Acta Univ. Sapientiae, Mathematica, **3**, 2 (2011) 115–128

On the distribution of q-additive functions under some conditions III.

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Abstract. The existence of the limit distribution of a q-additive function over the set of integers characterized by the sum of digits is investigated.

1 Introduction

Notation

$\mathbb{N}, \mathbb{R}, \mathbb{C}$, as usual denote the set of natural, real and complex numbers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

q-additive and q-multiplicative functions

Let $q \geq 2$ be an integer, the q-ary expansion of $n \in \mathbb{N}_0$ is defined as

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j, \tag{1}$$

where the digits $\varepsilon_j(n)$ are taken from $\mathbb{A}_q = \{0, 1, \ldots, q-1\}$. It is clear that the right hand side of (1) is finite.

Let $\mathcal{A}_q$ be the set of q-additive, and $\mathcal{M}_q$ be the set of q-multiplicative functions.

2010 Mathematics Subject Classification: 11K65, 11P99, 11N37
Key words and phrases: q-additive functions, distribution function

$f : \mathbb{N}_0 \to \mathbb{R}$ belongs to $\mathcal{A}_q$ if $f(0) = 0$ and

$$f(n) = \sum_{j=0}^{\infty} f\left(\varepsilon_j(n)\, q^j\right) \quad (n \in \mathbb{N}_0). \tag{2}$$

We say that $g : \mathbb{N}_0 \to \mathbb{C}$ belongs to $\mathcal{M}_q$, if $g(0) = 1$,

$$g(n) = \prod_{j=0}^{\infty} g\left(\varepsilon_j(n)\, q^j\right) \quad (n \in \mathbb{N}_0). \tag{3}$$

Let $\bar{\mathcal{M}}_q \subseteq \mathcal{M}_q$ be the set of those q-multiplicative functions g, for which $|g(n)| = 1 \quad (n \in \mathbb{N}_0)$.

Let $\beta_h(n) = \sum_{\varepsilon_j(n)=h} 1 \quad (h = 1, \dots, q-1)$, $\alpha(n) = \sum_{j=0}^{\infty} \varepsilon_j(n)$. We say that $f \in \mathcal{A}_q$ has a limit distribution, if

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x | f(n) < y\} \quad (= G(y)) \tag{4}$$

exists for almost all y, and G is a distribution function, i.e. it is monotonic, furthermore $\lim_{y\to-\infty} G(y) = 0$, $\lim_{y\to\infty} G(y) = 1$.

H. Delange [1] proved that $f \in \mathcal{A}_q$ has a limit distribution if and only the series

$$\sum_{j=0}^{\infty} \sum_{a \in A_q} f\left(aq^j\right), \tag{5}$$

$$\sum_{j=0}^{\infty} \sum_{a \in A_q} f^2\left(aq^j\right) \tag{6}$$

are convergent. He proved that for some $g \in \bar{\mathcal{M}}_q$, the limit

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \le x} g(n) = M(g)$$

exists and $M(g) \neq 0$, if and only if

$$m_j := \frac{1}{q} \sum_{c \in A_q} g\left(cq^j\right) \neq 0 \quad (j = 0, 1, 2, \dots) \tag{7}$$

and

$$\sum_{j=0}^{\infty}(1-m_j)=\sum_{j=0}^{\infty}\frac{1}{q}\left(\sum_{c\in A_q}\left(1-g\left(cq^j\right)\right)\right) \tag{8}$$

is convergent. Furthermore,

$$M(g)=\prod_{j=0}^{\infty}m_j, \tag{9}$$

if (7) holds and (8) is convergent.

Distribution of q-additive functions under the conditions that $\beta_h(n)$ are fixed.

For some fixed N, let $r_1,\dots,r_{q-1}$ be such nonnegative integers for which $r_1+\dots+r_{q-1}\le N$. Let $r_0=N-(r_1+\dots+r_{q-1})$, $\underline{r}=(r_1,r_2,\dots,r_{q-1})$.

