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On the metric dimension of strongly annihilating-ideal graphs of commutative rings

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Abstract. Let R be a commutative ring with identity and $A(R)$ be the set of ideals with non-zero annihilator. The strongly annihilating-ideal graph of R is defined as the graph $SAG(R)$ with the vertex set $A(R)^* = A(R) \setminus \{0\}$ and two distinct vertices I and J are adjacent if and only if $I \cap Ann(J) \neq (0)$ and $J \cap Ann(I) \neq (0)$. In this paper, we study the metric dimension of $SAG(R)$ and some metric dimension formulae for strongly annihilating-ideal graphs are given.

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1 Introduction

The problem of finding the metric dimension of a graph was first studied by Harary and Melter [7]. Determining the metric dimension of a graph as an NP-complete problem has attracted many graph theorists and it has appeared in various applications of graph theory, for example pharmaceutical chemistry [5], robot navigation [8], combinatorial optimization [14] and so on. Recently, there was much work done in computing the metric dimension of graphs associated with algebraic structures. Calculating the metric dimension for the commuting graph of a dihedral group was done in [1], for the zero-divisor graphs of commutative rings in [9, 10, 12], for the compressed zero-divisor graphs of commutative rings in [13], for total graphs of finite commutative rings in [6], for some graphs of modules in [11] and for annihilator graphs of commutative rings in [15]. Motivated by these papers, we study the metric dimension of another graph associated with a commutative ring.

Throughout this paper, all rings are assumed to be commutative with identity. The sets of all zero-divisors, nilpotent elements and maximal ideals are denoted by $Z(R)$, $\mathrm{Nil}(R)$ and $\mathrm{Max}(R)$, respectively. For a subset T of a ring R we let $T^* = T\setminus\{0\}$. An ideal with non-zero annihilator is called an *annihilating-ideal*. The set of annihilating-ideals of R is denoted by $A(R)$. For every subset I of R, we denote the *annihilator* of I by $\mathrm{Ann}(I)$. A non-zero ideal I of R is called *essential* if I has a non-zero intersection with every other non-zero ideal of R. The set of essential annihilating-ideal ideals of R is denoted by $\mathrm{Ess}(R)$. The ring R is said to be *reduced* if it has no non-zero nilpotent element. Some more definitions about commutative rings can be find in [2, 4].

We use the standard terminology of graphs following [18]. Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. We recall that a graph is *connected* if there exists a path connecting any two distinct vertices. The *distance* between two distinct vertices x and y, denoted by $d(x, y)$, is the length of the shortest path connecting them (if such a path does not exist, then we set $d(x, y) = \infty$). The *diameter* of a connected graph G, denoted by $\mathrm{diam}(G)$, is the maximum distance between any pair of vertices of G. For a vertex x in G, we denote the set of all vertices adjacent to x by $N(x)$ and $N[x] = N(x) \cup \{x\}$. A k-*partite* graph is one whose vertex set can be partitioned into k subsets so that an edge has both ends in no subset. A *complete* k-*partite* graph is a k-partite graph in which each vertex is adjacent to every vertex that is not in the same subset. The complete bipartite (i.e., 2-partite) graph with part sizes m and n is denoted by $K_{m,n}$. If $m = 1$, then the bipartite graph is called star. A graph in which each pair of vertices is joined by an edge is called a *complete* graph and use K_n to denote it with n vertices and its complement is denoted by $\overline{K}_n$ (possibly n is zero). Also, a cycle of order n is denoted by C_n. A subset of vertices $S \subseteq V(G)$ resolves a graph G, and S is a *resolving set of* G, if every vertex is uniquely determined by its vector of distances to the vertices of S. In general, for an ordered subset $S = \{v_1, v_2, \ldots, v_k\}$ of vertices in a connected graph G and a vertex $v \in V(G) \setminus S$ of G, the metric representation of v with respect to S is the k-vector $D(v|S) = (d(v, v_1), d(v, v_2), \ldots, d(v, v_k))$. The set S is a resolving set for G if $D(u|S) = D(v|S)$ implies that $u = v$, for all pair of vertices, $v, u \in V(G) \setminus S$. A resolving set S of minimum cardinality is the *metric basis for* G, and the number of elements in the resolving set of minimum cardinality is the *metric dimension of* G. We denote the metric dimension of a graph G by $\dim_M(G)$. Let G be a connected graph such that $|V(G)| \geq 2$. Two distinct vertices u and v are *distance similar*, if $d(u, x) = d(v, x)$, for all $x \in V(G) \setminus \{u, v\}$. It can be easily checked that two distinct vertices u and v are distance similar if either $u - v \notin E(G)$ and $N(u) = N(v)$ or $u - v \in E(G)$ and $N[u] = N[v]$.

