. Sai Prasad [11]. We shall further give some immediate properties of this structure.

Proposition 1 *If* $(M^n, \Phi, \xi, \eta, g)$ *is a para-Kenmotsu manifold, then* [11]*:*

$$S(X, \xi) = -(n-1)\eta(X), \tag{10}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{11}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{12}$$

where S *is the Ricci curvature tensor and* R *is the Riemannian curvature tensor.*

In view of (12), one can easily bring out the followings:

$$\begin{aligned} g(C(X, Y)Z, \xi) &= \eta(C(X, Y)Z)) \\ &= \frac{1}{n-2}\Big[\Big(\frac{r}{n-1} + 1)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\Big) \\ &- \Big(S(Y, Z)\eta(X) - S(X, Z)\eta(Y)\Big)\Big], \end{aligned} \tag{13}$$

$$\begin{aligned} g(H(X, Y)Z, \xi) &= \eta(H(X, Y)Z)) \\ &= \frac{1}{n-2}[g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \\ &- (S(Y, Z)\eta(X) - S(X, Z)\eta(Y))], \end{aligned} \tag{14}$$

$$\begin{aligned} g(\tilde{C}(X, Y)Z, \xi) &= \eta(\tilde{C}(X, Y)Z)) \\ &= \left(\frac{r}{n(n-1)} + 1\right)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \end{aligned} \tag{15}$$

$$\begin{aligned} g(P(X, Y)Z, \xi) &= \eta(P(X, Y)Z) \\ &= g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \\ &- \frac{1}{n-1}[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)]. \end{aligned} \tag{16}$$

Definition 2 [10] *An almost paracontact Riemannian manifold* M^n *is said to be an* η*-Einstein manifold if the Ricci curvature tensor* S *is of the form*

$$S = ag + b\eta \otimes \eta,$$

where a *and* b *are smooth functions on* M^n *and* η *is a* 1*-form.*

In particular, if $b = 0$, then M^n is said to be an *Einstein manifold.*

3 Main results

In this section we consider different types of semi-symmetric para-Kenmotsu manifolds, namely, para-Kenmotsu manifolds belonging to the classes G_i ($i = 1, 2, ..., 8$).

3.1 Para-Kenmotsu manifolds belonging to the class G_1

We consider para-Kenmotsu manifolds admitting the condition

$$(R(X,Y) \cdot R)(Z,U)V = 0,$$

which implies

$$g(R(\xi,Y)R(Z,U)V,\xi) - g(R(R(\xi,Y)Z,U)V,\xi) \\ -g(R(Z,R(\xi,Y)U)V,\xi) - g(R(Z,U)R(\xi,Y)V,\xi) = 0. \tag{17}$$

Putting $Y = Z = e_i$ in (17), where $\{e_1, e_2, e_3, ..., e_{n-1}, e_n = \xi\}$ is an orthonormal basis of the tangent space at each point of the manifold M^n and taking the summation over i, $1 \le i \le n$, we get

$$\sum_{i=1}^{n} g(R(\xi,e_i)R(e_i,U)V,\xi) - \sum_{i=1}^{n} g(R(R(\xi,e_i)e_i,U)V,\xi) \\ -\sum_{i=1}^{n} g(R(e_i,R(\xi,e_i)U)V,\xi) - \sum_{i=1}^{n} g(R(e_i,U)R(\xi,e_i)V,\xi) = 0 \tag{18}$$

Using (10)-(12) we obtain

$$\sum_{i=1}^{n} g(R(\xi,e_i)R(e_i,U)V,\xi) = -g(U,V) + \eta(U)\eta(V) - S(U,V), \tag{19}$$

$$\sum_{i=1}^{n} g(R(R(\xi,e_i)e_i,U)V,\xi) = -(n-1)[-g(U,V) + \eta(U)\eta(V)], \tag{20}$$

$$\sum_{i=1}^{n} g(R(e_i,R(\xi,e_i)U)V,\xi) = -g(U,V) + \eta(U)\eta(V), \tag{21}$$

$$\sum_{i=1}^{n} g(R(e_i,U)R(\xi,e_i)V,\xi) = (n-1)\eta(U)\eta(V). \tag{22}$$

By virtue of (19), (20), (21) and (22), the equation (18) yields

$$S(U,V) = -(n-1)g(U,V). \tag{23}$$

Thus, we state the following theorem.

