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On the connection between tridiagonal matrices, Chebyshev polynomials, and Fibonacci numbers

Carlos M. da Fonseca
Kuwait College of Science and Technology,
Doha District, Block 4, P.O. Box 27235,
Safat 13133, Kuwait
email: c.dafonseca@kcst.edu.kw

Abstract. In this note, we recall several connections between the determinant of some tridiagonal matrices and the orthogonal polynomials allowing the relation between Chebyshev polynomials of second kind and Fibonacci numbers. With basic transformations, we are able to recover some recent results on this matter, bringing them into one place.

1 Orthogonal polynomials and tridiagonal matrices

From the elementary orthogonal polynomials theory, we know that any monic sequence orthogonal polynomial sequence $\{P_n(x)\}$ is defined by the recurrence formula

$$P_n(x) = (x - c_n)\, P_{n-1}(x) - \lambda_n\, P_{n-2}(x)\ , \quad \text{for } n = 1, 2, \ldots \tag{1}$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$, and for complex numbers c_n's and λ_n's, if and only if $\lambda_n \neq 0$ [6, Theorem 4.4]. Moreover, (1) is equivalent

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to

$$P_n(x) = \begin{vmatrix} x-c_1 & 1 & & & \\ \lambda_2 & x-c_2 & 1 & & \\ & \lambda_3 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \lambda_n & x-c_n \end{vmatrix} \tag{2}$$

(cf. [6, Exercise 4.12, p.26]).

Setting $\lambda_n = 1$ and $c_n = 0$, for any positive integer n, in (1) or (2), we obtain $U_n(x) = P_n(2x)$, the Chebyshev polynomial of second kind of degree n (cf. [1, (22.7.5)] or [6, Exercise 4.9, p.25]). The Chebyshev polynomials of second kind satisfy the three-term recurrence relations

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)\,, \quad \text{for all } n = 1, 2, \ldots\,, \tag{3}$$

or, equivalently, we may state

$$U_n(x) = \begin{vmatrix} 2x & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2x \end{vmatrix}_{n\times n}.$$

Among the most important explicit representations for the $U_n(x)$ we have

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}\,, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi), \tag{4}$$

as we can see, for example, in [1, (22.3.16)] or [6, Exercise 1.2, p.5]. From de Moivre's formula we have

$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}}. \tag{5}$$

Another formula with some relevance is

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k} \tag{6}$$

which can be found for example in [1, (22.3.7)]. There are many other representations and relations for $U_n(x)$. As stated in [8, p.187], many of them are paraphrases of trigonometric identities, derivations from (4). For particular values of $U_n(x)$ the reader is referred to [1, (22.4.5)].

Regarding the generating function for $U_n(x)$, we have

$$\frac{1}{1-2zx+z^2} = \sum_{n=0}^{\infty} U_n(x)\, z^n \tag{7}$$

(cf. [1, (22.9.10)]).

From (3), we can easily deduce that

$$(\sqrt{ab})^n U_n\left(\frac{c}{2\sqrt{ab}}\right) = \begin{vmatrix} c & a & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & a \\ & & b & c \end{vmatrix}_{n\times n} . \tag{8}$$

provided $ab \neq 0$. Perhaps one of the most interesting applications of this identity is

$$F_n = (-\mathrm{i})^{n-1}\, U_{n-1}\left(\frac{\mathrm{i}}{2}\right),$$

where F_n is the nth Fibonacci number (cf. for example [2, 5, 12]), setting $a = c = 1$ and $b = -1$.

Many of these results were recently recast. In this short note, we aim to bring them into attention and put into one place. At the same time, we provide shorter proofs to many of them.

2 Determinants of tridiagonal matrices

In [14], the authors guessed that the determinant $D_n(c)$ in (8), for $a = b = 1$, satisfies

$$D_n(c) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} c^{n-2k}.$$

This is repeatedly proved in [13] and in [16], while indeed it is an immediate consequence of (6). For that propose the authors in [13] proved (7) for $D_n(c)$.

Next, it is proved in [13] that

$$D_n(c) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta},$$

where $\alpha = \frac{1}{\beta} = \frac{c+\sqrt{c^2-4}}{2}$, and the trivial cases are ignored. In [16], three alternative proofs are provided to this fact. Nonetheless, this equality is immediate from (5), if one notices that $\beta = \frac{c-\sqrt{c^2-4}}{2}$. A particular case of (8) is also claimed to be proved by induction when $c = a + b$, using (5).