Let

$$S_N(\underline{r})=\left\{n<q^N|\beta_h(n)=r_h,\ h=1,\dots,q-1\right\}. \tag{10}$$

Then

$$M(N|\underline{r})=\#S_N(\underline{r})=\frac{N!}{r_0!r_1!\dots r_{q-1}!}. \tag{11}$$

In [2] we proved the following

Lemma 1 *Let* $f\in\mathcal{A}_q$, $E_N=\sum_{b\in A_q}\frac{r_b}{N}\sum_{j=0}^{N-1}f\left(bq^j\right)$,

$$\Delta_N(\underline{r})=\frac{1}{M(N|\underline{r})}\sum_{n\in S_N(\underline{r})}(f(n)-E_N)^2. \tag{12}$$

Then

$$\Delta_N(\underline{r})<c\sum_{j=0}^{N-1}\sum_{b=0}^{q-1}f^2\left(bq^j\right), \tag{13}$$

c *is a constant which may depend only on* q.

We shall prove

Theorem 1 *Let* $g\in\bar{\mathcal{M}}_q$, *assume that*

$$\sum_{j=0}^{\infty}\sum_{b\in A_q}\left(1-g\left(bq^j\right)\right) \tag{14}$$

is convergent. Let $\lambda_0, \lambda_1, \ldots, \lambda_{q-1}$ *be positive numbers, such that* $\lambda_0 + \cdots + \lambda_{q-1} = 1$*. Let*

$$H\left(g|\lambda_0, \ldots, \lambda_{q-1}\right) := \prod_{j=0}^{\infty} \left(\sum_{b \in A_q} \lambda_j g\left(bq^j\right) \right). \tag{15}$$

If $\underline{r}^{(N)} = \left(r_1^{(N)}, \ldots r_{q-1}^{(N)}\right)$ *is such a sequence for which* $\frac{r_j^{(N)}}{N} \to \lambda_j$ $(j = 1, \ldots, q-1)$*, then*

$$\lim_{N \to \infty} \frac{1}{M\left(N|\underline{r}^{(N)}\right)} \sum_{\substack{n < q^N \\ n \in S_N\left(\underline{r}^{(N)}\right)}} g(n) = H\left(g|\lambda_0, \ldots, \lambda_{q-1}\right). \tag{16}$$

Hence we obtain

Theorem 2 *Let* $f \in \mathcal{A}_q$*, assume that* (5),(6) *are convergent. Let* $\lambda_0, \ldots, \lambda_{q-1}$ *be positive numbers such that* $\lambda_0 + \cdots + \lambda_{q-1} = 1$*. Let* $\eta_0, \eta_1 \ldots$ *be independent random variables,* $P\left(\eta_l = f\left(bq^l\right)\right) = \lambda_b \quad (b \in A_q)$*.*

Let

$$\Theta = \sum_{l=0}^{\infty} \eta_l, \tag{17}$$

$$F_{\underline{\lambda}}(y) := P(\Theta < y), \ \underline{\lambda} = (\lambda_1, \ldots, \lambda_{q-1}). \tag{18}$$

From the 3 series theorem of Kolmogorov it follows that the sum (17) *is convergent with probability 1, thus* $F_{\underline{\lambda}}(y)$ *exists.*

If $\frac{r_j^{(N)}}{N} \to \lambda_j \quad (j = 0, \ldots, q-1)$*, then*

$$\lim_{N \to \infty} \frac{1}{M\left(N|\underline{r}^{(N)}\right)} \# \left\{ n < q^N | n \in S_N\left(\underline{r}^{(N)}\right), \ f(n) < y \right\} = F_\lambda(y),$$

if y *is a continuity point of* F_λ*.*

F_λ *is continuous, if* $f\left(bq^j\right) \neq 0$ *holds for infinitely many elements of* $\left\{bq^j | j = 0, 1, 2, \ldots, b \in A_q\right\}$*.*

In [2] we proved Theorem 1 for $\lambda_1 = \ldots = \lambda_{q-1} = \frac{1}{q}$, and in the case $q = 2$ for $0 < \lambda_1 < 1$.

Furthermore, in [2] we proved the following assertion.

