Acta Univ. Sapientiae, Mathematica, **12**, 2 (2020) 272–279

DOI: 10.2478/ausm-2020-0018

On topological properties of the set of maldistributed sequences

József Bukor
Department of Informatics,
J. Selye University,
Komárno, Slovakia
email: bukorj@ujs.sk

János T. Tóth
Department of Mathematics,
J. Selye University,
Komárno, Slovakia
email: tothj@ujs.sk

Abstract. The real sequence (x_n) is maldistributed if for any non-empty interval I, the set $\{n \in \mathbb{N} : x_n \in I\}$ has upper asymptotic density 1. The main result of this note is that the set of all maldistributed real sequences is a residual set in the set of all real sequences (i.e., the maldistribution is a typical property in the sense of Baire categories). We also generalize this result.

1 Introduction

Following the concept of statistical convergence for real sequences, J. A. Fridy [2] introduced the concept of statistical cluster points of a sequence (x_n). A number α is called a statistical cluster point of the sequence (x_n) provided that for every $\varepsilon > 0$ the set $\{n \in \mathbb{N} : |x_n - \alpha| < \varepsilon\}$ has a positive upper asymptotic density.

G. Myerson [7] calls a sequence (x_n) maldistributed if for any non-empty interval I the set $\{n \in \mathbb{N} : x_n \in I\}$ has upper asymptotic density 1. In [12] the maldistribution property is characterized by one-jump distribution functions. Examples of maldistributed sequences are given in [12] and [3]. Using the idea from [4] (Example VII) for the generalization of the concept of statistical

2010 Mathematics Subject Classification: 54E52
Key words and phrases: maldistributed sequence, weighted density, Baire category

convergence, we can extend the maldistribution property of sequences with the help of weighted densities.

The concept of weighted density as a generalization of asymptotic density was introduced in [1] and [10]. Let $f : \mathbb{N} \to (0, \infty)$ be a weight function with the properties

$$\sum_{n=1}^{\infty} f(n) = \infty, \qquad \lim_{n\to\infty} \frac{f(n)}{\sum_{a\le n} f(a)} = 0. \tag{1}$$

For $A \subset \mathbb{N}$ define by

$$\underline{d}_f(A) = \liminf_{n\to\infty} \frac{\sum_{a\le n,\ a\in A} f(a)}{\sum_{a\le n} f(a)} \quad \text{and} \quad \overline{d}_f(A) = \limsup_{n\to\infty} \frac{\sum_{a\le n,\ a\in A} f(a)}{\sum_{a\le n} f(a)}$$

the lower and upper f-densities of A, respectively. Note that the asymptotic densities correspond to $f(n) = 1$ and the logarithmic densities to $f(n) = \frac{1}{n}$. It is well-known that each set which has asymptotic density also has the logarithmic one but a set may have a logarithmic density without having an asymptotic one.

The main tool to compare weighted densities is the classical result of C. T. Rajagopal (cf. [9], Theorem 3) which, in terms of weighted densities, says the following.
Let $f, g : \mathbb{N} \to (0, \infty)$ be weight functions with properties (1). If $\frac{f(n)}{g(n)}$ is decreasing, then for any $A \subset \mathbb{N}$ we have

$$\underline{d}_g(A) \le \underline{d}_f(A) \le \overline{d}_f(A) \le \overline{d}_g(A). \tag{2}$$

Now we give a generalization of maldistributed sequences.

Definition 1 *Let $f : \mathbb{N} \to (0, \infty)$ be a weight function with properties (1). The sequence (x_n) is said to be f-maldistributed, if for any non-empty interval I the set $\{n \in \mathbb{N} : x_n \in I\}$ has upper f-density 1.*

Comparing to asymptotic density, logarithmic density is less sensitive to certain perturbations. For example, if a sequence is maldistributed, then it is not necessary f-maldistributed for $f(n) = \frac{1}{n}$ (which defines the logarithmic density).

Let us denote by $\mathcal{M}_f$ the set of all f-maldistributed sequences. The purpose of this note is to show that for any weight function f satisfying (1) the set $\mathcal{M}_f$ is residual in the Fréchet metric space of all real sequences.

Let $\mathbf{s}$ be the Fréchet metric space of all sequences of real numbers with the metric

$$\rho(\mathbf{x},\mathbf{y}) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1+|x_k-y_k|},$$

where $\mathbf{x} = (x_k)$, $\mathbf{y} = (y_k)$. It is known that $(\mathbf{s}, \rho)$ is a complete metric space.

In [5] it was proved that the set of all uniformly distributed sequences is a dense subset of the first Baire category in $\mathbf{s}$. The same is true for the set of all statistically convergent sequences of real numbers (cf. [11]).