Let R be a commutative ring with identity and $A(R)$ be the set of ideals with non-zero annihilator. The *strongly annihilating-ideal graph of* R is defined as the graph SAG(R) with the vertex set $A(R)^* = A(R) \setminus \{0\}$ and two distinct vertices I and J are adjacent if and only if $I \cap \mathrm{Ann}(J) \neq (0)$ and $J \cap \mathrm{Ann}(I) \neq (0)$. This graph was first introduced and studied in [16, 17]. It is worthy to mention that strongly annihilating-ideal graph is a generalization of annihilating-ideal graph. The *annihilating-ideal graph* of R, denoted by $\mathbb{AG}(R)$, is a graph with the vertex set $A(R)^*$ and two distinct vertices I and J are adjacent if and only if $IJ = 0$ (see [3] for more details). In this paper, we study the metric dimension of SAG(R) and we provide some metric dimension formulas for SAG(R).

2 Metric dimension of a strongly annihilating-ideal graph of a reduced ring

Let R be a commutative ring. In this section, we provide a metric dimension formula for a strongly annihilating-ideal graph when R is reduced.

Lemma 1 *Let R be a ring which is not an integral domain. Then $\dim_M(\mathrm{SAG}(R))$ is finite if and only if R has only finitely many ideals.*

Proof. One side is clear. To prove the other side, suppose that $\dim_M(\mathrm{SAG}(R))$ is finite and let $W = \{I_1, I_2, \ldots, I_n\}$ be the metric basis for SAG(R), where n

is a non-negative. By [16, Theorem 2.1], $diam(SAG(R)) \leq 2$ and so for every $I \in A(R)^* \backslash W$, there are $(2+1)^n$ possibilities for $D(I|W)$. Thus $|A(R)^*| \leq 3^n + n$ and hence R has only finitely many ideals. □

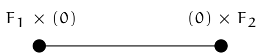
If R is a reduced ring with finitely many ideals, then by [2, Theorem 8.7], R is a direct product of finitely many fields. Using this fact, we prove the following result.

Theorem 1 *Let R be a reduced ring which is not an integral domain. If $dim_M(SAG(R))$ is finite, then:*
(1) *If $|Max(R)| \leq 3$, then $dim_M(SAG(R)) = |Max(R)| - 1$.*
(2) *If $|Max(R)| \geq 4$, then $dim_M(SAG(R)) = |Max(R)|$.*

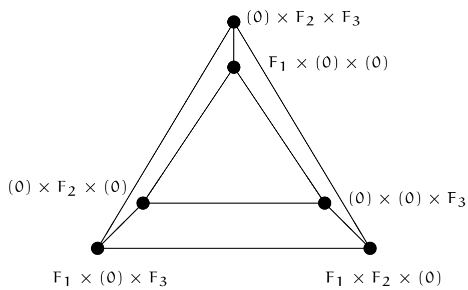
Proof. (1) Since $dim_M(SAG(R))$ is finite, R has only finitely many ideals, by Lemma 1. Also, since R is not an integral domain, $|Max(R)| \neq 1$. Hence $|Max(R)| = 2$ or 3. If $|Max(R)| = 2$, then $R \cong F_1 \times F_2$, where F_i is a field. Thus $SAG(R) = K_2$ and so $dim_M(SAG(R)) = 1$. If $|Max(R)| = 3$, then $R \cong F_1 \times F_2 \times F_3$, where F_i is a field for every $1 \leq i \leq 3$. Let $W = \{F_1 \times (0) \times F_3, F_1 \times F_2 \times (0)\}$. By the following figure, one may easily get

$D((0) \times F_2 \times (0)|W) = (1, 2)$,
$D(F_1 \times (0) \times (0)|W) = (2, 2)$,
$D((0) \times (0) \times F_3|W) = (2, 1)$,
$D((0) \times F_2 \times F_3|W) = (1, 1)$.