Theorem 1 *A para-Kenmotsu manifold belonging to the class* G_1 *is always an Einstein manifold with the Ricci curvature tensor given by* (23).

3.2 Para-Kenmotsu manifolds belonging to the class G_2

We consider para-Kenmotsu manifolds admitting the condition

$$(C(X,Y)\cdot R)(Z,U)V = 0,$$

which implies

$$\begin{aligned} &g(C(\xi,Y)R(Z,U)V,\xi) - g(R(C(\xi,Y)Z,U)V,\xi) \\ &-g(R(Z,C(\xi,Y)U)V,\xi) - g(R(Z,U)C(\xi,Y)V,\xi) = 0. \end{aligned} \tag{24}$$

Putting $Y = Z = e_i$ in (24) and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} g(C(\xi,e_i)R(e_i,U)V,\xi) - \sum_{i=1}^{n} g(R(C(\xi,e_i)e_i,U)V,\xi) \\ &-\sum_{i=1}^{n} g(R(e_i,C(\xi,e_i)U)V,\xi) - \sum_{i=1}^{n} g(R(e_i,U)C(\xi,e_i)V,\xi) = 0. \end{aligned} \tag{25}$$

Using (10)-(12) and (1) we obtain

$$\begin{aligned} \sum_{i=1}^{n} g(C(\xi,e_i)R(e_i,U)V,\xi) &= \frac{1}{n-2}\Big[\Big(\frac{r}{n-1}+1\Big)S(U,V) - S^2(U,V) \\ &- \Big(\frac{r}{n-1}+n\Big)\eta(R(\xi,U)V)\Big], \end{aligned} \tag{26}$$

$$\sum_{i=1}^{n} g(R(C(\xi,e_i)e_i,U)V,\xi) = 0, \tag{27}$$

$$\begin{aligned} \sum_{i=1}^{n} g(R(e_i,C(\xi,e_i)U)V,\xi) &= \frac{1}{n-2}\Big[-\Big(\frac{r}{n-1}+1\Big)\eta(R(\xi,U)V) \\ &- S(U,V) - (n-1)\eta(U)\eta(V)\Big], \end{aligned} \tag{28}$$

$$\sum_{i=1}^{n} g(R(e_i, U)C(\xi, e_i)V, \xi) = 0. \tag{29}$$

By virtue of (26), (27), (28), (29) and using (12), the equation (25) yields

$$\left(\frac{r}{n-1} + 2\right) S(U, V) = -(n-1)g(U, V) + S^2(U, V). \tag{30}$$

Thus, we state the following theorem.

Theorem 2 *The Ricci curvature tensor of a para-Kenmotsu manifold belonging to the class* G_2 *satisfies* (30).

3.3 Para-Kenmotsu manifolds belonging to the class G_3

We consider para-Kenmotsu manifolds admitting the condition

$$(\tilde{C}(X, Y) \cdot R)(Z, U)V = 0,$$

which implies

$$\begin{aligned} & g(\tilde{C}(\xi, Y)R(Z, U)V, \xi) - g(R(\tilde{C}(\xi, Y)Z, U)V, \xi) \\ & -g(R(Z, \tilde{C}(\xi, Y)U)V, \xi) - g(R(Z, U)\tilde{C}(\xi, Y)V, \xi) = 0. \end{aligned} \tag{31}$$

Putting $Y = Z = e_i$ in (31) and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} & \sum_{i=1}^{n} g(\tilde{C}(\xi, e_i)R(e_i, U)V, \xi) - \sum_{i=1}^{n} g(R(\tilde{C}(\xi, e_i)e_i, U)V, \xi) \\ & - \sum_{i=1}^{n} g(R(e_i, \tilde{C}(\xi, e_i)U)V, \xi) - \sum_{i=1}^{n} g(R(e_i, U)\tilde{C}(\xi, e_i)V, \xi) = 0. \end{aligned} \tag{32}$$