Theorem A *Let* $f \in \mathcal{A}_2,\ f\left(2^j\right) = \mathcal{O}(1) \quad (j \in \mathbb{N}),\ \eta_N = \frac{1}{N}\sum_{j=0}^{N-1} f\left(2^j\right),$

$$B_N^2 := \frac{1}{4}\sum_{j=0}^{N-1}\left(f\left(2^j\right) - \eta_N\right)^2.$$

Assume that $B_N \to \infty$. *Let* $\rho_N \to 0$.

Then

$$\lim_{N\to\infty} \frac{1}{\binom{N}{k}} \# \left\{ n < 2^N \left| \frac{f(n) - k\eta_N}{B_N} < y, \alpha(n) = k \right.\right\} = \Phi(y)$$

holds uniformly as $N \to \infty,\ k = k^{(N)}$ *satisfies*

$$\left|\frac{k}{N} - \frac{1}{2}\right| < \rho_N.$$

In [3] we mentioned that we are able to prove that under the conditions of Theorem A

$$\lim_{n\to\infty} \sup_{\frac{k}{N}\in[\delta,1-\delta]} \sup_{y\in\mathbb{R}} \left| \frac{1}{\binom{N}{k}} \# \left\{ n < 2^N, \alpha(n) = k, \frac{f(n) - k\eta_N}{2B_N\sqrt{(1-\eta)\eta}} < y \right\} - \Phi(y) \right|.$$

This assertion is not true, the correct assertion is

Theorem 3 *Let* $f \in \mathcal{A}_2,\ f\left(2^j\right) = \mathcal{O}(1)\ (j = 0, 1, 2, \ldots)$. *Let* $m_N = \sum_{j=0}^{N-1} f\left(2^j\right)$, $\sigma_N^2 = \sum_{j=0}^{N-1}\left(f\left(2^j\right) - \frac{m_N}{N}\right)^2$. *Let* $0 < \lambda < 1$,

$$F_{r,N}(y) = \frac{1}{\binom{N}{r}} \# \left\{ n < 2^N,\ \alpha(n) = r,\ \frac{f(n) - \frac{r}{N} m_N}{\sigma_N} < y \right\}.$$

Furthermore, let $F_\lambda(y)$ *be the distribution the characteristic function* $\varphi_\lambda(\tau) = \sum_{l=0}^{\infty} \alpha_l \frac{(i\tau)^l}{l!}$ *of which is given by the following formulas:*

$$\begin{aligned} \alpha_l &= 0, \quad \textit{if} \quad l \textit{ is odd},\ \alpha_0 = 1, \\ \alpha_{2k} &= \sum_{t=1}^{2k} \frac{\lambda^t}{t!2^t} \sum_{2m\le t} (-1)^m \binom{t}{2m} \cdot 2^{2m}(2m-1)!! \quad (k = 1, 2, \ldots). \end{aligned}$$

Since α_{2k} is bounded as $k \to \infty$, therefore the series defining $\varphi_\lambda(\tau)$ is absolutely convergent in $|\tau| < 1$.

We have

$$\lim_{\substack{\frac{r}{N} \to \lambda \\ N \to \infty}} F_{r,N}(y) = F_\lambda(y).$$

2 Proof of Theorem 1 and 2

Let us define $f\left(bq^j\right)$ as the argument of $g\left(bq^j\right)$, i.e. $g\left(bq^j\right) = e^{if\left(bq^j\right)}$. The condition (8) implies the convergence of (5) and (6). We can extend f as a q-additive function. Then $g(n) = e^{if(n)}$.

Let $g_M(n) = \prod_{j=0}^{M-1} g\left(\varepsilon_j(n) q^j\right)$. Thus $g_M\left(nq^M\right) = 1$ $(n \in \mathbb{N}_0)$. Let $f_M(n) = \sum_{j=0}^{M-1} f\left(\varepsilon_j(n) q^j\right)$; $h_M(n) = \sum_{j \geq M} f\left(\varepsilon_j(n) q^j\right)$.

Let M be fixed, and consider the integers $n < q^{N+M}$. Let $\delta > 0$ be an arbitrary (small) number. We shall estimate the number of those $n \in S_{N+M}\left(\underline{r}^{(N+M)}\right)$ for which $|g(n) - g_M(n)| \geq \delta$. If n is such an integer, then $|h_M(n)| \geq \delta_k$.