So for every $x, y \in V(SAG(R)) \backslash W$, $D(x|W) \neq D(y|W)$ and hence $dim_M(SAG(R)) = 2$.



$SAG(F_1 \times F_2)$



$SAG(F_1 \times F_2 \times F_3)$

(2) Assume that $|Max(R)| = n \geq 4$. By Lemma 1, $R \cong F_1 \times \cdots \times F_n$, where

F_i is a field for every $1 \leq i \leq n$. We show that $\mathrm{dim}_M(\mathrm{SAG}(R)) = n$. Indeed, we have the following claims:

Claim 1. $\mathrm{dim}_M(\mathrm{SAG}(R)) \geq n$.

Since $R \cong F_1 \times \cdots \times F_n$, by Lemma 1, $\mathrm{dim}_M(\mathrm{SAG}(R))$ is finite. Let $W = \{I_1, I_2, \dots, I_k\}$ be the metric basis for $\mathrm{SAG}(R)$, where k is a positive integer. On the other hand, by [16, Theorem 2.1], $\mathrm{diam}(\mathrm{SAG}(R)) \in \{1, 2\}$, and so for every $I \in A(R)^* \setminus W$, there are 2^k possibilities for $D(I|W)$. This implies that $|A(R)^*| - k \leq 2^k$. Since $|A(R)^*| = 2^n - 2$, $2^n - 2 - k \leq 2^k$ and hence $2^n \leq 2^k + 2 + k$. Since $n \geq 4$, we deduce that $k \geq n$. Therefore $\mathrm{dim}_M(\mathrm{SAG}(R)) \geq n$.

Claim 2. $\mathrm{dim}_M(\mathrm{SAG}(R)) \leq n$.

For every $1 \leq i \leq n$, let $(F_1, \dots, F_{i-1}, 0, F_{i+1}, \dots, F_n) = \mathfrak{m}_i \in A(R)^*$. Put $W = \{\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n\}$ (in fact $W = \mathrm{Max}(R)$). We show that W is the resolving set for $\mathrm{SAG}(R)$. To see this, let $I, J \in V(\mathrm{SAG}(R)) \setminus W$ and $I \neq J$. We need only to show that $D(I|W) \neq D(J|W)$. Let $I = (I_1, I_2, \dots, I_n)$ and $J = (J_1, J_2, \dots, J_n)$. Since $I \neq J$, $I_i = 0$ and $J_i = F_i$ or $I_i = F_i$ and $J_i = 0$, for some $1 \leq i \leq n$. Without loss of generality, assume that $I_1 = 0$ and $J_1 = F_1$. It is easy to see that $d(I, \mathfrak{m}_1) = 1$ and $d(J, \mathfrak{m}_1) = 2$. This clearly shows that $D(I|W) \neq D(J|W)$. Therefore $\mathrm{dim}_M(\mathrm{SAG}(R)) \leq n$.

Now, by Claims 1, 2, $\mathrm{dim}_M(\mathrm{SAG}(R)) = n$, for $n \geq 4$. □

3 Metric dimension of a strongly annihilating-ideal graph of a non-reduced ring

In this section, we discuss the metric dimension of strongly annihilating-ideal graphs for non-reduced rings. First we need to recall two lemmas from [16].