Another topic discussed in [13] is the inverse of

$$A = \begin{pmatrix} c & a & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & a \\ & & b & c \end{pmatrix}_{n\times n},$$

for $a = b = 1$. We remark that this matrix is occasionally called in [13] “diagonal matrix”, but this is obviously not appropriate. It is well-known that if A is nonsingular, i.e., $U_n(d) \neq 0$, with $d = c/(2\sqrt{ab})$, then its inverse is given by

$$(A^{-1})_{ij} = \begin{cases} (-1)^{i+j} \dfrac{a^{j-i}}{\left(\sqrt{ab}\right)^{j-i+1}} \dfrac{U_{i-1}(d)\, U_{n-j}(d)}{U_n(d)} & \text{if } i \leq j \\ (-1)^{i+j} \dfrac{b^{i-j}}{\left(\sqrt{ab}\right)^{i-j+1}} \dfrac{U_{j-1}(d)\, U_{n-i}(d)}{U_n(d)} & \text{if } i > j\,. \end{cases}$$

This can be found in [10, 19, 20, 21] or easily deduced from [11, p.28]. Yet, in [13] we can see this inverse for the particular case of $a = b = 1$. Moreover, in [16] the authors provide a detailed proof for it, while this particular case has been studied for the last 75 years (cf. [9, 17, 18]).

We also want to remark that the eigenpairs of A in [13] are wrong. Indeed, it is a standard result that, for example, the eigenvalues of A are

$$\lambda_k = c + 2\sqrt{ab}\cos\left(\frac{\ell\pi}{n+1}\right), \quad \text{for } \ell = 1, \ldots, n\,.$$

On contrary to what is suggested in [13], the real entries of the tridiagonal matrix A are absolutely irrelevant for this formula.

Finally, [13] provides an intricate proof for the equality

$$\begin{vmatrix} -c & 1 & & & \\ 2 & -2c & 1 & & \\ & 6 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & n(n-1) & -nc \end{vmatrix} = (-1)^n n! \begin{vmatrix} c & 1 & & & \\ 1 & c & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & c \end{vmatrix}. \tag{9}$$

In [16], among others, the authors claim that (9) is "neither trivial nor obvious". Next, using elementary matrix theory, we show that (9) is both trivial and obvious (see also [3]). In fact,

$$\begin{vmatrix} -c & 1 & & & \\ 2 & -2c & 1 & & \\ & 6 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & n(n-1) & -nc \end{vmatrix} = (-1)^n \begin{vmatrix} c & 1 & & & \\ 2 & 2c & 2 & & \\ & 3 & \ddots & \ddots & \\ & & \ddots & \ddots & n-1 \\ & & & n & nc \end{vmatrix}$$

$$= (-1)^n n! \begin{vmatrix} c & 1 & & & \\ 1 & c & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & c \end{vmatrix} .$$

We observe that the same identity can be found and proved using an involved approach in [15].

3 Central Delannoy numbers

Our final comment goes to the last section of the paper [13].

The Legendre polynomials $P_n(x)$ are a particular family of the ultraspherical polynomials [4, p.899]. Their generating function is

$$\frac{1}{\sqrt{1-2tx+t^2}}$$

and they can be defined, for example, by the contour integral

$$P_n(x) = \frac{1}{2\pi \mathrm{i}} \oint \frac{1}{\sqrt{1-2xt+t^2}} \frac{1}{t^{n+1}}\, dt$$

where the contour encloses the origin and is traversed in a counterclockwise direction [4, p.416]. The central Delannoy numbers $D(n)$ are defined as ([7, p.81])

$$D(n) = P_n(3) .$$

The authors recover a generalization for the central Delannoy numbers, $D_{a,b}(n)$, with generating function

$$\frac{1}{\sqrt{(x+a)(x+b)}} = \sum_{k=0}^{\infty} D_{a,b}(k)\, x^k .$$

However, next they claim that squaring both sides of the previous identity one gets

$$\frac{1}{(x+a)(x+b)} = \sum_{k=0}^{\infty} D_{a,b}(k)\, x^k ,$$

which is obviously not true.

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