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Oscillation of fast growing solutions of linear differential equations in the unit disc

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Abstract. In this paper, we investigate the relationship between solutions and their derivatives of the differential equation $f^{(k)} + A(z) f = 0$, $k \geq 2$, where $A(z) \not\equiv 0$ is an analytic function with finite iterated p-order and analytic functions of finite iterated p-order in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Instead of looking at the zeros of $f^{(j)}(z) - z$ $(j = 0, .., k)$, we proceed to a slight generalization by considering zeros of $f^{(j)}(z) - \varphi(z)$ $(j = 0, .., k)$, where φ is a small analytic function relative to f such that $\varphi^{(k-j)}(z) \not\equiv 0$ $(j = 0, ..., k)$, while the solution f is of infinite iterated p-order. This paper improves some very recent results of T. B. Cao and G. Zhang, A. Chen.

1 Introduction and statement of results

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's theory on the complex plane and in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ (see [13, 21, 23, 25, 26]). Many authors have investigated the growth and oscillation of the solutions of complex linear differential equations in $\mathbb{C}$ (see [2, 3, 4, 6, 9, 12, 16, 17, 18, 20, 24]). In the unit disc, there already exist many results [7, 8, 10, 11, 14, 15, 19, 22, 28], but the

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study is more difficult than that in the complex plane, because the efficient tool of Wiman-Valiron theory which doesn't hold in the unit disc.

We need to give some definitions and discussions. Firstly, let us give two definitions about the degree of small growth order of functions in Δ as polynomials on the complex plane $\mathbb{C}$. There are many types of definitions of small growth order of functions in Δ (i.e., see [10, 11]).

Definition 1 *Let* f *be a meromorphic function in* Δ*, and*

$$D(f) = \overline{\lim_{r\to 1^-}} \frac{T(r,\ f)}{-\log\ (1-r)} = b.$$

If $b < \infty$*, we say that* f *is of finite* b *degree (or is non-admissible); if* $b = \infty$*, we say that* f *is of infinite degree (or is admissible), both defined by characteristic function* $T(r,f)$.

Definition 2 *Let* f *be an analytic function in* Δ*, and*

$$D_M(f) = \overline{\lim_{r\to 1^-}} \frac{\log^+ M(r,\ f)}{-\log\ (1-r)} = a < \infty \ \ (\textit{or } a = \infty),$$

then we say that f *is a function of finite* a *degree (or of infinite degree) defined by maximum modulus function* $M(r,\ f) = \max_{|z|=r} |f(z)|$*. Moreover, for* $F \subset [0,1)$*, the upper and lower densities of* F *are defined by*

$$\overline{dens}_\Delta F = \overline{\lim_{r\to 1^-}} \frac{m(F\cap[0,r))}{m([0,r))}, \quad \underline{dens}_\Delta F = \underline{\lim_{r\to 1^-}} \frac{m(F\cap[0,r))}{m([0,r))}$$

respectively, where $m(G) = \int_G \frac{dt}{1-t}$ *for* $G \subset [0,1)$.

Now we give the definitions of iterated order and growth index to classify generally the functions of fast growth in Δ as those in $\mathbb{C}$ (see [5, 16, 17]). Let us define inductively, for $r \in [0,1)$, $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large in $(0,1)$, $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 3 (*see* [7, 8, 15]) *Let* f *be a meromorphic function in* Δ*. Then the iterated* p*-order of* f *is defined by*

$$\rho_p(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p^+ T(r,\ f)}{-\log\ (1-r)} \quad (p \geq 1 \ \ \textit{is an integer}),$$

where $\log_1^+ x = \log^+ x = \max\{\log x, 0\}$, $\log_{p+1}^+ x = \log^+ \log_p^+ x$. *For* $p = 1$, *this notation is called order and for* $p = 2$ *hyper-order* [14, 19].

Remark 1 *If* f *is analytic in* Δ, *then the iterated* p*-order of* f *is defined by*

$$\rho_{M,p}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_{p+1}^+ M(r,\ f)}{-\log\ (1-r)} \quad (p \geq 1 \textit{ is an integer}).$$

Remark 2 *It follows by M. Tsuji* [23, *p.* 205] *that if* f *is an analytic function in* Δ, *then we have the inequalities*

$$\rho_1(f) \leqslant \rho_{M,1}(f) \leqslant \rho_1(f) + 1,$$

which are the best possible in the sense that there are analytic functions g *and* h *such that* $\rho_{M,1}(g) = \rho_1(g)$ *and* $\rho_{M,1}(h) = \rho_1(h) + 1$, *see* [11]. *However, it follows by Proposition 2.2.2 in* [17] *that* $\rho_{M,p}(f) = \rho_p(f)$ *for* $p \geq 2$.

