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On some spaces of Cesàro sequences of fuzzy numbers associated with λ-convergence and Orlicz function

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Abstract. In the present paper we shall introduce some generalized difference Cesàro sequence spaces of fuzzy real numbers defined by Musielak-Orlicz function and λ-convergence. We make an effort to study some topological and algebraic properties of these sequence spaces. Furthermore, some inclusion relations between these sequence spaces are establish.

1 Introduction and preliminaries

Fuzzy set theory as compared to other mathematical theories is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool. The concepts of fuzzy sets and fuzzy set

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operations were first introduced by Zadeh [23] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [7] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties.

A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $X : \mathbb{R}^n \to [0, 1]$ which satisfies the following four conditions:

1. X is normal, i.e., there exist an $x_0 \in \mathbb{R}^n$ such that $X(x_0) = 1$,
2. X is fuzzy convex, i.e., for $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1, X(\lambda x + (1-\lambda)y) \geq \min[X(x), X(y)]$,
3. X is upper semi-continuous; i.e., if for each $\epsilon > 0$, $X^{-1}([0, a+\epsilon))$ for all $a \in [0, 1]$ is open in the usual topology of $\mathbb{R}^n$,
4. The closure of $\{x \in \mathbb{R}^n : X(x) > 0\}$, denoted by $[X]^0$, is compact.

Let $C(\mathbb{R}^n) = \{A \subset \mathbb{R}^n : A \text{ is compact and convex}\}$. The spaces $C(\mathbb{R}^n)$ has a linear structure induced by the operations

$$A + B = \{a + b, a \in A, b \in B\}$$

and

$$\lambda A = \{\lambda a : a \in A\}$$

for $A, B \in C(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. The Hausdorff distance between A and B of $C(\mathbb{R}^n)$ is defined as

$$\delta_\infty(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \ \sup_{b\in B} \inf_{a\in A} \|a - b\|\},$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete (non separable) metric space.

For $0 < \alpha \leq 1$, the α-level set, $X^\alpha = \{x \in \mathbb{R}^n : X(x) \geq \alpha\}$ is a non empty compact convex, subset of $\mathbb{R}^n$, as is the support X^0. Let $L(\mathbb{R}^n)$ denote the set of all fuzzy numbers. The linear structure of $L(\mathbb{R}^n)$ induces addition $X+Y$ and scalar multiplication λX, $\lambda \in \mathbb{R}$, in terms of α-level sets by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha$$

and

$$[\lambda X]^\alpha = \lambda [X]^\alpha.$$

Define for each $1 \leq q < \infty$

$$d_q(X, Y) = \left\{ \int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q d\alpha \right\}^{1/q}$$

and $d_\infty(X, Y) = \sup_{0<\alpha\leq 1} \delta_\infty(X^\alpha, Y^\alpha)$. Clearly $d_\infty(X, Y) = \lim_{q\to\infty} d_q(X, Y)$ with $d_q \leq d_r$ if $q \leq r$. Moreover $(L(\mathbb{R}^n), d_\infty)$ is a complete, separable and locally compact metric space. We denote by $w(f)$ the set of all sequences $X = (X_k)$ of fuzzy numbers. For more details about sequence spaces and fuzzy sequence spaces one can refer to [14, 15, 16, 17, 22].

Mursaleen and Noman (see [9, 10]) introduced the notion of λ-convergent and λ-bounded sequences as follows:
Let w be the set of all complex sequences $x = (x_k)$. Let $\lambda = (\lambda_k)_{k=1}^\infty$ be strictly increasing sequence of positive real numbers tending to infinity as

$$0 < \lambda_0 < \lambda_1 < .... \text{ and } \lambda_k \to \infty \text{ as } k \to \infty$$

and said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x if $\Lambda_m(x) \to L$ as $m \to \infty$, where

$$\Lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=1}^m (\lambda_k - \lambda_{k-1}) x_k.$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [11] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \frac{1}{\lambda_m} \left(\sum_{k=1}^m (\lambda_k - \lambda_{k-1}) |x_k - a| \right) = 0.$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m |\frac{1}{\lambda_m} \sum_{k=1}^m (\lambda_k - \lambda_{k-1})(x_k - a)| = 0$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a.