Assume that M is so large that for

$$E_M^{(N+M)} := \sum_{j=M}^{M+N-1} \sum_{b \in A_q} f\left(bq^j\right)$$

$\left|E_M^{(N+M)}\right| < \frac{\delta}{4}$. We shall apply (12), (13) for $h_M(n)$ and $E_M^{(N+M)}$. Then, in the right hand side of (13)

$$\sum_{j=M}^{N+M-1} \sum_{b \in A_q} f^2\left(bq^j\right)$$

tends to zero as $M \to \infty$. Consequently, the following assertion is true.

Let $\delta > 0, \varepsilon > 0$ be arbitrary constants. Then there exists such an M for which

$$\limsup_{N \to \infty} \frac{1}{M\left(N + M|\underline{r}^{(N+M)}\right)} \# \left\{ n \in S_{N+M}\left(\underline{r}^{(N+M)}\right) \middle| |g(n) - g_M(n)| > \delta \right\} < \varepsilon.$$

Now we estimate

$$\frac{1}{M\left(N + M|\underline{r}^{(N+M)}\right)} \sum_{n \in S_{N+M}\left(\underline{r}^{(n+M)}\right)} g_M(n).$$

Let us subdivide the integers $n \in S_{N+M}\left(\underline{r}^{(N+M)}\right)$ according to the digits $\varepsilon_0(n), \ldots, \varepsilon_{M-1}(n)$. Let $n = t + m \cdot q^M$. Then $n \in S_{N+M}\left(\underline{r}^{(N+M)}\right)$, if and only if

$$m \in S_N\left(r_1^{(N+M)} - \beta_1(t), \ldots, r_{q-1}^{(N+M)} - \beta_{q-1}(t)\right). \tag{19}$$

For fixed t the number of the m satisfying the condition (19) is

$$(\Psi_N(t) :=) \frac{N!}{\prod_{i=0}^{q-1}\left(r_i^{(N+M)} - \beta_i(t)\right)!},$$

where $\beta_0(t)$ is so defined that $\sum\limits_{i=0}^{q-1} \beta_i(t) = M$.

Let $\frac{r_b^{(N+M)}}{N+M} \to \lambda_b$. Then

$$\begin{aligned}
\frac{\Psi_N(t)}{S_{N+M}\left(\underline{r}^{(N+M)}\right)} &= \frac{1}{(N+1)\cdots(N+M)} \prod_{b=0}^{q-1} \frac{r_b^{(N+M)}!}{\left(r_b^{(N+M)} - \beta_b(t)\right)!} \\
&= \frac{1}{(N+1)\cdots(N+M)} \prod_{b=0}^{q-1} \prod_{l=0}^{\beta_b(t)-1} \left(r_b^{(N+M)} - l\right) \\
&= (1 + \mathcal{O}_N(1)) \prod_{b=0}^{q-1} \lambda_b^{\beta_b(t)},
\end{aligned}$$

and so

$$\lim_{N\to\infty} \frac{1}{M\left(N+M|\underline{r}^{(N+M)}\right)} \sum_{n\in S_{N+M}\left(\underline{r}^{(N+M)}\right)} g_M(n) = \prod_{j=0}^{M-1} \left\{\sum_b \lambda_b g\left(bq^j\right)\right\}.$$

Finally, let us to tend $M \to \infty$. Then (16) follows. Theorem 1 is proved.

Theorem 2 is a direct consequence of Theorem 1.

3 Some lemmas

Lemma 2 (Wintner, Frechet-Shohat) *Let* $F_n(z) \quad (n = 1, 2, \ldots)$ *be a sequence of distribution functions. For each non-negative integer* k *let*

$$\alpha_k = \lim_{n\to\infty} \int_{-\infty}^{\infty} z^k dF_n(z)$$

exist. Then there is a subsequence $F_{n_j}(z)$ $(n_1 < n_2 < \cdots)$ *which converges weakly to a limiting distribution* $F(z)$ *for which*

$$\alpha_k = \int_{-\infty}^{\infty} z^k dF(z) \quad (k = 0, 1, 2, \dots).$$

Moreover, if the set of moments α_k *determine* $F(z)$ *uniquely, then as* $n \to \infty$ *the distributions* $F_n(z)$ *converge weakly to* $F(z)$.