Definition 4 ([8]) *The growth index of the iterated order of a meromorphic function* $f(z)$ *in* Δ *is defined by*

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible,} \\ \min\{j \in \mathbb{N} : \rho_j(f) < +\infty\}, & \text{if } f \text{ is admissible,} \\ +\infty, & \text{if } \rho_j(f) = +\infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

For an analytic function f *in* Δ, *we also define*

$$i_M(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible,} \\ \min,\{j \in \mathbb{N} : \rho_{M,j}(f) < +\infty\}, & \text{if } f \text{ is admissible,} \\ +\infty, & \text{if } \rho_{M,j}(f) = +\infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

Remark 3 *If* $\rho_p(f) < \infty$ *or* $i(f) \leq p$, *then we say that* f *is of finite iterated* p*-order; if* $\rho_p(f) = \infty$ *or* $i(f) > p$, *then we say that* f *is of infinite iterated* p*-order. In particular, we say that* f *is of finite order if* $\rho(f) < \infty$ *or* $i(f) \leq 1$; f *is of infinite order if* $\rho(f) = \infty$ *or* $i(f) > 1$.

Definition 5 ([7, 28]) *Let* f *be a meromorphic function in* Δ. *Then the iterated exponent of convergence of the sequence of zeros of* $f(z)$ *is defined by*

$$\lambda_p(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p^+ N\left(r, \frac{1}{f}\right)}{-\log\ (1-r)}, \quad (p \geq 1 \quad \textit{is an integer}),$$

where $N\left(r,\frac{1}{f}\right)$ *is the counting function of zeros of* $f(z)$ *in* $\{|z|<r\}$*. For* $p=1$*, this notation is called exponent of convergence of the sequence of zeros and for* $p=2$ *hyper-exponent of convergence of the sequence of zeros.*

Similarly, the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ *is defined by*

$$\overline{\lambda}_p(f)=\overline{\lim_{r\to 1^-}}\frac{\log_p^+\overline{N}\left(r,\frac{1}{f}\right)}{-\log(1-r)},\quad (p\geq 1 \text{ is an integer}),$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ *is the counting function of distinct zeros of* $f(z)$ *in* $\{|z|<r\}$*. For* $p=1$*, this notation is called exponent of convergence of the sequence of distinct zeros and for* $p=2$ *hyper-exponent of convergence of the sequence of distinct zeros.*

Definition 6 ([7, 28]) *Let* f *be a meromorphic function in* Δ*. Then the iterated exponent of convergence of the sequence of fixed points of* $f(z)$ *is defined by*

$$\tau_p(f)=\lambda_p(f-z)=\overline{\lim_{r\to 1^-}}\frac{\log_p^+ N\left(r,\frac{1}{f-z}\right)}{-\log(1-r)}\ (p\geq 1 \text{ is an integer}).$$

For $p=1$*, this notation is called exponent of convergence of the sequence of fixed points and for* $p=2$ *hyper-exponent of convergence of the sequence of fixed points. Similarly, the iterated exponent of convergence of the sequence of distinct fixed points of* $f(z)$ *is defined by*

$$\overline{\tau}_p(f)=\overline{\lambda}_p(f-z)=\overline{\lim_{r\to 1^-}}\frac{\log_p^+\overline{N}\left(r,\frac{1}{f-z}\right)}{-\log(1-r)}\ (p\geq 1 \text{ is an integer}).$$

For $p=1$*, this notation is called exponent of convergence of the sequence of distinct fixed points and for* $p=2$ *hyper-exponent of convergence of the sequence of distinct fixed points. Thus* $\overline{\tau}_p(f)=\overline{\lambda}_p(f-z)$ *is an indication of oscillation of distinct fixed points of* $f(z)$*.*

For $k\geq 2$, we consider the linear differential equation

$$f^{(k)}+A(z)f=0, \tag{1}$$

where $A(z)\not\equiv 0$ is an analytic function in the unit disc of finite iterated p-order. Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades, see [27]. However, there are few studies on the fixed points of solutions of differential equations, specially in the unit disc. In [9], Z.-X. Chen firstly studied the problem on the fixed points and hyper-order of solutions of second order linear differential equations with entire coefficients. After that, there were some results which improve those of Z.-X. Chen, see [3, 4, 18, 20, 24]. In [7], T. B. Cao firstly investigated the fixed points of solutions of linear complex differential equations in the unit disc. Very recently in [28] , G. Zhang and A. Chen extended some results of [7] to the case of higher order linear differential equations with analytic coefficients and have obtained the following results.

