On the nullity of a family of tripartite graphs

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Abstract. The eigenvalues of the adjacency matrix of a graph form the spectrum of the graph. The multiplicity of the eigenvalue zero in the spectrum of a graph is called nullity of the graph. Fan and Qian (2009) obtained the nullity set of n-vertex bipartite graphs and characterized the bipartite graphs with nullity $n-4$ and the regular n-vertex bipartite graphs with nullity $n-6$. In this paper, we study similar problem for a class of tripartite graphs. As observed the nullity problem in tripartite graphs does not follow as an extension to that of the nullity of bipartite graphs, this makes the study of nullity in tripartite graphs interesting. In this direction, we obtain the nullity set of a class of n-vertex tripartite graphs and characterize these tripartite graphs with nullity $n-4$. We also characterize some tripartite graphs with nullity $n-6$ in this class.

1 Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. An edge joining a vertex v_i to a vertex v_j is denoted by $v_i v_j$ (or $v_j v_i$). The *adjacency matrix* of G is $A(G) = [a_{ij}]_{n \times n}$, where

$$a_{ij} = \left\{ \begin{array}{ll} 1 & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise} \end{array} \right. \quad (\forall v_i, v_j \in V(G)).$$

The eigenvalues of the graph G are the eigenvalues of $A(G)$ and the *spectrum* of G is the multiset of eigenvalues of G. The *nullity* of graph G, denoted by $\eta(G)$, is the multiplicity of the eigenvalue zero in the spectrum of G. The graph G is *singular* if $\eta(G) > 0$ and is *non-singular* if $\eta(G) = 0$. For graph theoretical terminology, we refer [10].

Collatz and Sinogowitz [2] posed the problem of characterizing singular graphs and since then, the theory of nullity of graphs has stimulated much research because of its importance in mathematics and chemistry. In literature, we find characterization of trees, unicyclic graphs, bicyclic graphs and bipartite graphs with their nullity.

Fiorini et al. [5] determined the greatest nullity among n-vertex trees in which no vertex has degree greater than a fixed positive integer D and gave a method of constructing the respective trees. Li and Chang [7] showed that there are some other trees with maximum nullity which can not be constructed by the method devised by Fiorini et al. [5] and modified the method of constructing the respective trees. Tan and Liu [11] found the nullity set of n-vertex unicyclic graphs with $n \geq 5$. They also characterized the unicyclic graphs with maximal nullity. The unicyclic graphs with minimum nullity are characterized by Li and Chang [8]. Li et al. [9] found the nullity set of n-vertex bicyclic graphs and gave a characterization of the bicyclic graphs with maximum nullity. Hu et al. [6] also found the nullity set of n-vertex bicyclic graphs, $n \geq 6$. Furthermore, the authors also characterized bicyclic graphs with extremal nullity. In Fan et al. [3], the authors found the nullity of unicyclic signed graphs and characterized n-vertex unicyclic signed graphs. Fan and Qian [4] introduced nullity of n-vertex bipartite graphs and presented characterization of bipartite graphs with nullity.

We find that the nullity of tripartite graphs does not follow as the extension of that of bipartite graphs. Thus it became interesting to characterize family of tripartite graphs with nullity. In this paper, we consider a class of n-vertex tripartite graphs, $n \geq 3$ and give nullity set of these tripartite graphs and characterize them with nullity $n - 4$, $n \geq 4$. Furthermore, we discuss some tripartite graphs with nullity $n - 6$, $n \geq 6$.