2 Main results

The main result of this paper is as follows.

Theorem 1 *Let $f : \mathbb{N} \to (0,\infty)$ be a weight function with properties (1). Then the set of all f-maildistributed sequences $\mathcal{M}_f$ is residual in the the Fréchet metric space of all sequences of real numbers $\mathbf{s}$.*

For the proof of the theorem we shall use the following lemma.

Lemma 1 *For the interval $I = [a,b]$ denote by $\mathcal{A}(I,\alpha)$ the set of all $\mathbf{x} = (x_k) \in \mathbf{s}$ for which*

$$\overline{d}_f\big(\{n \in \mathbb{N} : x_n \in I\}\big) \le \alpha\,,$$

where $\alpha \in (0,1)$. Then $\mathcal{A}(I,\alpha)$ is a set of the first Baire category in $\mathbf{s}$.

Proof of Lemma 1. We define a continuous function $h : \mathbb{R} \to [0,1]$ by

$$h(x) = \begin{cases} \frac{2x-2a}{b-a} & \text{for} \quad x \in \left[a, \frac{a+b}{2}\right] \\ \frac{2b-2x}{b-a} & \text{for} \quad x \in \left[\frac{a+b}{2}, b\right] \\ 0 & \text{for} \quad x \in \mathbb{R} \setminus [a,b] \end{cases}$$

We choose an arbitrary real number $\beta \in (\alpha, 1)$. Using the function h we define for $\mathbf{x} = (x_k) \in \mathbf{s}$ and fixed n the function $g_n : \mathbf{s} \to [0,1]$ in the following way:

$$g_n(\mathbf{x}) = \max\left\{ \beta,\ \frac{\sum\limits_{k=1}^{n} h(x_k).f(k)}{\sum\limits_{k=1}^{n} f(k)} \right\}.$$

Denote $\mathcal{A}^*(I,\alpha)$ the set of all $\mathbf{x} = (x_k) \in \mathbf{s}$ for which there exists the limit $\lim_{n\to\infty} g_n(\mathbf{x})$.

One can easily check that for each $\mathbf{x} = (x_k) \in \mathbf{s}$ and natural number n we have

$$\frac{\sum_{k=1}^{n} h(x_k).f(k)}{\sum_{k=1}^{n} f(k)} \le \frac{\sum_{k\le n,\ x_k\in I} f(k)}{\sum_{k\le n} f(k)}. \tag{3}$$

For any $\mathbf{x} \in \mathcal{A}(I,\alpha)$, the right hand side of (3) does not exceed α if n is large enough. Therefore $\lim_{n\to\infty} g_n(\mathbf{x}) = \beta$, and then $\mathcal{A}(I,\alpha) \subset \mathcal{A}^*(I,\alpha)$.

Put $g(\mathbf{x}) = \lim_{n\to\infty} g_n(\mathbf{x})$ for $\mathbf{x} \in \mathcal{A}^*(I,\alpha)$. We shall prove that

(a) the function g_n $(n = 1,2,\dots)$ is a continuous function on $\mathbf{s}$,

(b) g is discontinuous at each point of $\mathcal{A}^*(I,\alpha)$.

(a) Let $\mathbf{x}^0 = (x_k^0)_{k=1}^{\infty}$, $\mathbf{x}^{(j)} = (x_k^{(j)})_{k=1}^{\infty} \in \mathbf{s}$ $(j = 1,2,\dots)$ and $\mathbf{x}^{(j)} \to \mathbf{x}^0$ (for $j \to \infty$).

Then from the convergence in the space $\mathbf{s}$ for each fixed k we have $\lim_{j\to\infty} x_k^{(j)} = x_k^0$. The continuity of function h implies $\lim_{j\to\infty} g_n(\mathbf{x}^{(j)}) = g_n(\mathbf{x}^0)$. Thus g_n $(n = 1,2,\dots)$ is continuous on $\mathbf{s}$.

(b) Let $\mathbf{y} = (y_k) \in \mathcal{A}^*(I,\alpha)$. We have the following two possibilities.

(1) $g(\mathbf{y}) < 1$,

(2) $g(\mathbf{y}) = 1$.

In case (1) we choose a positive ε such that $\varepsilon < 1 - g(\mathbf{y})$. It is suffice to prove that in each ball $K(\mathbf{y},\delta) = \{\mathbf{x} \in \mathcal{A}^*(I,\alpha),\ \rho(\mathbf{x},\mathbf{y}) < \delta\}$ $(\delta > 0)$ of the subspace $\mathcal{A}^*(I,\alpha)$ of $\mathbf{s}$ there exists an element $\mathbf{x} = (x_k) \in \mathbf{s}$ with $g(\mathbf{x}) - g(\mathbf{y}) > \varepsilon$.