Theorem 1 ([28]) *Let* H *be a set of complex numbers satisfying* $\overline{dens}_{\Delta}\{|z| : z \in H \subseteq \Delta\} > 0$, *and let* $A(z) \not\equiv 0$ *be an analytic function in* Δ *such that* $\rho_{M,p}(A) = \sigma < +\infty$ *and for real number* $\alpha > 0$, *we have for all* $\varepsilon > 0$ *sufficiently small,*

$$|A(z)| \geq \exp_p\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\} \tag{2}$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of equation (1) satisfies*

$$\tau_p\left(f^{(j)}\right) = \overline{\tau}_p\left(f^{(j)}\right) = \rho_p(f) = +\infty \ \ (j = 0, \dots, k), \tag{3}$$

$$\tau_{p+1}\left(f^{(j)}\right) = \overline{\tau}_{p+1}\left(f^{(j)}\right) = \rho_{p+1}(f) = \rho_{M,p}(A) = \sigma \ \ (j = 0, \dots, k). \tag{4}$$

Theorem 2 ([28]) *Let* H *be a set of complex numbers satisfying* $\overline{dens}_{\Delta}\{|z| : z \in H \subseteq \Delta\} > 0$, *and let* $A(z) \not\equiv 0$ *be an analytic function in* Δ *such that* $\rho_p(A) = \sigma < +\infty$ *and for real number* $\alpha > 0$, *we have for all* $\varepsilon > 0$ *sufficiently small,*

$$T(r, A(z)) \geq \exp_{p-1}\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\} \tag{5}$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of equation (1) satisfies*

$$\tau_p\left(f^{(j)}\right) = \overline{\tau}_p\left(f^{(j)}\right) = \rho_p(f) = +\infty \ \ (j = 0, \dots, k), \tag{6}$$

$$\rho_{M,p}(A) \geq \tau_{p+1}\left(f^{(j)}\right) = \overline{\tau}_{p+1}\left(f^{(j)}\right) = \rho_{p+1}(f) \geq \sigma \ \ (j = 0, \dots, k). \tag{7}$$

In the present paper, we continue to study the oscillation of solutions of equation (1) in the unit disc. The main purpose of this paper is to study the relation between solutions and their derivatives of the differential equation (1) and analytic functions of finite iterated p-order. We obtain an extension of Theorems 1-2. In fact, we prove the following results:

Theorem 3 *Assume that the assumptions of Theorem 1 hold. If* $\varphi(z)$ *is an analytic function in* Δ *such that* $\varphi^{(k-j)}(z) \not\equiv 0$ $(j = 0, ..., k)$ *with finite iterated* p- *order* $\rho_p(\varphi) < +\infty$, *then every solution* $f(z) \not\equiv 0$ *of (1), satisfies*

$$\lambda_p\left(f^{(j)} - \varphi\right) = \overline{\lambda}_p\left(f^{(j)} - \varphi\right) = \rho_p(f) = +\infty \ \ (j = 0, \ldots, k), \tag{8}$$

$$\lambda_{p+1}\left(f^{(j)} - \varphi\right) = \overline{\lambda}_{p+1}\left(f^{(j)} - \varphi\right) = \rho_{p+1}(f) = \rho_{M,p}(A) = \sigma \ \ (j = 0, \ldots, k). \tag{9}$$

Theorem 4 *Assume that the assumptions of Theorem 2 hold. If* $\varphi(z)$ *is an analytic function in* Δ *such that* $\varphi^{(k-j)}(z) \not\equiv 0$ $(j = 0, \ldots, k)$ *with finite iterated* p- *order* $\rho_p(\varphi) < +\infty$, *then every solution* $f(z) \not\equiv 0$ *of (1), satisfies*

$$\lambda_p\left(f^{(j)} - \varphi\right) = \overline{\lambda}_p\left(f^{(j)} - \varphi\right) = \rho_p(f) = +\infty \ \ (j = 0, \ldots, k), \tag{10}$$

$$\rho_{M,p}(A) \geq \lambda_{p+1}\left(f^{(j)} - \varphi\right) = \overline{\lambda}_{p+1}\left(f^{(j)} - \varphi\right) = \rho_{p+1}(f) \geq \sigma \ \ (j = 0, ..., k). \tag{11}$$

2 Auxiliary lemmas

We need the following lemmas in the proofs of our theorems.