Using (10)-(12) and (3) we obtain

$$\sum_{i=1}^{n} g(\tilde{C}(\xi, e_i)R(e_i, U)V, \xi) = \left(\frac{r}{n(n-1)} + 1\right)[-g(U, V) + \eta(U)\eta(V) - S(U, V)], \tag{33}$$

$$\sum_{i=1}^{n} g(R(\tilde{C}(\xi, e_i)e_i, U)V, \xi) = \left(\frac{r}{n(n-1)} + 1\right)(n-1)[g(U, V) - \eta(U)\eta(V)], \tag{34}$$

$$\sum_{i=1}^{n} g(R(e_i, \tilde{C}(\xi, e_i)U)V, \xi) = \left(\frac{r}{n(n-1)} + 1\right)[-g(U, V) + \eta(U)\eta(V)], \quad (35)$$

$$\sum_{i=1}^{n} g(R(e_i, U)\tilde{C}(\xi, e_i)V, \xi) = \left(\frac{r}{n(n-1)} + 1\right)(n-1)\eta(U)\eta(V). \quad (36)$$

By virtue of (33), (34), (35) and (36), the equation (32) yields

$$\left(\frac{r}{n(n-1)} + 1\right)[S(U, V) + (n-1)g(U, V)] = 0. \quad (37)$$

Thus, we state the following theorem.

Theorem 3 *The Ricci curvature tensor of a para-Kenmotsu manifold belonging to the class* G_3 *satisfies* (37).

3.4 Para-Kenmotsu manifolds belonging to the class G_4

We consider para-Kenmotsu manifolds admitting the condition

$$(H(X, Y) \cdot R)(Z, U)V = 0,$$

which implies

$$\begin{aligned} &g(H(\xi, Y)R(Z, U)V, \xi) - g(R(H(\xi, Y)Z, U)V, \xi) \\ &-g(R(Z, H(\xi, Y)U)V, \xi) - g(R(Z, U)H(\xi, Y)V, \xi) = 0. \end{aligned} \quad (38)$$

Putting $Y = Z = e_i$ in (38) and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} g(H(\xi, e_i)R(e_i, U)V, \xi) - \sum_{i=1}^{n} g(R(H(\xi, e_i)e_i, U)V, \xi) \\ &-\sum_{i=1}^{n} g(R(e_i, H(\xi, e_i)U)V, \xi) - \sum_{i=1}^{n} g(R(e_i, U)H(\xi, e_i)V, \xi) = 0. \end{aligned} \quad (39)$$

Using (10)-(12) and (2) we obtain

$$\begin{aligned} \sum_{i=1}^{n} g(H(\xi, e_i)R(e_i, U)V, \xi) &= \frac{1}{n-2}[ng(U, V) - n\eta(U)\eta(V) \\ &+ S(U, V) - S^2(U, V)], \end{aligned} \quad (40)$$

$$\sum_{i=1}^{n} g(R(H(\xi, e_i)e_i, U)V, \xi) = \frac{1}{n-2}[rg(U,V) - r\eta(U)\eta(V)], \tag{41}$$

$$\sum_{i=1}^{n} g(R(e_i, H(\xi, e_i)U)V, \xi) = \frac{1}{n-2}[g(U,V) - n\eta(U)\eta(V) - S(U,V)], \tag{42}$$

$$\sum_{i=1}^{n} g(R(e_i, U)H(\xi, e_i)V, \xi) = \frac{1}{n-2} r\eta(U)\eta(V). \tag{43}$$

By virtue of (40), (41), (42) and (43), the equation (39) yields

$$S(U,V) = -\frac{n-r-1}{2} g(U,V) + \frac{1}{2} S^2(U,V). \tag{44}$$

Thus, we state the following theorem.

Theorem 4 *The Ricci curvature tensor of a para-Kenmotsu manifold belonging to the class* G_4 *satisfies* (44).