2   Preliminaries

In this section, we give some definitions, terminologies and known results which will be used later. Let G be an n-vertex graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G)$, the *neighbor set* of S in G, denoted as $N(S)$, is a set containing those vertices of G that are adjacent to some vertex in S. If $S = \{v\}$, we write $N(v)$ for $N(\{v\})$. If S is nonempty, we denote by $G[S]$ the *induced subgraph* of G. The *rank* of the graph G, denoted by $\mathrm{rank}(G)$, is the rank of its adjacency matrix $A(G)$, that is, $\mathrm{rank}(G) = \mathrm{rank}(A(G))$. It is known that $\eta(G) = n - \mathrm{rank}(G)$. The graph G is said to be an *expanded graph* if its vertex set $V(G)$ can be partitioned into $V_1, V_2 \ldots, V_k$, $k \geq 2$, such that $G[V_i]$ is an empty graph, for $1 \leq i \leq k$. If $G[V_i \cup V_j]$ is a nonempty graph, it is a complete bipartite graph for $1 \leq i, j \leq k$, $i \neq j$. If G is an expanded graph on $V_1, V_2, \ldots, V_k$, each V_i, for $1 \leq i \leq k$, is called an *expanded vertex* of order $|V_i|$. We observe that each simple graph can be viewed as an expanded graph.

    The n-vertex graph G is said to be an *expanded path* of length k if its vertex set $V(G)$ can be partitioned into $V_1, \ldots, V_k$, $k \geq 2$, such that

(i) $G[V_i]$ is an empty graph for $1 \leq i \leq k$,

(ii) $G[V_i \cup V_{i+1}]$ is a complete bipartite graph for $1 \leq i \leq k - 1$,

(iii) $G[V_i \cup V_j]$ is an empty graph for $1 \leq i, j \leq k$ with $j \neq i + 1$.

We use the notation $\mathbb{P}_k(V_1, \ldots, V_k)$ to denote an expanded path on $V_1, \ldots, V_k$ of length k. Similarly, an *expanded cycle* of length k, denoted by $\mathbb{C}_k(V_1, \ldots, V_k)$, is obtained from the expanded path $\mathbb{P}_k(V_1, \ldots, V_k)$ by adding edges between each vertex of V_1 and each of V_k. When there is no ambiguity, we simply write $\mathbb{P}_k$ and $\mathbb{C}_k$ respectively to represent an expanded path and an expanded cycle of length k. An *expanded decomposition* of the graph G is a list of expanded subgraphs such that each edge of G appears in exactly one expanded subgraph in the list.

    The graph G is *tripartite* if its vertex set can be partitioned into three subsets X, Y and Z such that $G[X]$, $G[Y]$ and $G[Z]$ are empty graphs; such a partition (X, Y, Z) is called a *tripartition*. For any $S \subseteq V(G)$, we denote by $N_X(S)$ the neighbors of S in X. Analogously, we can define $N_Y(S)$ and $N_Z(S)$. We consider a special class of tripartite graphs defined as follows. Let $\mathcal{T}_n$ be the family of those n-vertex tripartite graphs G, $n \geq 5$, whose tripartition (X, Y, Z) satisfies the following:

$$N_X(Y') \neq X \text{ and } N_Z(Y') \neq Z \;\; \forall \;\; Y' \subseteq Y, \tag{1}$$

$$G[X \cup Z] \text{ is complete bipartite.} \tag{2}$$

Lemma 1 (Fan [4]) *An expanded path of length* $k \geq 2$ *and order* n *has nullity* $n - k$, *if* k *is even and* $n - k + 1$, *if* k *is odd.*

Let K_{n_1,n_2} denote the complete bipartite graph, where n_1 and n_2 are the sizes of its partite sets. Also, K_1 denotes an isolated vertex and $r\,K_1$ denotes r copies of K_1. Then the following result characterizes the graphs with nullity $n - 2$.

Lemma 2 (Cheng [1]) *Let* G *be an* n*-vertex simple graph,* $n \geq 2$. *Then* $\eta(G) = n - 2$ *if and only if* $G \cong K_{n_1,n_2} \cup n_3 K_1$, *where* $n = n_1 + n_2 + n_3$, $n_1, n_2 > 0$ *and* $n_3 \geq 0$.