Lemma 3 *In the notations of Lemma 2 let the series*

$$\varphi(\tau) = \sum_{l=0}^{\infty} \alpha_l \frac{(i\tau)^l}{l!}$$

converge absolutely in a disc of complex τ *values in* $|\tau| < c$, $c > 0$. *Then the* α_k *determine the distribution function* $F(u)$ *uniquely. Moreover, the characteristic function* $\varphi(t)$ *of this distribution had the above representation in the disc* $|\tau| < t$, *and can be analytically continued into the strip* $|Im(t)| < \tau$.

The proof of Lemma 2 can be found in [5] while the proof of Lemma 3 is given in [6]. (Vol. I., page 60).

4 Proof of Theorem 3

Let

$$m_N = \sum_{j=0}^{N-1} f\left(2^j\right), \tag{20}$$

$$F\left(2^j\right) = f\left(2^j\right) - \frac{m_N}{N}, \tag{21}$$

$$\sigma_N^2(f) = \sum_{j=0}^{N-1} F^2\left(2^j\right), \tag{22}$$

$$G\left(2^j\right) = \frac{F(2^j)}{\sigma_N(f)}. \tag{23}$$

Then

$$\sigma_N^2(G) = \sum_{j=0}^{N-1} G^2\left(2^j\right) = 1. \tag{24}$$

Let

$$T_k := \frac{1}{\binom{N}{r}} \sum_{\substack{n<2^N \\ \alpha(n)=r}} G^k(n)\,. \tag{25}$$

T_k depends on N and on r, also. Let

$$\alpha_k := \frac{1}{k!} \lim_{\substack{N\to\infty \\ \frac{r}{N}\to\infty}} T_k.$$

We shall prove that α_k exists for every $k \in \mathbb{N}$, and that the function $\varphi(\tau)$ in Lemma 3 with these α_k is regular in a circle $|\tau| < c,\ c > 0$. It is enough to prove that α_k is bounded. The theorem will follow from Lemma 2, 3 immediately.

It is clear that $T_1 = 0$ and so $\alpha_1 = 0$.

We observe that

$$\sum_{l_1,\dots,l_t \in \{0,1,\dots,N-1\}} G\left(2^{l_1}\right)^{j_1} \dots G\left(2^{l_t}\right)^{j_t} \kappa\binom{l_1,\dots,l_t}{j_1,\dots,j_t} \tag{26}$$

$$= \begin{cases} \mathcal{O}(1)\,, & \text{if} \quad \min j_l \geq 2, \\ o_N(1)\,, & \text{if} \quad \min j_l \geq 2 \quad \text{and} \quad \max j_l \geq 3, \end{cases}$$

if $0 \leq \kappa\binom{l_1,\dots,l_t}{j_1,\dots,j_t} \leq 1$.

Since $\max_l |G(2^l)| \leq \frac{c}{\sigma_N(f)} \to 0 \quad (N \to \infty)\,,\ \sigma_N^2(G) = 1$, this assertion is clear.

Let $D_N := \{0, 1, \dots, N-1\}$.

Let us consider sums of type

$$A_\nu := \sum_{\substack{l_1,\dots,l_t \\ j_1,\dots,j_t \\ u_1,\dots,u_\nu}} B\binom{l_1,\dots,l_t}{j_1,\dots,j_t} G(2^{u_1}) \cdots G(2^{u_\nu}) \tag{27}$$

where $l_1, \dots, l_t, u_1, \dots, u_\nu$ run over all possible distinct choices of $l_1, \dots, l_t$, $u_1, \dots, u_\nu \in D_N$, $\min\limits_{l=1,\dots,t} j_l \geq 2$

$$B\binom{l_1,\dots,l_t}{j_1,\dots,j_t} = G\left(2^{l_1}\right)^{j_1} \cdots G\left(2^{l_t}\right)^{j_t} \kappa\binom{l_1,\dots,l_t}{j_1,\dots,j_t}, \tag{28}$$

$0 \leq \kappa\binom{l_1,\dots,l_t}{j_1,\dots,j_t}$.