Definition 1 *A fuzzy real number* X *is a fuzzy set on* R, *i.e. a mapping* $X : R \to I(= [0, 1])$ *associating each real number* t *with its grade of membership* $X(t)$.

Definition 2 *A fuzzy real number* X *is called convex if* $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, *where* $s < t < r$.

Definition 3 *If there exists* $t_0 \in R$ *such that* $X(t_0) = 1$, *then the fuzzy real number* X *is called normal.*

Definition 4 *A fuzzy real number* X *is said to be upper semi continuous if for each* $\epsilon > 0$, $X^{-1}([0, a + \epsilon))$, *for all* $a \in I$, *is open in the usual topology of* R.

The class of all upper semi-continuous, normal, convex fuzzy real numbers is denoted by $R(I)$.

Definition 5 *For* $X \in R(I)$, *the* α*-level set* X^α, *for* $0 < \alpha \leq 1$ *is defined by* $X^\alpha = \{t \in R : X(t) \geq \alpha\}$. *The* 0*-level, i.e.* X^0 *is the closure of strong* 0*-cut, i.e.* $X^0 = cl\{t \in R : X(t) > 0\}$.

Definition 6 *The absolute value of* $X \in R(I)$, *i.e.* $|X|$ *is defined by*

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{for } t \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Definition 7 *For* $r \in R$, $\bar{r} \in R(I)$ *is defined as*

$$\bar{r}(t) = \begin{cases} 1, & \text{if } t = r \\ 0, & \text{if } t \neq r. \end{cases}$$

Definition 8 *The additive identity and multiplicative identity of* $R(I)$ *are denoted by* $\bar{0}$ *and* $\bar{1}$ *respectively. The zero sequence of fuzzy real numbers is denoted by* $\bar{\theta}$.

Definition 9 *Let* D *be the set of all closed bounded intervals* $X = [X^L, X^R]$. *Define* $d : D \times D \longrightarrow R$ *by* $d(X, Y) = \max\{|X^L - Y^L|, |X^R - Y^R|\}$. *Then clearly* (D, d) *is a complete metric space.*
Define $\bar{d} : R(I) \times R(I)$ *by* $\bar{d}(X, Y) = \sup_{0<\alpha\leq 1} d(X^\alpha, Y^\alpha)$, *for* $X, Y \in R(I)$. *Then it is well known that* $(R(I), \bar{d})$ *is a complete metric space.*

Definition 10 *A sequence* $X = (X_k)$ *of fuzzy numbers is said to be convergent to a fuzzy number* X_0, *if for every* $\epsilon > 0$ *there exists a positive integer* k_0 *such that* $\bar{d}(X_k, X_0) < \epsilon$, *for all* $k \geq k_0$.

Definition 11 *A sequence* $X = (X_k)$ *of fuzzy numbers is said to be bounded if the set* $\{X_k : k \in \mathbb{N}\}$ *of fuzzy numbers is bounded.*

Definition 12 *A sequence space* E *is said to be solid(or normal) if* $(Y_n) \in E$ *whenever* $(X_n) \in E$ *and* $|Y_n| \leq |X_n|$ *for all* $n \in \mathbb{N}$.

Definition 13 *Let* $X = (X_n)$ *be a sequence, then* $S(X)$ *denotes the set of all permutations of the elements of* (X_n) *i.e.* $S(X) = \{(X_{\pi(n)}) : \pi$ *is a permutation of* $N\}$. *A sequence space* E *is said to be symmetric if* $S(X) \subset E$ *for all* $X \in E$.

Definition 14 *A sequence space* E *is said to be convergence-free if* $(Y_n) \in E$ *whenever* $(X_n) \in E$ *and* $X_n = \bar{0}$ *implies* $Y_n = \bar{0}$.

Definition 15 *A sequence space* E *is said to be monotone if* E *contains the canonical pre-images of all its step spaces.*

Lemma 1 [3] *A sequence space* E *is normal implies* E *is monotone.*

The notion of difference sequence spaces was introduced by Kizmaz [4], who studied the difference sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [1] by introducing the spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [19] who studied the spaces $\ell_\infty(\Delta_m^n)$, $c(\Delta_m^n)$ and $c_0(\Delta_m^n)$. Let m, n be non-negative integers, then we have sequence spaces

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+1})$ and $\Delta_m^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv}. \tag{1}$$

Taking $m = 1$, we get the spaces $\ell_\infty(\Delta^n), c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et and Çolak [1]. Taking $m = n = 1$, we get the spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [4].