3 Main results

Let $G \in \mathcal{T}_n$ with tripartition (X, Y, Z). The adjacency matrix $A(G)$ of G is defined by

$$A(G) = \begin{array}{c} \\ X \\ Z \\ Y \end{array} \begin{array}{c} X \quad\; Z \quad\; Y \\ \left[\begin{array}{ccc} \mathbf{0} & J & C_1 \\ J^t & \mathbf{0} & C_2 \\ C_1^t & C_2^t & \mathbf{0} \end{array} \right], \end{array}$$

where J and $\mathbf{0}$ respectively denote the matrices with all entries 1 and 0. Let C and B denote the matrices defined as follows.

$$C = \left[\begin{array}{c} C_1 \\ C_2 \end{array} \right] \quad \text{and} \quad B = \left[\begin{array}{cc} \mathbf{0} & J \\ J^t & \mathbf{0} \end{array} \right].$$

The matrix $A(G)$ can be viewed as

$$A(G) = \left[\begin{array}{cc} B & C \\ C^t & \mathbf{0} \end{array} \right]. \tag{3}$$

Let

$$U = [B \quad C], \quad L = [C^t \quad \mathbf{0}]. \tag{4}$$

Then $A(G)$ can be written as $A(G) = \left[\begin{array}{c} U \\ L \end{array} \right]$.

For each $v \in X \cup Z$, we denote by U_v the row of $A(G)$ corresponding to the vertex v. Similarly, for each $v \in Y$, the row of $A(G)$ corresponding to the

vertex v is denoted by L_v. Let $S \subseteq X \cup Z$. Then from the matrix $A(G)$, we see that

$$\sum_{v \in S} k_v U_v = [b_1 \ b_2 \ c], \tag{5}$$

where b_1, b_2 are constant row matrices respectively of dimension $1 \times |X|$ and $1 \times |Z|$, c is row vector of dimension $1 \times |Y|$, and k_v's are real constants. Similarly, for any $Y' \subseteq Y$, we can write

$$\sum_{v \in Y'} k_v' L_v = [c_1 \ c_2 \ \mathbf{0}], \tag{6}$$

where c_1, c_2 and $\mathbf{0}$ are row vectors respectively of dimension $1 \times |X|$, $1 \times |Z|$ and $1 \times |Y|$, and k_v''s are real constants.

The following result gives information about the rank of a tripartite graph in $\mathcal{T}_n$.

Lemma 3 *Let* $G \in \mathcal{T}_n$ *with tripartition* (X, Y, Z) *and adjacency matrix defined by* (3). *Then*

$$\text{rank}(G) = \text{rank}(U) + \text{rank}(L), \tag{7}$$

where U *and* L *are defined by* (4).

Proof. Let S and Y' be arbitrary subsets, respectively of $X \cup Z$ and Y. To prove (7), it is enough to show that $\sum_{v \in S} k_v U_v \neq \sum_{v \in Y'} k_v' L_v$ whenever $\sum_{v \in S} k_v U_v \neq 0$ and $\sum_{v \in Y'} k_v' L_v \neq 0$, and k_v's and k_v''s are real constants.

We can write $\sum_{v \in S} k_v U_v = [b_1 \ b_2 \ c]$ and $\sum_{v \in Y'} k_v' L_v = [c_1 \ c_2 \ \mathbf{0}]$, where b_1, b_2, c, c_1, c_2 and $\mathbf{0}$ are defined in (5) and (6). By condition (1), there exists a vertex in X which is not adjacent to any vertex in Y. Similarly, there exists a vertex in Z which is not adjacent to any vertex in Y. Thus there are at least two zero columns in C^t corresponding to a vertex in X and to a vertex in Z. That is, there are zero entries in vectors c_1 and c_2. Now, if $\sum_{v \in S} k_v U_v = \sum_{v \in Y'} k_v' L_v$ then $[b_1 \ b_2 \ c] = [c_1 \ c_2 \ \mathbf{0}]$. As b_1 and b_2 are constant vectors, the vectors b_1, b_2, c, c_1, c_2 are all zero vectors. This completes the proof. $\square$

Corollary 4 *Let* $G \in \mathcal{T}_n$ *with tripartition* (X, Y, Z) *and the adjacency matrix* $A(G)$ *defined by* (3). *Then* $\text{rank}(G) = 2(1 + \text{rank}(C))$.