Assume that $\nu = 1$. Let us sum $G(2^{u_1})$ over all possible values, $u_1 \in D_N \setminus \{l_1, \dots, l_t\}$.
We have

$$A_1 = -\sum_{j=1}^{t} \sum_{\substack{l_1,\dots,l_t \\ j_1,\dots,j_t}} B\binom{l_1, \dots, l_t}{j_1, \dots, j_t} G\left(2^{l_j}\right),$$

and so $A_\nu \to 0 \quad (N \to \infty)$ follows from (26). Let now $\nu = 2$.
We obtain that

$$\sum_{\substack{u_2 \notin \{l_1,\dots,l_t\} \\ u_2 \neq u_1}} G(2^{u_2}) = -G\left(2^{l_1}\right) - \dots - G\left(2^{l_t}\right) - G(2^{u_1})$$

and so

$$A_2 = \sum_{\substack{l_1,\dots,l_t,u_1 \\ j_1,\dots,j_t}} B\binom{l_1, \dots, l_t}{j_1, \dots, j_t} G^2(2^{u_1}) + o_N(1).$$

Let $\nu > 2$. For fixed $l_1, \dots, l_t, u_1, \dots, u_{\nu-1}$ the variable u_ν run over $D_N \setminus (\{l_1, \dots, l_t\} \cup \{u_1, \dots, u_{\nu-1}\})$. Since

$$\sum_{u_\nu} G(2^{u_\nu}) = -\sum_{j=1}^{t} G\left(2^{l_j}\right) - G(2^{u_1}) - \dots - G(2^{u_{\nu-1}}),$$

we have

$$A_\nu = -\sum_{\substack{l_1,\dots,l_t \\ j_1,\dots,j_t}} B\binom{l_1, \dots, l_t}{j_1, \dots, j_t} G(2^{u_1}) \dots G(2^{u_{\nu-1}}) \left(G(2^{u_1}) + \dots + G(2^{u_{\nu-1}})\right)$$
$$+ o_N(1),$$

and so

$$A_\nu = -(\nu - 1) \sum_{\substack{l_1,\dots,l_t,l_{t+1} \\ j_1,\dots,j_t,2 \\ u_1,\dots,u_{\nu-2}}} B\binom{l_1, \dots, l_t}{j_1, \dots, j_t} G^2\left(2^{l_{t+1}}\right) G(2^{u_1}) \dots G(2^{u_{\nu-2}})$$
$$+ o_N(1).$$

Thus the sum A_ν can be substituted by $(\nu - 1)$ sums of type $A_{\nu-2}$, with the error $o_N(1)$.

Let us continue the reduction. We obtain that $A_\nu = o_N(1)$, if ν is an odd number, furthermore, $A_\nu = o_N(1)$, if $\max\limits_{j=1,\dots,t} l_j \geq 3$.

We can write

$$T_k = \frac{1}{\binom{N}{r}} \sum_{\substack{\alpha(n)=r \\ n<2^N}} G^k(n) = \frac{1}{\binom{N}{r}} \sum_{\substack{n<2^n \\ \alpha(n)=r}} \left\{ \sum_{j=0}^{N-1} G\left(\varepsilon_j(n)\cdot 2^j\right)\right\}^k \tag{29}$$
$$= \sum_{t=1}^{k} \nu(t,N) \sum_{u_1,\dots,u_k}^{*} G(2^{u_1})\dots G(2^{u_k}),$$

where $*$ indicates that the summation is over those $u_1,\dots,u_k \in D_N$, for which the number of distinct element of $u_1,\dots,u_k$ is t, and $\nu(t,N) = \frac{r}{N}\cdot\frac{r-1}{N-1}\cdots\frac{r-(t-1)}{N-(t-1)}$. Thus $\nu(t,N) = \lambda^t + o_N(1)$.