Lemma 2 [16, Lemma 2.1] *Let R be a ring and $I, J \in A(R)^*$. Then the following statements hold.*

(1) *If $I - J$ is not an edge of $\mathrm{SAG}(R)$, then $\mathrm{Ann}(IJ) = \mathrm{Ann}(I)$ or $\mathrm{Ann}(IJ) = \mathrm{Ann}(J)$. Moreover, if R is a reduced ring, then the converse is also true.*

(2) *If $I - J$ is an edge of $\mathbb{AG}(R)$, then $I - J$ is an edge of $\mathrm{SAG}(R)$.*

(3) *If $\mathrm{Ann}(I) \not\subseteq \mathrm{Ann}(J)$ and $\mathrm{Ann}(J) \not\subseteq \mathrm{Ann}(I)$, then $I - J$ is an edge of $\mathrm{SAG}(R)$. Moreover if R is a reduced ring, then the converse is also true.*

(4) *Let* $n \geq 1$ *be a positive integer. Suppose that* $R \cong R_1 \times \cdots \times R_n$*, where* R_i *is a ring, for every* $1 \leq i \leq n$*, and* $I = (I_1, \ldots, I_n)$ *and* $J = (J_1, \ldots, J_n)$ *are two vertices of* $\mathrm{SAG}(R)$*. If* $I_i \cap \mathrm{Ann}(J_i) \neq (0)$ *and* $J_j \cap \mathrm{Ann}(I_j) \neq (0)$*, for some* $1 \leq i, j \leq n$*, then* $I - J$ *is an edge of* $\mathrm{SAG}(R)$*. In particular, if* $I_i - J_i$ *is an edge of* $\mathrm{SAG}(R_i)$ *or* $I_i = J_i$ *and* $I_i \cap \mathrm{Ann}(I_i) \neq (0)$*, for some* $1 \leq i \leq n$*, then* $I - J$ *is an edge of* $\mathrm{SAG}(R)$*.*

(5) *If* $I, J \in \mathrm{Ess}(R)$ *or* $\mathrm{Ann}(I), \mathrm{Ann}(J) \in \mathrm{Ess}(R)$*, then* I *is adjacent to* J*.*

(6) *If* $d_{\mathbb{AG}(R)}(I, J) = 3$ *for some distinct* $I, J \in A(R)^*$*, then* $I - J$ *is an edge of* $\mathrm{SAG}(R)$*.*

(7) *If* $I - J$ *is not an edge of* $\mathrm{SAG}(R)$ *for some distinct* $I, J \in A(R)^*$*, then* $d_{\mathbb{AG}(R)}(I, J) = 2$*.*

Lemma 3 [16, Lemma 2.2] *Let* R *be a non-reduced ring and* I *be an ideal of* R *such that* $I^n = (0)$*, for some positive integer* n*. Then* $\mathrm{Ann}(I)$ *is an essential ideal of* R*.*

Remark 1 *Let* G *be a connected graph and* $V_1, V_2, \ldots, V_k$ *be a partition of* $V(G)$ *such that for every* $1 \leq i \leq k$*, if* $x, y \in V_i$*, then* $N(x) = N(y)$*. Then* $\dim_M(G) \geq |V(G)| - k$*.*

Next, we provide some formulas for the metric dimension of strongly annihilating-ideal graphs for non-reduced rings.

Theorem 2 *Suppose that* $R \cong R_1 \times \cdots \times R_n$*, where* R_i *is an Artinian local ring such that for every* $1 \leq i \leq n$*,* $|A(R_i)^*| = 1$*. Then* $\dim_M(\mathrm{SAG}(R)) = 2n$*.*

Proof. Assume that $X = (R_1, 0, \ldots, 0)$ and $Y = (I_1, 0, \ldots, 0)$, where $I_1 \in A(R_1)^*$. By Part 4 of Lemma 2, it is easy to see that $N(X) = N(Y)$. This implies that if W is the metric basis for $\mathrm{SAG}(R)$, then $X \in W$ or $Y \in W$. Without loss of generality, we may assume that $X \in W$. Similarly, we may assume that $W_1 \subseteq W$, where $W_1 = \{(R_1, 0, \ldots, 0), (0, R_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, R_n)\}$.