Proof. By the construction of the matrix $A(G)$ and by the arguments used in Lemma 3, we see that $\text{rank}(U) = \text{rank}(B) + \text{rank}(C)$. Since $\text{rank}(B) = 2$ and $\text{rank}(L) = \text{rank}(C) = \text{rank}(C^t)$, we get from (7) that $\text{rank}(G) = 2(1 + \text{rank}(C))$. $\square$

Let $\mathbb{C}_k(\overline{e})$ denote an expanded cycle of length k with an expanded chord $\overline{e}$ joining two non-adjacent expanded vertices of the cycle $\mathbb{C}_k$. We have the following observation.

Lemma 5 *If* $G = \mathbb{C}_5(\overline{e}) \cup kK_1$ *is a graph of order* n *shown in Figure 1,* $k \geq 0$, *then* $G \in \mathcal{T}_n$ *and* $\eta(G) = n - 4$.

Proof. Let $X = X_1 \cup X'$, $Z = Z_1 \cup Z'$ and $Y = Y_1 \cup Y'$, where Y' is possibly empty. Then we see that the graph G is a tripartite graph with tripartition (X, Y, Z). Moreover, G satisfies (1) because $N_{X'}(Y) = \emptyset$ and $N_{Z'}(Y) = \emptyset$. Also, $G[X \cup Z] = \mathbb{P}(X, Z)$, that is, G satisfies (2). Thus $G \in \mathcal{T}_n$. Let $A(G)$ be the adjacency matrix of G defined by (3). By the construction of G, we see that all rows of C^t are identical and therefore $\text{rank}(C) = 1$. By Corollary 4, we conclude that $\eta(G) = n - 4$. $\qquad\square$
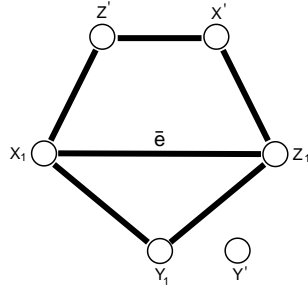

Figure 1: An expanded graph $\mathbb{C}_5(\overline{e}) \cup kK_1$

The next result gives the nullity set of the graphs in $\mathcal{T}_n$.

Theorem 6 *Let* m_1, m_2 *and* m_3 *be positive integers such that* $n = m_1 + m_2 + m_3$. *Then for each integer* $k \in \{0, 1, \ldots, \min\{m_1 + m_3 - 2, m_2\}\}$, *there is a graph* $G \in \mathcal{T}_n$ *with tripartition* (X, Y, Z) *such that* $|X| = m_1$, $|Y| = m_2$, $|Z| = m_3$ *and* $\eta(G) = n - 2(k + 1)$. *Conversely, if* $G \in \mathcal{T}_n$ *with tripartition* (X, Y, Z) *then* $\eta(G) = n - 2(1 + k)$, *where* $k \in \{0, 1, \ldots, \min\{|X| + |Z| - 2, |Y|\}\}$.

Proof. First, we prove that for each $k \in \{0, 1, \ldots, \min\{m_1 + m_3 - 2, m_2\}\}$, there is a graph $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) such that $|X| = m_1$, $|Y| = m_2$, $|Z| = m_3$ and $\eta(G) = n - 2(k+1)$. We take three non-empty sets $X = \{x_1, x_2, \ldots, x_{m_1}\}$, $Y = \{y_1, y_2, \ldots, y_{m_2}\}$ and $Z = \{z_1, z_2, \ldots, z_{m_3}\}$. If $k = 0$, we construct a graph $G = P(X, Z) \cup m_2 K_1$. Clearly, $G \in \mathcal{T}_n$. By using Lemma 2, $\eta(G) = n - 2$. If $k > 0$, we consider following two cases.