3.5 Para-Kenmotsu manifolds belonging to the class G_5

We consider para-Kenmotsu manifolds admitting the condition

$$(R(X,Y) \cdot C)(Z,U)V = 0,$$

which implies

$$\begin{aligned} &g(R(\xi, Y)C(Z,U)V, \xi) - g(C(R(\xi, Y)Z, U)V, \xi) \\ &-g(C(Z, R(\xi, Y)U)V, \xi) - g(C(Z,U)R(\xi, Y)V, \xi) = 0. \end{aligned} \tag{45}$$

Putting $Y = Z = e_i$ in (45) and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} g(R(\xi, e_i)C(e_i, U)V, \xi) - \sum_{i=1}^{n} g(C(R(\xi, e_i)e_i, U)V, \xi) \\ &- \sum_{i=1}^{n} g(C(e_i, R(\xi, e_i)U)V, \xi) - \sum_{i=1}^{n} g(C(e_i, U)R(\xi, e_i)V, \xi) = 0. \end{aligned} \tag{46}$$

Using (10)-(12) and (1) we obtain

$$\sum_{i=1}^{n} g(R(\xi, e_i)C(e_i, U)V, \xi) = \eta(C(\xi, U)V), \tag{47}$$

$$\sum_{i=1}^{n} g(C(R(\xi, e_i)e_i, U)V, \xi) = -(n-1)\eta(C(\xi, U)V), \tag{48}$$

$$\sum_{i=1}^{n} g(C(e_i, R(\xi, e_i)U)V, \xi) = \eta(C(\xi, U)V), \tag{49}$$

$$\sum_{i=1}^{n} g(C(e_i, U)R(\xi, e_i)V, \xi) = 0. \tag{50}$$

By virtue of (47), (48), (49), (50) and using (13), the equation (46) yields

$$S(U, V) = \left(\frac{r}{n-1} + 1\right) g(U, V) - \left(\frac{r}{n-1} + n\right) \eta(U)\eta(V). \tag{51}$$

Thus, we state the following theorem.

Theorem 5 *A para-Kenmotsu manifold belonging to the class* G_5 *is always an* η*-Einstein manifold with the Ricci curvature tensor given by* (51).

3.6 Para-Kenmotsu manifolds belonging to the class G_6

We consider para-Kenmotsu manifolds admitting the condition

$$(C(X, Y) \cdot H)(Z, U)V = 0,$$

which implies

$$\begin{aligned} &g(C(\xi, Y)H(Z, U)V, \xi) - g(H(C(\xi, Y)Z, U)V, \xi) \\ &-g(H(Z, C(\xi, Y)U)V, \xi) - g(H(Z, U)C(\xi, Y)V, \xi) = 0. \end{aligned} \tag{52}$$

Putting $Y = Z = e_i$ in (52) and taking the summation over i, $1 \le i \le n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} g(C(\xi, e_i)H(e_i, U)V, \xi) - \sum_{i=1}^{n} g(H(C(\xi, e_i)e_i, U)V, \xi) \\ &- \sum_{i=1}^{n} g(H(e_i, C(\xi, e_i)U)V, \xi) - \sum_{i=1}^{n} g(H(e_i, U)C(\xi, e_i)V, \xi) = 0. \end{aligned} \tag{53}$$

Using (10)-(12), (1) and (2) we obtain

$$\sum_{i=1}^{n} g(C(\xi, e_i)H(e_i, U)V, \xi) \tag{54}$$

$$= \frac{1}{(n-2)^2}\Big[-r\Big(\frac{r}{n-1}+1\Big)g(U,V) - nS^2(U,V) + rS(U,V)$$
$$+ \parallel Q \parallel^2 g(U,V) - (n-2)\Big(\frac{r}{n-1}+n\Big)\eta(H(\xi,U)V)\Big],$$

$$\sum_{i=1}^{n} g(H(C(\xi,e_i)e_i,U)V,\xi) = 0, \tag{55}$$

$$\sum_{i=1}^{n} g(H(e_i,C(\xi,e_i)U)V,\xi) \tag{56}$$
$$= \frac{1}{(n-2)^2}\Big[-(n-2)\Big(\frac{r}{n-1}+1\Big)\eta(H(\xi,U)V) - S^2(U,V)$$
$$+ S(U,V) + n(n-1)\eta(U)\eta(V)\Big],$$

$$\sum_{i=1}^{n} g(H(e_i,U)C(\xi,e_i)V,\xi) \tag{57}$$
$$= -\frac{1}{(n-2)^2}\Big[\frac{r^2}{n-1} + 2r - \parallel Q \parallel^2 + n(n-1)\Big]\eta(U)\eta(V).$$