The sum $\sum\limits_{u_1,\dots,u_k}^{*}$ can be rewritten in the form $\sum\limits_{\substack{l_1<\dots<l_t \\ j_1,\dots,j_t}}$, where the multiplicity of the occurrence of l_h is j_h, thus $j_1+\dots+j_t = k$. It is clear that $G\left(2^{l_1}\right)^{j_1}\dots G\left(2^{l_t}\right)^{j_t}$ occurs for

$$\binom{k}{j_1}\binom{k-j_1}{j_2}\cdots\binom{k-(j_1+\dots+j_{t-1})}{j_t}$$
$$= \frac{k!}{j_1!\,(k-j_1)!}\cdot\frac{(k-j_1)!}{(k-(j_1+j_2))!\,j_2!}\cdots\frac{(k-(j_1+\dots+j_{t-1}))!}{j_t!}$$
$$= \frac{k!}{j_1!j_2!\dots j_t!}$$

distinct choices of $u_1,\dots,u_k$ as $G(2^{u_1})\dots G(2^{u_k})$. Thus

$$T_k = \textstyle\sum_{t=1}^{k} \nu(t,N)\, k! \sum_{\substack{l_1<\dots<l_t \\ j_1,\dots,j_t}} \frac{G(2^{l_1})^{j_1}}{j_1!}\cdots\frac{G(2^{l_t})^{j_t}}{j_t!} \tag{30}$$
$$= k! \textstyle\sum_{t=1}^{k} \frac{\nu(t,N)}{t!} \sum_{\substack{l_1<\dots<l_t \\ j_1,\dots,j_t}} \frac{G(2^{l_1})^{j_1}}{j_1!}\cdots\frac{G(2^{l_t})^{j_t}}{j_t!}.$$

In the last sum $l_1,\dots,l_t$ run over all those elements of D_N for which $l_u \neq l_v$, if $u \neq v$.

Let $E(j_1,\dots,j_t) = \sum\limits_{\substack{l_1,\dots,l_t \\ j_1,\dots,j_t}} G\left(2^{l_1}\right)^{j_1}\dots G\left(2^{l_t}\right)^{j_t}$. As we have seen earlier, $E(j_1,\dots,j_t) \to 0$ if $\max j_u \geq 3$, or if $\#\{u|j_u = 1\}$ = odd number. Hence we obtain that $T_k \to 0$ if k is odd. Thus $\alpha_k = 0$ for odd k. Let us write now $2k$ into the place of k.

Then

$$T_{2k}=(2k)!\sum_{t=1}^{2k}\frac{\nu(t,N)}{t!}\sum_{j_1+\cdots+j_t=2k}{}^{*}\frac{E(j_1,\ldots,j_t)}{j_1!\ldots j_t!}+o_N(1)$$

where $*$ indicates that we have to sum over those $j_1,\ldots,j_t$ for which $j_\nu=1,2$. It is clear that $E(j_1,\ldots,j_t)$ is symmetric in the variables, i.e. $E(j_{m_1},\ldots,j_{m_t})=E(j_1,\ldots,j_t)$ if $m_1,\ldots,m_t$ is a permutation of $\{1,\ldots,t\}$.

Let

$$\sigma_{h,m}=E\left(\overbrace{2,\ldots,2}^{h},\ \overbrace{1,\ldots,1}^{m}\right).$$

If $j_1+\cdots+j_t=2k$, then $2h+m=2k,\ t=h+m$, thus

$$T_{2k}=(2k)!\sum_{t=1}^{2k}\frac{\nu(t,N)}{t!}\sum_{h\le t}\binom{t}{h}\frac{1}{2^h}\sigma_{h,t-h}+o_N(1). \tag{31}$$

It is clear that

$$\sigma_{h,0}=\sum_{l_1,\ldots,l_h}G\left(2^{l_1}\right)^2\ldots G\left(2^{l_h}\right)^2$$
$$=\left\{\sum G^2\left(2^l\right)\right\}^h=1+o_N(1).$$

Furthermore, as we observed earlier, $\sigma_{h,m}\to 0\quad(N\to\infty)$ if m =odd.