Case 1. When $k \leq m_1 - 1$. Since $k \leq m_2$, we construct a tripartite graph G with tripartition (X, Y, Z) that satisfies the following.

(i) $G[X \cup Z] = \mathbb{P}(X, Z)$,

(ii) $|N_X(y_i)| = 1$ and $N_Z(y_i) = \emptyset$ for $1 \leq i \leq k$,

(iii) $N_X(y_i) \cap N_X(y_j) = \emptyset$ for $i \neq j$ and $1 \leq i, j \leq k$,

(iv) $d(y_i) = 0$ for $k + 1 \leq i \leq m_2$.

Then $G \in \mathcal{T}_n$. Moreover, the adjacency matrix of G is given by (3), where

$$C = \left[\begin{array}{c|c} I_{k \times k} & \mathbf{0}_{k \times (m_2 - k)} \\ \hline \mathbf{0}_{(m_1 + m_3 - k) \times k} & \mathbf{0}_{(m_1 + m_3 - k) \times (m_2 - k)} \end{array} \right].$$

Then $\mathrm{rank}(C) = k$. By Corollary 4, we get $\eta(G) = n - 2(1 + k)$.

Case 2. When $k > m_1 - 1$. Since $k \leq m_2$ and $k - (m_1 - 1) \leq m_3 - 1$, we construct a tripartite graph G with tripartition (X, Y, Z) that satisfies the following.

(i) $G[X \cup Z] = \mathbb{P}(X, Z)$,

(ii) $|N_X(y_i)| = 1$ for $1 \leq i \leq m_1 - 1$,

(iii) $|N_Z(y_i)| = 1$ for $m_1 \leq i \leq k$,

(iv) $N_X(y_i) \cap N_X(y_j) = \emptyset$ for $i \neq j$ and $1 \leq i, j \leq m_1 - 1$,

(v) $N_Z(y_i) \cap N_Z(y_j) = \emptyset$ for $i \neq j$ and $m_1 \leq i, j \leq k$,

(vi) $d(y_i) = 0$ for $k + 1 \leq i \leq m_2$.

Then $G \in \mathcal{T}_n$. Moreover, the adjacency matrix of G is given by (3), where

$$C =$$

$$\left[\begin{array}{c|c|c} I_{(m_1-1) \times (m_1-1)} & \mathbf{0}_{(m_1-1) \times (k-m_1+1)} & \mathbf{0}_{(m_1-1) \times (m_2-k)} \\ \hline \mathbf{0}_{1 \times (m_1-1)} & \mathbf{0}_{1 \times (k-m_1+1)} & \mathbf{0}_{1 \times (m_2-k)} \\ \hline \mathbf{0}_{(k-m_1+1) \times (m_1-1)} & I_{(k-m_1+1) \times (k-m_1+1)} & \mathbf{0}_{(k-m_1+1) \times (m_2-k)} \\ \hline \mathbf{0}_{(m_1+m_2-1-k) \times (m_1-1)} & \mathbf{0}_{(m_1+m_2-1-k) \times (k-m_1+1)} & \mathbf{0}_{(m_1+m_2-1-k) \times (m_2-k)} \end{array} \right].$$

Then $\mathrm{rank}(C) = k$. Corollary 4 gives $\eta(G) = n - 2(1 + k)$.

Conversely, we show that if $G \in \mathcal{T}_n$ with tripartition (X, Y, Z), then $\eta(G) = n - 2(1+k)$ where $k \in \{0, 1, \ldots, \min\{|X|+|Z|-2, |Y|\}\}$. By Corollary 4, $\operatorname{rank}(G) = 2(1+\operatorname{rank}(C))$. By (1), there are at least two zero rows in C. This implies that $\operatorname{rank}(C) \leq \min\{|X|+|Z|-2, |Y|\}$. The result is true by setting $\operatorname{rank}(C) = k$. □

From Corollary 4, for each graph $G \in \mathcal{T}_n$ with $A(G)$ defined by (3), we can write

$$\eta(G) = n - 2(1 + \operatorname{rank}(C)). \tag{8}$$

The next result is a direct consequence of Lemma 2.