Let $\delta > 0$. Choose a positive integer m such that $\sum_{k=m+1}^{\infty} 2^{-k} < \delta$, and define the sequence $\mathbf{x} = (x_k)$ in the following way:

$$x_k = \begin{cases} y_k, & \text{if } k \le m, \\ \frac{a+b}{2}, & \text{if } k > m. \end{cases}$$

Hence $\rho(\mathbf{x},\mathbf{y}) < \delta$, further $h(x_k) = 1$ for $k > m$. Then

$$\frac{\sum_{k=1}^{n} h(x_k).f(k)}{\sum_{k=1}^{n} f(k)} \ge \frac{\sum_{k=m+1}^{n} f(k)}{\sum_{k=1}^{n} f(k)} = 1 - \frac{\sum_{k=1}^{m} f(k)}{\sum_{k=1}^{n} f(k)} \to 1 \quad \text{for } n \to \infty,$$

and therefore $g(\mathbf{x}) = \lim_{n\to\infty} g_n(\mathbf{x}) = 1$. Then immediately follows

$$g(\mathbf{x}) - g(\mathbf{y}) = 1 - g(\mathbf{y}) > \varepsilon.$$

In case (2) we have $g(\mathbf{y}) = 1$. Let $\delta, m, \mathbf{x}$ have the previous meaning. Put

$$x_k = \begin{cases} y_k, & \text{if } k \le m, \\ a, & \text{if } k > m. \end{cases}$$

Then, clearly $\rho(\mathbf{x}, \mathbf{y}) < \delta$, and $h(x_k) = 0$ for $k > m$. Then

$$\frac{\sum_{k=1}^{n} h(x_k).f(k)}{\sum_{k=1}^{n} f(k)} \le \frac{\sum_{k=1}^{m} f(k)}{\sum_{k=1}^{n} f(k)} \to 0 \quad \text{for} \quad n \to \infty.$$

So, we have $g(\mathbf{x}) = \lim_{n\to\infty} g_n(\mathbf{x}) = \beta$, and therefore $g(\mathbf{y}) - g(\mathbf{x}) = 1 - \beta > 0$. Hence the discontinuity of g at $\mathbf{y} \in \mathcal{A}^*(I, \alpha)$ has been proved.

The function g is a limit function (on $\mathcal{A}^*(I, \alpha)$) of the sequence of continuous functions $(g_n)_{n=1}^{\infty}$ on $\mathcal{A}^*(I, \alpha)$. Then the function g is a function in the first Baire class on $\mathcal{A}^*(I, \alpha)$. According to the well-known fact that the set of discontinuity points of an arbitrary function of the first Baire class is a set of the first Baire category (cf. [8], p. 32), we see that the set $\mathcal{A}^*(I, \alpha)$ is of the first Baire category in $\mathcal{A}^*(I, \alpha)$ Thus $\mathcal{A}^*(I, \alpha)$ is in $\mathbf{s}$, too. Since $\mathcal{A}(I, \alpha) \subset \mathcal{A}^*(I, \alpha)$, the assertion follows. □

Proof of Theorem 1. Denote by $\mathbb{Q}$ the set of all rational numbers. Denote by $\mathcal{H}$ the set of all $\mathbf{x} = (x_k) \in \mathbf{s}$ for which there exists an interval I with

$$\overline{d}_f\big(\{n \in \mathbb{N} : x_n \in I\}\big) \le \alpha$$

for some $\alpha \in (0, 1)$. Combining Lemma 1 and the fact that for each interval I there exist rational numbers a, b such that $I \subset [a, b]$, we have

$$\mathcal{H} \subset \bigcup_{a,b\in\mathbb{Q},\ a<b} \ \bigcup_{i\in\mathbb{N},\ i\ge 2} A\left([a, b], 1 - \frac{1}{i}\right)$$

from which follows at once that $\mathcal{H}$ is a meager set. But $\mathcal{M}_f = \mathbf{s} \setminus \mathcal{H}$ and therefore the assertion of theorem follows. Hence the property of f-maldistribution is a typical property of real sequences from the topological point of view. □

We now introduce the concept of f-maldistributed integer sequences.

Definition 2 *Let* $f : \mathbb{N} \to (0, \infty)$ *be a weight function with properties* (1)*. The sequence* (x_n) *of positive integers is said to be* f*-maldistributed, if for any positive integers* $m \geq 2$ *and* $j \in \{0, 1, \dots, m-1\}$ *the set* $\{n \in \mathbb{N} : x_n \equiv j \pmod{m}\}$ *has upper* f*-density 1.*

Let $\mathbf{S}$ be the Baire's space of all sequences of positive integers with the metric ρ' defined in the following way.