Now, assume that $X = (0, R_2, \ldots, R_n)$ and $Y = (I_1, R_2, \ldots, R_n)$, where $I_1 \in A(R_1)^*$. It is easy to see that $N(X) = N(Y)$ and so if W is the metric basis for $\mathrm{SAG}(R)$, then $X \in W$ or $Y \in W$. Without loss of generality, we may assume that $X \in W$. Similarly, we may assume that $W_2 \subseteq W$, where

$$W_2 = \{(0, R_2, \ldots, R_n), (R_1, 0, R_3, \ldots, R_n), \ldots, (R_1, \ldots, R_{n-1}, 0)\}.$$

Since $|W_1| = |W_2| = n$ and $W_1 \cup W_2 \subseteq W$, $|W| \geq 2n$. We show that $|W| \leq 2n$. For this, it is enough to show that W is a resolving set and consequently it is the metric basis for the graph SAG(R). Let $X, Y \notin W$, $X \neq Y$, $X = (I_1, \ldots, I_n)$ and $Y = (J_1, \ldots, J_n)$. We show that $D(X|W) \neq D(Y|W)$. Since $X \neq Y$, for some $1 \leq i \leq n$, we conclude that $I_i \neq J_i$. Without loss of generality, one may assume that $I_1 \supset J_1$. We have the following cases:

Case 1. $I_1 = R_1$.

Subcase 1. For some $2 \leq j \leq n$, $J_j \neq 0$. In this case, $Z - Y$ is an edge of SAG(R) but $Z - X$ is not an edge of SAG(R), where $Z = (R_1, 0, \ldots, 0)$. Since $Z \in W$, we deduce that $D(X|W) \neq D(Y|W)$.

Subcase 2. For every $2 \leq j \leq n$, $J_j = 0$. Since $I_1 = R_1$ and $(R_1, 0, \ldots, 0) \in W$, for some $2 \leq i \leq n$, $I_i \neq 0$. If $I_i = R_i$, for some $2 \leq i \leq n$, then $Z - Y$ is an edge of SAG(R) but $Z - X$ is not an edge of SAG(R), where $Z = (0, \ldots, 0, R_i, 0 \ldots, 0)$. So we can let for every $2 \leq i \leq n$, $I_i \neq R_i$. Now, without loss of generality, we may assume that $I_2 \neq 0$. Obviously, $Z - X$ is an edge of SAG(R) but $Z - Y$ is not an edge of SAG(R), where $Z = (R_1, 0, R_3 \ldots, R_n)$. Since $Z \in W$, $D(X|W) \neq D(Y|W)$.

Case 2. $I_1 \neq R_1$. Since $I_1 \neq R_1$, $J_1 \neq R_1$. Also, since $X \neq Y$, we may let $I_1 \in A(R_1)^*$ and $J_1 = 0$. If $I_i \neq R_i$, for some $2 \leq i \leq n$, then $Z - X$ is an edge of SAG(R) but $Z - Y$ is not an edge of SAG(R), where $Z = (0, R_2, R_3 \ldots, R_n)$. Since $Z \in W$, $D(X|W) \neq D(Y|W)$. So let $X = (I_1, R_2, \ldots, R_n)$. Since $J_1 = 0$ and $Y \notin W$, for some $2 \leq i \leq n$, $J_i \in A(R_1)^*$. Without loss of generality, we may assume that $J_2 \in A(R_2)^*$. If $J_i \neq 0$, for some $3 \leq i \leq n$, then we put $Z = (0, R_2, \ldots, R_{i-1}, 0, R_{i+1}, \ldots, R_n)$. It is not hard to check that $Z - Y$ is an edge of SAG(R) but $Z - X$ is not an edge of SAG(R). If for every $3 \leq i \leq n$, $J_i = 0$, then we put $Z = (R_1, R_2, \ldots, 0, \ldots, 0)$. In both cases we have that $D(X|W) \neq D(Y|W)$. Therefore, $|W| \leq 2n$. □

Theorem 3 *Suppose that* $R \cong R_1 \times \cdots \times R_n$, *where* R_i *is an Artinian local ring such that for every* $1 \leq i \leq n$, $|A(R_i)^*| \geq 2$. *Then* $\dim_M(\mathrm{SAG}(R)) = |A(R)^*| - 3^n + 2$.

Proof. If R is local, then Lemma 3 implies that SAG(R) is complete and hence $\dim_M(\mathrm{SAG}(R)) = |A(R)^*| - 1$. So let $R \cong R_1 \times \cdots \times R_n$ and $n \geq 2$. Assume that

$X = (I_1, \ldots, I_n)$, $Y = (J_1, \ldots, J_n)$ are vertices of SAG(R). Define the relation $\sim$ on V(SAG(R)) as follows: $X \sim Y$, whenever, the following two conditions hold.