Theorem 7 *For a graph $G \in \mathcal{T}_n$ with tripartition (X, Y, Z), $\eta(G) = n - 2$ if and only if $G = \mathbb{P}(X, Z) \cup |Y|K_1$.*

Proof. Let $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) and $\eta(G) = n - 2$. Then from equation (8), we have $\operatorname{rank}(C) = 0$. That is, $d(y) = 0$ for all $y \in Y$. Thus $G = \mathbb{P}(X, Z) \cup |Y|K_1$. Conversely, suppose that $G = \mathbb{P}(X, Z) \cup |Y|K_1$. Using Lemma 1, we see that $\eta(G) = n - 2$. □

Theorem 8 *Let $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) and $n \geq 4$. Then $\eta(G) = n - 4$ if and only if G is a graph H possibly with some isolated vertices, where H is an expanded path of length 4 or the expanded graph $\mathbb{C}_5(\bar{e})$.*

Proof. Let $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) and $\eta(G) = n - 4$. Let $A(G)$ be the adjacency matrix of G defined by (3). Then by (8), we have $\operatorname{rank}(C) = 1$, that is, $\operatorname{rank}(L) = 1$. This means that there is only one independent row, say, L_{y_0} in L, where $y_0 \in Y$ is the vertex corresponding to L_{y_0}. Then for each $y \in Y$, either $N(y) = N(y_0)$ or $N(y) = \emptyset$. Let $Y_1 \subseteq Y$ is the set of all vertices of Y with non-zero degree. We have the following three cases.

Case 1. When $N_Z(Y_1) = \emptyset$. In this case, $N(Y_1) \subseteq X$. By condition (1), $N(Y_1) \neq X$. We partition X, Y and Z into Y_1, $N(Y_1)$, Z and $X \setminus N(Y_1)$. Then G can be drawn as an expanded path $\mathbb{P}(Y_1, N(Y_1), Z, X \setminus N(Y_1))$ possibly with some isolated vertices in $Y \setminus Y_1$.

Case 2. When $N_X(Y_1) = \emptyset$. In this case, $N(Y_1) \subseteq Z$. By condition (1), $N(Y_1) \neq Z$. We partition X, Y and Z into Y_1, $N(Y_1)$, X and $Z \setminus N(Y_1)$. Then G can be drawn as an expanded path $\mathbb{P}(Y_1, N(Y_1), X, Z \setminus N(Y_1))$ possibly with some isolated vertices in $Y \setminus Y_1$.

Case 3. When $N_X(Y_1) \neq \emptyset$ and $N_Z(Y_1) \neq \emptyset$. We can partition X into X_1 and X', such that $X_1 = N_X(Y_1)$ and $X' = X \setminus X_1$. Similarly, we can partition Z into Z_1 and Z', such that $Z_1 = N_Z(Y_1)$ and $Z' = Z \setminus Z_1$. Then, using condition (2), one can draw the graph G as an expanded graph $\mathbb{C}_5(\bar{e})$ (shown in Figure 1).

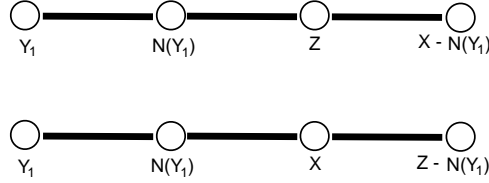There are possibly some isolated vertices in Y', where $Y' = Y \setminus Y_1$.