Lemma 1 ([8]) *If* f *and* g *are meromorphic functions in* Δ, $p \geq 1$ *is an integer, then we have*
(i) $\rho_p(f) = \rho_p(1/f)$, $\rho_p(a.f) = \rho_p(f)$ $(a \in \mathbb{C} - \{0\})$;
(ii) $\rho_p(f) = \rho_p\left(f'\right)$;
(iii) $\max\{\rho_p(f+g), \rho_p(fg)\} \leq \max\{\rho_p(f), \rho_p(g)\}$;
(iv) if $\rho_p(f) < \rho_p(g)$, *then* $\rho_p(f+g) = \rho_p(g)$, $\rho_p(fg) = \rho_p(g)$.

Lemma 2 ([14]) *Let* f *be a meromorphic function in the unit disc* Δ, *and let* $k \geq 1$ *be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f), \tag{12}$$

where $S(r,f) = O\left(\log^+ T(r,f)) + \log \frac{1}{1-r}\right)$, *possibly outside a set* $E \subset [0,1)$ *with* $\int_E \frac{dr}{1-r} < +\infty$. *If f is of finite order (namely, finite iterated* 1*-order) of growth, then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log(\frac{1}{1-r})\right). \tag{13}$$

Lemma 3 *Let* f *be a meromorphic function in the unit disc* Δ *for which* $i(f) = p \geq 1$ *and* $\rho_p(f) = \beta < +\infty$, *and let* $k \geq 1$ *be an integer. Then for any* $\varepsilon > 0$,

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right) \tag{14}$$

holds for all r *outside a set* $E \subset [0,1)$ *with* $\int_E \frac{dr}{1-r} < +\infty$.

Proof. First for $k = 1$. Since $\rho_p(f) = \beta < +\infty$, we have for all $r \to 1^-$

$$T(r,f) \leq \exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}. \tag{15}$$

By Lemma 2, we have

$$m\left(r, \frac{f'}{f}\right) = O\left(\ln^+ T(r,f) + \ln\frac{1}{1-r}\right) \tag{16}$$

holds for all r outside a set $E \subset [0,1)$ with $\int_E \frac{dr}{1-r} < +\infty$. Hence, we have

$$m\left(r, \frac{f'}{f}\right) = O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right), \quad r \notin E. \tag{17}$$

Next, we assume that we have

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right), \quad r \notin E \tag{18}$$

for some integer $k \geq 1$. Since $N\left(r, f^{(k)}\right) \leq (k+1)\,N(r,f)$, it holds that

$$\begin{aligned} T\left(r, f^{(k)}\right) &= m\left(r, f^{(k)}\right) + N\left(r, f^{(k)}\right) \leq \\ &\leq m\left(r, \frac{f^{(k)}}{f}\right) + m(r,f) + (k+1)\,N(r,f) \end{aligned}$$

$$\leq O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right)+(k+1)\,T\,(r,f)=O\left(\exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right). \tag{19}$$

By (16) and (19) we again obtain

$$m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right)=O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right),\ r\notin E \tag{20}$$

and hence,

$$\begin{aligned} m\left(r,\frac{f^{(k+1)}}{f}\right) &\leq m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right)+m\left(r,\frac{f^{(k)}}{f}\right)= \\ &= O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right),\ r\notin E. \end{aligned} \tag{21}$$

■

Lemma 4 ([1]) *Let* $g:(0,1)\to\mathbb{R}$ *and* $h:(0,1)\to\mathbb{R}$ *be monotone increasing functions such that* $g(r)\leq h(r)$ *holds outside of an exceptional set* $E\subset[0,1)$ *of finite logarithmic measure. Then there exists a* $d\in(0,1)$ *such if* $s(r)=1-d(1-r)$, *then* $g(r)\leq h(s(r))$ *for all* $r\in[0,1)$.

Lemma 5 *Let* $A_0,\ A_1,\dots,\ A_{k-1},\ F\not\equiv 0$ *be finite iterated* p- *order analytic functions in the unit disc* Δ. *If* f *is a solution with* $\rho_p(f)=+\infty$ *and* $\rho_{p+1}(f)=\rho<+\infty$ *of the*

$$f^{(k)}+A_{k-1}f^{(k-1)}+\dots+A_1f'+A_0f=F, \tag{22}$$

then $\overline{\lambda}_p(f)=\lambda_p(f)=\rho_p(f)=+\infty$ *and* $\overline{\lambda}_{p+1}(f)=\lambda_{p+1}(f)=\rho_{p+1}(f)=\rho$.