(1) "$I_i = 0$ if and only if $J_i = 0$" for every $1 \leq i \leq n$.

(2) "$0 \neq I_i \subseteq \mathrm{Nil}(R_i)$ if and only if $0 \neq J_i \subseteq \mathrm{Nil}(R_i)$" for every $1 \leq i \leq n$.

It is easily seen that $\sim$ is an equivalence relation on $V(\mathrm{SAG}(R))$. By [X], we mean the equivalence class of X. Let X_1 and X_2 be two elements of [X]. Since $X_1 \sim X_2$, by Part 4 of Lemma 2, $N(X_1) = N(X_2)$. This, together with the fact that the number of equivalence classes is $3^n - 2$ and Remark 1, implies that

$$\dim_M(\mathrm{SAG}(R)) \geq |A(R)^*| - (3^n - 2) = |A(R)^*| - 3^n + 2.$$

We show that

$$\dim_M(\mathrm{SAG}(R)) \leq |A(R)^*| - 3^n + 2.$$

Let

$A = \{(I_1, \ldots, I_n) \in V(\mathrm{SAG}(R)) \,|\, I_i \in \{0, \mathrm{Nil}(R_i), \ldots, R_i\} \text{ for every } 1 \leq i \leq n\}$ and $W = A(R)^* \setminus A$.

It is shown that W is a resolving set and consequently it is the metric basis for the graph $\mathrm{SAG}(R)$. To see this, let $X, Y \in A$ and $X \neq Y$. We show that $D(X|W) \neq D(Y|W)$. Let $X = (I_1, \ldots, I_n)$ and $Y = (J_1, \ldots, J_n)$. Since $X \neq Y$, for some $1 \leq i \leq n$, $I_i \neq J_i$. Without loss of generality, we may assume that $I_1 \supset J_1$. We have the following cases:

Case 1. $I_1 = R_1$.

Subcase 1. $J_1 = 0$. In this case $Z - X$ is an edge of $\mathrm{SAG}(R)$ but $Z - Y$ is not an edge of $\mathrm{SAG}(R)$, where $Z = (I'_1, R_2, \ldots, R_n)$ and $I'_1 \in A(R_1)^* \setminus \{\mathrm{Nil}(R_1)\}$. Since $Z \in W$, $D(X|W) \neq D(Y|W)$. **Subcase 2.** $J_1 = \mathrm{Nil}(R_1)$. In this case $Z - Y$ is an edge of $\mathrm{SAG}(R)$ but $Z - X$ is not an edge of $\mathrm{SAG}(R)$, where $Z = (J'_1, 0, \ldots, 0)$, $J'_1 \in A(R_1)^*$ and $J'_1 \neq \mathrm{Nil}(R_1)$. Since $Z \in W$, $D(X|W) \neq D(Y|W)$.

Case 2. $I_1 = \mathrm{Nil}(R_1)$.

Since $I_1 \neq J_1$ and $I_1 \supseteq J_1$, $J_1 = 0$. Hence $Z - X$ is an edge of $\mathrm{SAG}(R)$ but $Z - Y$ is not an edge of $\mathrm{SAG}(R)$, where $Z = (J'_1, R_2, \ldots, R_n)$ and $J'_1 \in A(R_1)^*$ and $J'_1 \neq \mathrm{Nil}(R_1)$. Since $Z \in W$, $D(X|W) \neq D(Y|W)$. Therefore,

$$\dim_M(\mathrm{SAG}(R)) \leq |W|.$$

Since $|A| = 3^n - 2$, $|W| = |A(R)^*| - (3^n - 2) = |A(R)^*| - 3^n + 2$. Therefore,

$$\dim_M(\mathrm{SAG}(R)) \leq |A(R)^*| - 3^n + 2.$$

□

Next, we provide some upper and lower bounds for the metric dimension of strongly annihilating-ideal graphs for some other classes of non-reduced rings.