Figure 2: Two expanded paths $\mathbb{P}(Y_1, N(Y_1), Z, X \setminus N(Y_1))$ and $\mathbb{P}(Y_1, N(Y_1), X, Z \setminus N(Y_1))$ of length 4

Conversely, let G be drawn as an expanded path of length 4 possibly with some isolated vertices. Then Lemma 1 yields that $\eta(G) = n - 4$. Furthermore, if G can be drawn as $\mathbb{C}_5(\overline{e})$ with some isolated vertices, then using Lemma 5, we get $\eta(G) = n - 4$. □

4   Some graphs in $\mathcal{T}_n$ with nullity $n - 6$

In this section, we consider some graphs in $\mathcal{T}_n$, $n \geq 6$, with nullity $n - 6$. Let $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) and let $X' = X \setminus N_X(Y)$. Note that $X' \neq \emptyset$ by (1). We assume that

$$G[N_X(Y) \cup Y] = \mathbb{P}(N_X(Y), Y). \tag{9}$$

The following result gives a characterization of a graph G in $\mathcal{T}_n$, $n \geq 6$ satisfying (9) and $\eta(G) = n - 6$.

Theorem 9 *Let* $G \in \mathcal{T}_n$ *with tripartition* (X, Y, Z), $n \geq 6$. *Assume that* G *satisfies* (9) *and* $\eta(G) = n - 6$. *Then* G *has one of the following expanded decomposition.*

(1) $\mathbb{C}_5(\overline{e})$, $\mathbb{P}_2$,

(2) $\mathbb{C}_5(\overline{e})$, $\mathbb{C}_3$, $\mathbb{P}_2$,

(3) $2\mathbb{C}_5(\overline{e})$, $2\mathbb{P}_2$,

(4) $\mathbb{C}_5(\overline{e})$, $\mathbb{C}_3$, $2\mathbb{P}_2$.

Proof. Let $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) satisfying (9) and $\eta(G) = n - 6$, $n \geq 6$. Let $X' = X \setminus N_X(Y)$ and $Z' = Z \setminus N_Z(Y)$. From (1), we see that X' and Z' are nonempty. Let $A(G)$ be the adjacency matrix of G defined by (3). Since $\eta(G) = n - 6$, using Corollary 4, we have $\text{rank}(L) = 2$. This implies that L has two independent rows, say, L_{y_1} and L_{y_2}, where $y_1, y_2 \in Y$. Using (1) and (9), the columns of L corresponding to the vertices of X are constant. Then for each $y \in Y$, either $L_y = L_{y_1}$ or $L_y = L_{y_2}$. Thus we partition Y into Y_1 and Y_2, where

$$Y_1 = \{y \in Y \mid L_y = L_{y_1}\}, Y_2 = \{y \in Y \mid L_y = L_{y_2}\}.$$

Note that $N_Z(y) = N_Z(Y_1)$ for each $y \in Y_1$, and $N_Z(y) = N_Z(Y_2)$ for each $y \in Y_2$. Since $\text{rank}(L) = 2$, either $N_Z(Y_1) \neq \emptyset$ or $N_Z(Y_2) \neq \emptyset$. Without loss of generality, assume that $N_Z(Y_1) \neq \emptyset$ and $N_Z(Y_1) \not\subseteq N_Z(Y_2)$. The following three cases are possible.

Case 1. When $N_Z(Y_1) \cap N_Z(Y_2) = \emptyset$.
If $N_Z(Y_2) = \emptyset$, then $N_Z(Y) = N_Z(Y_1)$. We draw the graph G as an expanded graph on six expanded vertices $X_1 = N_X(Y)$, X', $Z_1 = N_Z(Y)$, Z', Y_1 and Y_2. Here Y' is possibly empty. The graph is shown in Figure 3 (i). In this case, we can decompose the graph G into $\mathbb{C}_5(\bar{e})$ and $\mathbb{P}_2$.