By virtue of (54), (55), (56), (57) and using (14), the equation (53) yields

$$(n+r-2)S(U,V) \tag{58}$$
$$= \Big[r\Big(\frac{r}{n-1}+1\Big) + n - 1 - \parallel Q \parallel^2\Big]g(U,V)$$
$$+ \Big[\parallel Q \parallel^2 - r\Big(\frac{r}{n-1}+2\Big) - n(n-1)\Big]\eta(U)\eta(V) + (n-1)S^2(U,V).$$

Thus, we state the following theorem.

Theorem 6 *The Ricci curvature tensor of a para-Kenmotsu manifold belonging to the class* G_6 *satisfies* (58).

3.7 Para-Kenmotsu manifolds belonging to the class G_7

We consider para-Kenmotsu manifolds admitting the condition

$$(\tilde{C}(X,Y)\cdot H)(Z,U)V = 0,$$

which implies

$$g(\tilde{C}(\xi, Y)H(Z, U)V, \xi) - g(H(\tilde{C}(\xi, Y)Z, U)V, \xi) \\ -g(H(Z, \tilde{C}(\xi, Y)U)V, \xi) - g(H(Z, U)\tilde{C}(\xi, Y)V, \xi) = 0. \tag{59}$$

Putting $Y = Z = e_i$ in (59) and taking the summation over i, $1 \leq i \leq n$, we get

$$\sum_{i=1}^{n} g(\tilde{C}(\xi, e_i)H(e_i, U)V, \xi) - \sum_{i=1}^{n} g(H(\tilde{C}(\xi, e_i)e_i, U)V, \xi) \\ -\sum_{i=1}^{n} g(H(e_i, \tilde{C}(\xi, e_i)U)V, \xi) - \sum_{i=1}^{n} g(H(e_i, U)\tilde{C}(\xi, e_i)V, \xi) = 0. \tag{60}$$

Using (10)-(12), (2) and (3) we obtain

$$\sum_{i=1}^{n} g(\tilde{C}(\xi, e_i)H(e_i, U)V, \xi) = \left(\frac{r}{n(n-1)} + 1\right)\left[\eta(H(\xi, U)V) + \frac{r}{n-2}g(U, V)\right], \tag{61}$$

$$\sum_{i=1}^{n} g(H(\tilde{C}(\xi, e_i)e_i, U)V, \xi) = -\left(\frac{r}{n(n-1)} + 1\right)(n-1)\eta(H(\xi, U)V), \tag{62}$$

$$\sum_{i=1}^{n} g(H(e_i, \tilde{C}(\xi, e_i)U)V, \xi) = \left(\frac{r}{n(n-1)} + 1\right)\eta(H(\xi, U)V), \tag{63}$$

$$\sum_{i=1}^{n} g(H(e_i, U)\tilde{C}(\xi, e_i)V, \xi) = \left(\frac{r}{n(n-1)} + 1\right)\frac{r}{n-2}\eta(U)\eta(V). \tag{64}$$

By virtue of (61), (62), (63), (64) and using (14), the equation (60) yields

$$\left(\frac{r}{n(n-1)} + 1\right)[S(U, V) - \left(\frac{r}{n-1} + 1\right)g(U, V) - \left(\frac{r}{n-1} - n\right)\eta(U)\eta(V)] = 0. \tag{65}$$

Thus, we state the following theorem.

Theorem 7 *The Ricci curvature tensor of a para-Kenmotsu manifold belonging to the class* G_7 *satisfies* (65).