Theorem 4 *Suppose that* $R \cong R_1 \times \cdots \times R_n \times F_{n+1} \times \cdots \times F_{n+m}$, *where* R_i *is an Artinian local ring such that* $|A(R_i)| = 2$ *for every* $1 \leq i \leq n$ *and* F_i *is a field for every* $1+n \leq i \leq n+m$. *Then* $n+m \leq \dim_M(\mathrm{SAG}(R)) \leq 2^{n+m}-2$.

Proof. Suppose that $W = \{I_1, I_2, \ldots, I_k\}$ be the metric basis for SAG(R), for some non-negative integer k. Since $\mathrm{diam}(\mathrm{SAG}(R)) \leq 2$, there are exactly $(2)^k$ possibilities for $D(I|W)$, for every $I \in A(R)^* \setminus W$. On the other hand, since $|A(R)^*| = 3^n 2^m - 2$, we must have $3^n 2^m - 2 - k \leq 2^k$. This implies that $n+m \leq k$. Hence $n+m \leq \dim_M(\mathrm{SAG}(R))$. It is shown that $\dim_M(\mathrm{SAG}(R)) \leq 2^{n+m} - 2$. Let

$W = \{(I_1, \ldots, I_{n+m}) \in V(\mathrm{SAG}(R)) \mid I_i \in \{0, R_1, \ldots, R_n, F_1, \ldots, F_m\}$ for every $1 \leq i \leq n+m\}$.

We show that W is a resolving set for SAG(R). For this, let $X, Y \in A(R)^* \setminus W$ and $X \neq Y$. We show that $D(X|W) \neq D(Y|W)$. Let $X = (I_1, \ldots, I_{n+m})$ and $Y = (J_1, \ldots, J_{n+m})$. Since $X \neq Y$, $I_i \neq J_i$, for some $1 \leq i \leq n+m$.

We have the following cases:

Case 1. For some $n+1 \leq i \leq n+m$, $I_i \neq J_i$.

Without loss of generality, we may assume that $i = n+m$, $I_{n+m} = F_{n+m}$ and $J_{n+m} = 0$. Now, put $Z = (R_1, \ldots, R_n, F_{n+1} \ldots, F_{n+m-1}, 0)$. Since for some $1 \leq i \leq n$, $I_i \in A(R_i)^*$, one may easily see that $Z - X$ is an edge of SAG(R) but $Z - Y$ is not an edge of SAG(R). Since $Z \in W$, $D(X|W) \neq D(Y|W)$.

Case 2. For every $n+1 \leq i \leq n+m$, $I_i = J_i$.

Since $I_i \neq J_i$, for some $1 \leq i \leq n$, one can let $J_1 \subset I_1$. Thus we have the following subcases:

Subcase 1. $J_1 = 0$ and $I_1 \in A(R_1)^*$.

Since $J_1 = 0$, for some $2 \leq i \leq n$, $J_i \in A(R_i)^*$. Hence one can let $J_2 \in A(R_2)^*$. If for some $2 \leq i \leq n$, $I_i \neq R_i$ or for some $1+m \leq i \leq n+m$, $I_i \neq F_i$, then put $Z = (0, R_2, R_3 \ldots, R_n, F_{n+1}, \ldots, F_{n+m})$. $Z - X$ is an edge of SAG(R) but $Z - Y$ is not an edge of SAG(R). Since $Z \in W$, $D(X|W) \neq D(Y|W)$. So we let $X = (I_1, R_2 \ldots, R_n, F_{n+1}, \ldots, F_{n+m})$ Similarly, if for some $3 \leq i \leq n$, $J_i \neq R_i$ or for some $1+m \leq j \leq n+m$, $J_i \neq F_i$, then without loss of generality, we may assume that $J_3 \neq R_3$. Then put $Z = (0, 0, R_3 \ldots, R_n, F_{n+1}, \ldots, F_{n+m})$. Thus $Z - Y$ is an edge of SAG(R) but $Z - X$ is not an edge of SAG(R). Since $Z \in W$, $D(X|W) \neq D(Y|W)$. Now, let $X = (I_1, R_2 \ldots, R_n, F_{n+1}, \ldots, F_{n+m})$ and $Y = (0, J_2, R_3, \ldots, R_n, F_{n+1}, \ldots, F_{n+m})$. Put $Z = (0, R_2, 0 \ldots, 0, 0, \ldots, 0)$. Therefore, $Z - Y$ is an edge of SAG(R) but $Z - X$ is not an edge of SAG(R). Since $Z \in W$, $D(X|W) \neq D(Y|W)$.