If $N_Z(Y_2) \neq \emptyset$, we partition $N_Z(Y)$ into $Z_1 = N_Z(Y_1)$ and $Z_2 = N_Z(Y_2)$. We draw the graph G as an expanded graph on seven expanded vertices $X_1 = N_X(Y)$, X', $Z_1 = N_Z(Y_1)$, $Z_2 = N_Z(Y_2)$, Z', Y_1 and Y_2. The graph is shown in Figure 3 (ii). In this case, the graph can be decomposed into $\mathbb{C}_5(\bar{e})$, $\mathbb{C}_3$ and $\mathbb{P}_2$.

Case 2. When $N_Z(Y_1) \cap N_Z(Y_2) \neq \emptyset$ and $N_Z(Y_2) \not\subseteq N_Z(Y_1)$.
Let $Z_1 = N_Z(Y_1) \cap N_Z(Y_2)$, $Z_2 = N_Z(Y_1) \setminus N_Z(Y_2)$ and $Z_3 = N_Z(Y_2) \setminus N_Z(Y_1)$. Then Z', Z_1, Z_2 and Z_3 form a partition of Z. The graph can be drawn as an expanded graph on eight expanded vertices $X_1 = N_X(Y)$, X', Z_1, Z_2, Z_3, Z', Y_1 and Y_2. The graph is shown in Figure 4 (i). In this case, the graph G can be decomposed into $2\mathbb{C}_5(\bar{e})$ and $2\mathbb{P}_2$.

Case 3. When $N_Z(Y_1) \cap N_Z(Y_2) \neq \emptyset$ and $N_Z(Y_2) \subseteq N_Z(Y_1)$.
We draw the graph G as an expanded graph on seven expanded vertices $X_1 = N_X(Y)$, X', $Z_1 = N_Z(Y_2)$, $Z_2 = N_Z(Y_1) \setminus N_Z(Y_2)$, Z', Y_1 and Y_2. The graph is shown in Figure 4 (ii). The graph G can be decomposed into $\mathbb{C}_5(\bar{e})$, $\mathbb{C}_3$ and $2\mathbb{P}_2$. $\qquad \square$
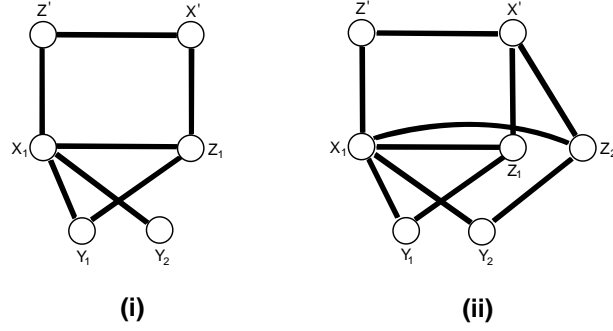
Figure 3: Graphs drawn in Case 1 with $N(Y_1) \cap N(Y_2) = \emptyset$
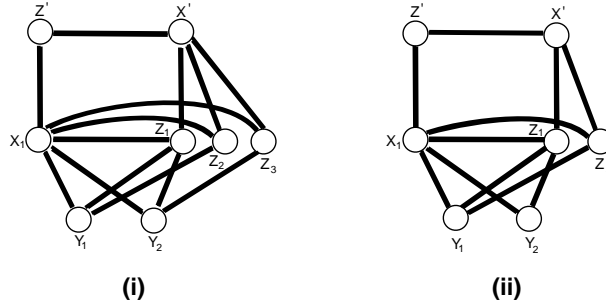
Figure 4: Graphs drawn in Case 2 and Case 3 with $N(Y_1) \cap N(Y_2) \neq \emptyset$

5   Conclusion

We studied n-vertex tripartite graphs satisfying (1) and (2). We obtained the nullity set of this class of n-vertex tripartite graphs and characterized them with nullity $n - 4$. It will be interesting to consider a more general class of n-vertex tripartite graphs and to characterize them with their nullity. In Theorem 9, we characterized those n-vertex tripartite graphs whose nullity is $n - 6$ and that satisfy (1), (2) and (9). We are not sure about the converse of Theorem 9 and it is left as an open problem.

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