Let $m=2$. We have

$$\sigma_{h,2}=\sum_{l_1,\ldots,l_h,u_1,u_2}G^2\left(2^{l_1}\right)\ldots G^2\left(2^{l_h}\right)G\left(2^{u_1}\right)G\left(2^{u_2}\right)$$
$$=-\sum_{l_1,\ldots,l_h,u_1}G^2\left(2^{l_1}\right)\ldots G^2\left(2^{l_h}\right)G^2\left(2^{u_1}\right)+o_N(1)$$
$$=-\sigma_{h+1,0}+o_N(1)=-1+o_N(1).$$

Let $m=2\nu,\nu\ge 2$.

$$\sigma_{h,2\nu}=\sum_{\substack{l_1,\ldots,l_h\\u_1,\ldots,u_{2\nu}}}G^2\left(2^{l_1}\right)\ldots G^2\left(2^{l_h}\right)G\left(2^{u_1}\right)\ldots G\left(2^{u_{2\nu}}\right).$$

Since $G\left(2^{u_{2\nu}}\right)$ should be summed over $D_N \setminus \{l_1, \dots, l_h\} \cup \{u_1 \dots, u_{2\nu-1}\}$, and so $\sum_{u_{2\nu}} G\left(2^{u_{2\nu}}\right) = -\sum G\left(2^{l_i}\right) - \sum_{1}^{2\nu-1} G\left(2^{u_j}\right)$, we obtain that

$$\sigma_{h,2\nu} = -\left(2\nu - 1\right)\sigma_{h+1,2(\nu-1)} + o_N(1) \quad (\nu = 1, 2, \dots).$$

Thus we have

$$\begin{aligned}
&\sigma_{h,0} = 1 + o_N(1), \quad \sigma_{h,2} = -1 + o_N(1), \\
&\sigma_{h,4} = -3 \cdot \sigma_{h+1,2} = 3 + o_N(1), \\
&\sigma_{h,6} = -5 \cdot \sigma_{h+1,4} = -3 \cdot 5 + o_N(1),
\end{aligned}$$

and in general

$$\sigma_{h,2\nu} = (-1)^{\nu}\left(2\nu - 1\right)!! + o_N(1).$$

Here $(2m-1)!! = (2m-1)(2m-3)\dots \cdot 3 \cdot 1$.

Let us write $t - h = 2m$ in (31). Then

$$\begin{aligned}
\binom{t}{h}\frac{1}{2^h}\sigma_{h,t-h} &= \binom{t}{2m}\frac{2^{2m}}{2^t}\sigma_{h,2m} \\
&= (-1)^m \binom{t}{2m}\frac{2^{2m}}{2^t}\left(2m-1\right)!! + o_N(1),
\end{aligned}$$

and so

$$T_{2k} = (2k)!\sum_{t=1}^{2k}\lambda^t \cdot \frac{1}{t!2^t}\sum_{2m\le t}(-1)^m\binom{t}{2m}2^{2m}\cdot\left(2m-1\right)!! + o_N(1).$$

Let us apply Lemma 3. In the notation of Lemma 3 we have

$$\begin{aligned}
\alpha_{2k} &= \lim_{N\to\infty}\frac{T_{2k}}{(2k)!} \\
&= \sum_{t=1}^{2k}\frac{\lambda^t}{t!2^t}\sum_{2m\le t}(-1)^m\binom{t}{2m}2^{2m}\cdot\left(2m-1\right)!!.
\end{aligned}$$

We shall prove that α_{2k} is bounded as $2k \to \infty$. Indeed

$$\frac{(2m-1)!!}{(2m)!} = \frac{1}{2^m m!}, \quad \frac{2^{2m}}{2^t} \le 1,$$

thus

$$|\alpha_{2k}| \leq \sum_{t=1}^{2k} \frac{\lambda^t}{t!} \sum_{2m \leq t} \frac{t!\,(2m-1)\,!!}{(2m)\,!\,(t-2m)\,!}$$
$$\leq \sum_{t=1}^{2k} \lambda^t \sum_{2m \leq t} \frac{1}{(t-2m)\,!\,(2^m m!)}.$$

Here $m = 0$ can be occur, $0! = 1$.
We obtain that

$$|\alpha_{2k}| < c\lambda$$

with some c, c may depend on λ.

Acknowledgement

The European Union and the European Special Fund have provided financial support to the project under the agreement TÁMOP-4.2.1/B-0.9/1/KMR-2010-0003.

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Received: August 1, 2011