3.8 Para-Kenmotsu manifolds belonging to the class G_8

We consider para-Kenmotsu manifolds admitting the condition

$$(H(X,Y)\cdot H)(Z,U)V = 0,$$

which implies

$$\begin{aligned} & g(H(\xi,Y)H(Z,U)V,\xi) - g(H(H(\xi,Y)Z,U)V,\xi) \\ & -g(H(Z,H(\xi,Y)U)V,\xi) - g(H(Z,U)H(\xi,Y)V,\xi) = 0. \end{aligned} \tag{66}$$

Putting $Y = Z = e_i$ in (66) and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} & \sum_{i=1}^{n} g(H(\xi,e_i)H(e_i,U)V,\xi) - \sum_{i=1}^{n} g(H(H(\xi,e_i)e_i,U)V,\xi) \\ & -\sum_{i=1}^{n} g(H(e_i,H(\xi,e_i)U)V,\xi) - \sum_{i=1}^{n} g(H(e_i,U)H(\xi,e_i)V,\xi) = 0. \end{aligned} \tag{67}$$

Using (10)-(12) and (2) we obtain

$$\begin{aligned} & \sum_{i=1}^{n} g(H(\xi,e_i)H(e_i,U)V,\xi) \\ = & \frac{1}{(n-2)^2}[-n(n-2)\eta(H(\xi,U)V) - nS^2(U,V) + rS(U,V) \\ + & \parallel Q \parallel^2 g(U,V) - rg(U,V)], \end{aligned} \tag{68}$$

$$\sum_{i=1}^{n} g(H(H(\xi,e_i)e_i,U)V,\xi) = -\frac{r}{n-2}\eta(H(\xi,U)V), \tag{69}$$

$$\begin{aligned} & \sum_{i=1}^{n} g(H(e_i,H(\xi,e_i)U)V,\xi) \\ = & \frac{1}{(n-2)^2}[-(n-2)\eta(H(\xi,U)V) - S^2(U,V) \\ & +S(U,V) + n(n-1)\eta(U)\eta(V)], \end{aligned} \tag{70}$$

$$\sum_{i=1}^{n} g(H(e_i,U)H(\xi,e_i)V,\xi) = -\frac{1}{(n-2)^2}[2r - \parallel Q \parallel^2 + n(n-1)]\eta(U)\eta(V). \tag{71}$$

By virtue of (68), (69), (70), (71) and (14), the equation (67) yields

$$\begin{aligned}(n-2)S(U,V) &= \left[(n-1)-\parallel Q\parallel^2\right]g(U,V) \\ -\left[n(n-1)-(n-2)r\right. &- \left.\parallel Q\parallel^2\right]\eta(U)\eta(V)+(n-1)S^2(U,V).\end{aligned} \quad (72)$$

Thus, we state the following theorem.

Theorem 8 *The Ricci curvature tensor of a para-Kenmotsu manifold belonging to the class* G_8 *satisfies* (72).

We can conclude the followings.

Remark 1 *Let* M^n *be a para-Kenmotsu manifold of dimension* $n > 3$.

i) *If* M^n *belongs to the class* G_1, *then* M^n *is an Einstein manifold of constant negative scalar curvature* $r = -n(n-1)$.

ii) *If* M^n *belongs to the class* G_2, *then the Ricci operator satisfies*

$$\parallel Q\parallel^2 \geq (n-1)^2.$$

iii) *If* M^n *belongs to the class* G_3, *then* M^n *is of constant negative scalar curvature* $r = -n(n-1)$.

iv) *If* M^n *belongs to the class* G_4, *then the scalar curvature satisfies*

$$r = \frac{n(n-1)}{n-2} - \frac{1}{n-2}\parallel Q\parallel^2 \leq \frac{n(n-1)}{n-2}.$$

v) *If* M^n *belongs to the class* G_5, *then* M^n *is an* η*-Einstein manifold.*

vi) *If* M^n *belongs to the class* G_7, *then* M^n *is of vanishing or constant negative scalar curvature* $r = -n(n-1)$.

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