Subcase 2. $J_1 = 0$ and $I_1 = R_1$.

Since $J_1 = 0$, for some $2 \leq i \leq n$, $J_i \in A(R_i)^*$. Hence one may let $J_2 \in A(R_2)^*$. Assume that $Z = (R_1, 0, \dots, 0)$. Thus $Z - Y$ is an edge of SAG(R) but $Z - X$ is not an edge of SAG(R) (note that since $Z \in W$, $Z \neq X$). This implies that $D(X|W) \neq D(Y|W)$.

Subcase 3. $J_1 \in A(R_1)^*$ and $I_1 = R_1$. If $J_i \neq 0$, for some $2 \leq i \leq n$, then one may assume that $J_2 \neq 0$. Suppose that $Z = (R_1, 0, \dots, 0)$. Then $Z-Y$ is an edge of SAG(R) but $Z-X$ is not an edge of SAG(R). Hence $D(X|W) \neq D(Y|W)$. Let $Y = (J_1, 0, \dots, 0)$. Since $X \notin W$, for some $2 \leq i \leq n$, $I_i \in A(R_i)^*$. So, we can let $I_2 \in A(R_2)^*$. If $I_i \neq 0$, for some $3 \leq i \leq n$, then we can assume that $I_3 \neq 0$. If we put $Z = (R_1, R_2, 0, \dots, 0)$, then we easily get $D(X|W) \neq D(Y|W)$. Finally, if $X = (R_1, I_2, 0, \dots, 0)$ and $Y = (J_1, 0, \dots, 0)$, then $D(X|W) \neq D(Y|W)$. Since $Z - X$ is an edge of SAG(R) but $Z - Y$ is not an edge of SAG(R), where $Z = (R_1, 0, R_3, 0 \dots, 0)$. Therefore, $\mathrm{dim}_M(\mathrm{SAG}(R)) \leq |W|$. Since $|W| = 2^{n+m} - 2$, $\mathrm{dim}_M(\mathrm{SAG}(R)) \leq 2^{n+m} - 2$. □

We end this paper with the following example.

Example 1 (1) *Let* $R = \mathbb{Z}_4 \times \mathbb{Z}_2$. *Then* $\mathrm{SAG}(R) = C_4$ *and hence* $\mathrm{dim}_M(\mathrm{SAG}(R)) = 2$. *Also, in Theorem 4,* $n = m = 1$, *and so* $\mathrm{dim}_M(\mathrm{SAG}(R)) = 2$.

(2) *Let* $R = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $\mathrm{dim}_M(\mathrm{SAG}(R)) = k$. *We show that* $3 \leq k \leq 6$. *Since* $\mathrm{diam}(\mathrm{SAG}(R)) \leq 2$ *and* $|A(R)^*| = 10$, $10 - k \leq 2^k$. *Thus* $k \geq 3$. *Let* $W = \{((2), \mathbb{Z}_2, \mathbb{Z}_2), ((2), 0, \mathbb{Z}_2), ((2), \mathbb{Z}_2, 0), ((2), 0, 0)\}$. *Then*

$D((\mathbb{Z}_4, 0, 0)|W) = (1, 1, 1, 2)$,

$D((\mathbb{Z}_4, \mathbb{Z}_2, 0))|W) = (1, 1, 2, 2)$,

$D((\mathbb{Z}_4, 0, \mathbb{Z}_2))|W) = (1, 2, 1, 2)$,

$D((0, \mathbb{Z}_2, \mathbb{Z}_2)|W) = (2, 1, 1, 1)$,

$D((0, \mathbb{Z}_2, 0)|W) = (2, 1, 2, 1)$,

$D((0, 0, \mathbb{Z}_2)|W) = (2, 2, 1, 1)$.

Therefore, W *is a resolving set for* SAG(R) *and hence* $k \leq 6$.

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