Proof. Since $A_0,\ A_1,\dots,\ A_{k-1},\ F\not\equiv 0$ are analytic in Δ, then all solutions of (22) are analytic in Δ (see [14]). By (22), we can write

$$\frac{1}{f}=\frac{1}{F}\left(\frac{f^{(k)}}{f}+A_{k-1}\frac{f^{(k-1)}}{f}+\dots+A_1\frac{f'}{f}+A_0\right). \tag{23}$$

If f has a zero at $z_0 \in \Delta$ of order $\gamma\,(> k)$, then F must have a zero at z_0 of order at least $\gamma - k$. Hence,

$$N\left(r,\frac{1}{f}\right) \le k\,\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{F}\right). \tag{24}$$

By (23), we have

$$m\left(r,\frac{1}{f}\right) \le \sum_{j=1}^{k} m\left(r,\frac{f^{(j)}}{f}\right) + \sum_{j=0}^{k-1} m\left(r,A_j\right) + m\left(r,\frac{1}{F}\right) + O\left(1\right). \tag{25}$$

Applying the Lemma 3, we have

$$m\left(r,\frac{f^{(j)}}{f}\right) = O\left(\exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right) \quad (j = 1,\ldots,k), \tag{26}$$

where $\rho_{p+1}(f) = \rho < +\infty$, holds for all r outside a set $E \subset [0,1)$ with $\int_E \frac{dr}{1-r} < +\infty$. By (24), (25) and (26) we get

$$\begin{aligned} T\left(r,f\right) &= T\left(r,\frac{1}{f}\right) + O\left(1\right) \le k\overline{N}\left(r,\frac{1}{f}\right) + \sum_{j=0}^{k-1} T\left(r,A_j\right) + T\left(r,F\right) + \\ &\quad + O\left(\exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right) \quad (|z| = r \notin E). \end{aligned} \tag{27}$$

Set

$$\mu = \max\{\rho_p\left(A_j\right) \quad (j = 0,\ldots,k-1), \rho_p\left(F\right)\}.$$

Then for $r \to 1^-$, we have

$$T\left(r,A_0\right) + \ldots + T\left(r,A_{k-1}\right) + T\left(r,F\right) \le (k+1)\exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\mu+\varepsilon}. \tag{28}$$

Thus, by (27) and (28), we have for $r \to 1^-$

$$\begin{aligned} T\left(r,f\right) &\le k\,\overline{N}\left(r,\frac{1}{f}\right) + (k+1)\exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\mu+\varepsilon} + \\ &\quad + O\left(\exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right) = \\ &= k\,\overline{N}\left(r,\frac{1}{f}\right) + O\left(\exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\eta}\right), \quad (|z| = r \notin E). \end{aligned} \tag{29}$$

where $\eta < +\infty$. Hence for any f with $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho$, by Lemma 4 and (29), we have

$$\lambda_p(f) \geq \overline{\lambda}_p(f) \geq \rho_p(f) = +\infty$$

and $\lambda_{p+1}(f) \geq \overline{\lambda}_{p+1}(f) \geq \rho_{p+1}(f)$. Since $\overline{\lambda}_{p+1}(f) \leq \lambda_{p+1}(f) \leq \rho_{p+1}(f)$, we have $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho$. ■

Lemma 6 ([7]) *Let* H *be a set of complex numbers satisfying* $\overline{dens}_\Delta\{|z| : z \in H \subseteq \Delta\} > 0$, *and let* $A(z) \not\equiv 0$ *be an analytic function in* Δ *such that* $\rho_{M,p}(A) = \sigma < +\infty$ *and for real number* $\alpha > 0$, *we have for all* $\varepsilon > 0$ *sufficiently small,*

$$|A(z)| \geq \exp_p\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}, \tag{30}$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of equation (1) satisfies* $\rho_p(f) = +\infty$ *and* $\rho_{p+1}(f) = \rho_{M,p}(A) = \sigma$.