Let $\mathbf{x} = (x_k) \in \mathbf{S}$, and $\mathbf{y} = (y_k) \in \mathbf{S}$. If $\mathbf{x} = \mathbf{y}$, then $\rho'(\mathbf{x}, \mathbf{y}) = 0$, otherwise

$$\rho'(\mathbf{x}, \mathbf{y}) = \frac{1}{\min\{n : x_n \neq y_n\}}.$$

The space $(\mathbf{S}, \rho')$ is a complete metric space. In [6] the topological properties of the set of all uniformly distributed sequences of positive integers in Baire's space were investigated.

The following auxilary result is similar to Lemma 1.

Lemma 2 *For the positive integers* $m \geq 2$ *and* $j \in \{0, 1, \dots, m-1\}$ *denote by* $\mathcal{A}(j, m, \alpha)$ *the set of all* $\mathbf{x} = (x_k) \in \mathbf{S}$ *for which*

$$\overline{d}_f\big(\{n \in \mathbb{N} : x_n \equiv j \pmod{m}\}\big) \leq \alpha\,,$$

where $\alpha \in (0, 1)$*. Then* $\mathcal{A}(j, m, \alpha)$ *is a set of the first Baire category in* $\mathbf{S}$*.*

The proof is analogous to the proof of Lemma 1. The crucial role is played by the function $g_n : \mathbf{S} \to [0, 1]$ given by

$$g_n(\mathbf{x}) = \max\left\{ \sqrt{\alpha},\ \frac{\sum\limits_{\substack{k \leq n \\ x_k \equiv j \pmod{m}}} f(k)}{\sum\limits_{k=1}^{n} f(k)} \right\}.$$

The following theorem says that the set of all f-maldistributed integer sequences form a residual set in Baire's space.

Theorem 2 *Let* $f : \mathbb{N} \to (0, \infty)$ *be a weight function with properties* (1)*. Denote by* $\mathcal{G}$ *the set of all* $\mathbf{x} = (x_k) \in \mathbf{S}$ *for which there exist* $m \geq 2$ *and* $j \in \{0, 1, \dots, m-1\}$ *such that*

$$\overline{d}_f\big(\{n \in \mathbb{N} : x_n \equiv j \pmod{m}\}\big) \leq \alpha$$

for some $\alpha \in (0, 1)$*. Then* $\mathcal{G}$ *is a set of the first Baire category in* $\mathbf{S}$*.*

Proof. Combining Lemma 2 with the fact that

$$\mathcal{G} \subset \bigcup_{m=2}^{+\infty} \bigcup_{j=0}^{m-1} \bigcup_{i=2}^{+\infty} A\left(j, m, 1 - \frac{1}{i}\right)$$

it immediately follows that $\mathcal{G}$ is a meager set in **S**. □

References

[1] R. Alexander, Density and multiplicative structure of sets of integers, *Acta Arith.*, **12** (1976), 321–332.

[2] J. A. Fridy, Statistical limit points, *Proc. Amer. Math. Soc.*, **118** (1993), 1187–1192.

[3] P. Kostyrko, M. Mačaj, T. Šalát, O. Strauch, On statistical limit points, *Proc. Amer. Math. Soc.*, **129** (2000), 2647–2654.

[4] P. Kostyrko, M. Mačaj, T. Šalát, O. Strauch, I-convergence and extremal I-limit points, *Math. Slovaca*, **55** (2005), 443–464.

[5] V. László, T. Šalát, The structure of some sequence spaces, and uniform distribution (mod 1), *Periodica Math. Hung.*, **10** (1979), 89–98.

[6] V. László, T. Šalát, Uniformly distributed sequences of positive integers in Baire's space, *Math. Slovaca*, **41** (1991), 277–281.

[7] G. Myerson, A sampler of recent developments in the distribution of sequences, *Lecture Notes in pure and applied Mathematics*, **147** (1993), 163–190.

[8] J. C. Oxtoby. *Measure and Category.* Graduate texts in Mathematics. Springer, 1980.

[9] C. T. Rajagopal, Some limit theorems, *Amer. J. Math.*, **70** (1948), 157–166.

[10] H. Rohrbach, B. Volkmann, Verallgemeinerte asymptotische Dichten, *J. Reine Angew. Math.*, **194** (1955), 195–209.

[11] T. Šalát, On statistically convergent sequences of real numbers, *Math. Slovaca*, **30** (1980), 139–150.

[12] O. Strauch, Uniformly maldistributed sequences in a strict sense, *Monatsh. Math.*, **120** (1995), 153–164.

Received: February 20, 2019