Lemma 7 ([7]) *Let* H *be a set of complex numbers satisfying* $\overline{dens}_\Delta\{|z| : z \in H \subseteq \Delta\} > 0$, *and let* $A(z) \not\equiv 0$ *be an analytic function in* Δ *such that* $\rho_p(A) = \sigma < +\infty$ *and for real number* $\alpha > 0$, *we have for all* $\varepsilon > 0$ *sufficiently small,*

$$T(r, A(z)) \geq \exp_{p-1}\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}, \tag{31}$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of equation (1) satisfies* $\rho_p(f) = +\infty$ *and* $\rho_{M,p}(A) \geq \rho_{p+1}(f) \geq \sigma$.

3 Proof of Theorem 3

Suppose that $f(z) \not\equiv 0$ is a solution of equation (1). Then by Lemma 6, we have $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho_{M,p}(A) = \sigma$. Set $w_j = f^{(j)} - \varphi$ $(j = 0, 1, \ldots, k)$. Since $\rho_p(\varphi) < +\infty$, then by Lemma 1, we have $\rho_p(w_j) = \rho_p(f) = +\infty$, $\rho_{p+1}(w_j) = \rho_{p+1}(f) = \rho_{M,p}(A) = \sigma$, $\lambda_p(w_j) = \lambda_p\left(f^{(j)} - \varphi\right)$, $\overline{\lambda}_p(w_j) = \overline{\lambda}_p\left(f^{(j)} - \varphi\right)$ $(j = 0, 1, \ldots, k)$. Differentiating both sides of $w_j = f^{(j)} - \varphi$ and replacing $f^{(k)}$ with $f^{(k)} = -Af$, we obtain

$$w_j^{(k-j)} = -Af - \varphi^{(k-j)} \quad (j = 0, 1, \ldots, k). \tag{32}$$

Then, we have

$$f = -\frac{w_j^{(k-j)} + \varphi^{(k-j)}}{A}. \tag{33}$$

Substituting (33) into equation (1), we get

$$\left(\frac{w_j^{(k-j)}}{A}\right)^{(k)} + w_j^{(k-j)} = -\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + \varphi^{(k-j)}\right). \tag{34}$$

By (34), we can write

$$\begin{aligned} &w_j^{(2k-j)} + \Phi_{2k-j-1} w_j^{(2k-j-1)} + \ldots + \Phi_{k-j} w_j^{(k-j)} \\ &= -A\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + A\left(\frac{\varphi^{(k-j)}}{A}\right)\right), \end{aligned} \tag{35}$$

where $\Phi_{k-j}(z), \ldots, \Phi_{2k-j-1}(z)$ $(j = 0, 1, \ldots, k)$ are analytic functions with

$$\rho_{M,p}(\Phi_{k-j}) \leq \sigma, \ldots, \rho_{M,p}(\Phi_{2k-j-1}) \leq \sigma \quad (j = 0, 1, \ldots, k).$$

By Lemma 1 we have $\rho_p\left(\frac{\varphi^{(k-j)}}{A}\right) < +\infty$. Thus, by $A \not\equiv 0$ $\varphi^{9k-j)} \not\equiv 0$, $(j = 0, \ldots, k)$ and Lemma 6, we obtain

$$-A\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + A\left(\frac{\varphi^{(k-j)}}{A}\right)\right) \not\equiv 0. \tag{36}$$

Hence, by Lemma 5, we have $\lambda_p(w_j) = \overline{\lambda}_p(w_j) = \rho_p(w_j) = +\infty$ and $\lambda_{p+1}(w_j) = \overline{\lambda}_{p+1}(w_j) = \rho_{p+1}(w_j) = \rho_{M,p}(A) = \sigma$ $(j = 0, 1, \ldots, k)$. Thus

$$\lambda_p\left(f^{(j)} - \varphi\right) = \overline{\lambda}_p\left(f^{(j)} - \varphi\right) = \rho_p(f) = +\infty \ \ (j = 0, 1, \ldots, k),$$

$$\lambda_{p+1}\left(f^{(j)} - \varphi\right) = \overline{\lambda}_{p+1}\left(f^{(j)} - \varphi\right) = \rho_{p+1}(f) = \rho_{M,p}(A) = \sigma \ \ (j = 0, 1, \ldots, k).$$

4 Proof of Theorem 4

Suppose that $f(z) \not\equiv 0$ is a solution of equation (1). Then by Lemma 7, we have $\rho_p(f) = +\infty$ and $\rho_{M,p}(A) \geq \rho_{p+1}(f) \geq \sigma$. By using similar reasoning as in the proof of Theorem 3, we obtain Theorem 4.

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