Definition 16 *Ng and Lee* [12] *defined the Cesàro sequence spaces* X_p *of non-absolute type as follows:*

$$x = (x_k) \in X_p \textit{ if and only if } \sigma(x) \in \ell_p, 1 \leq p < \infty,$$

where $\sigma(x) = \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right)_{n=1}^{\infty}$.

Orhan [13] defined the Cesàro difference sequence spaces $X_p(\Lambda)$, for $1 \leq p < \infty$ and studied their different properties and proved some inclusion results. He also obtained the duals of these sequence spaces.

Musaleen et al. [8] defined the second difference Cesàro sequence spaces $X_p(\Lambda^2)$, for $1 \leq p < \infty$ and studied their different topological properties and proved some inclusion results. They also calculated their duals sequence spaces.

Later on, Tripathy et al. [20] further introduced new types of difference Cesàro sequence spaces as $C_\infty(\Delta_m^n), O_\infty(\Delta_m^n), C_p(\Delta_m^n), O_p(\Delta_m^n)$ and $\ell_\infty(\Delta_m^n)$, for $1 \leq p < \infty$.

For $m = 1$, the spaces $C_p(\Delta^n)$ and $C_\infty(\Delta_m^n)$ are studied by Et [2].

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. An Orlicz function M is said to satisfy Δ_2-condition for all values of x, if there exists a constant $K > 0$, $M(Lx) \leq KLM(x)$, for all $x > 0$ and for $L > 1$. If convexity of the Orlicz function is replaced by subadditivity i.e. $M(x+y) \leq M(x)+M(y)$, then this function is called as modulus function [18].

Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \left\{ x = (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

is known as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Also it was shown in [5] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is said to be Musielak-Orlicz function (see [6]).

Let $m, n \geq 0$ be fixed integers, $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. In this paper we define the following generalized difference Cesàro sequence spaces of fuzzy real numbers:

$$C^F(\mathcal{M}, \Lambda, \Delta_m^n, p) =$$
$$\left\{ X = (X_k) \in w(F) : \sum_{i=1}^{\infty} \left(\frac{1}{i} \sum_{k=1}^{i} \left(M_k \left(\frac{\bar{d}(\Lambda_k \Delta_m^n X_k, \bar{0})}{\rho} \right) \right) \right)^{p_k} < \infty, \text{ for some } \rho > 0 \right\},$$

$C^F_\infty(\mathcal{M},\Lambda,\Delta^n_m,p) =$
$$\left\{X=(X_k)\in w(F): \sup_i \frac{1}{i}\left(\sum_{k=1}^{i} M_k\left(\frac{\overline{d}(\Lambda_k\Delta^n_m X_k,\overline{0})}{\rho}\right)\right)^{p_k}<\infty, \text{ for some } \rho>0\right\},$$

$\ell^F(\mathcal{M},\Lambda,\Delta^n_m,p) =$
$$\left\{X=(X_k)\in w(F): \sum_{k=1}^{\infty}\left(M_k\left(\frac{\overline{d}(\Lambda_k\Delta^n_m X_k,\overline{0})}{\rho}\right)\right)^{p_k}<\infty, \text{ for some } \rho>0\right\},$$

$O^F(\mathcal{M},\Lambda,\Delta^n_m,p) =$
$$\left\{X=(X_k)\in w(F): \sum_{i=1}^{\infty}\frac{1}{i}\left(\sum_{k=1}^{i}\left(M_k\left(\frac{\overline{d}(\Lambda_k\Delta^n_m X_k,\overline{0})}{\rho}\right)\right)\right)^{p_k}<\infty, \text{for some } \rho>0\right\},$$

$O^F_\infty(\mathcal{M},\Lambda,\Delta^n_m,p) =$
$$\left\{X=(X_k)\in w(F): \sup_i \frac{1}{i}\sum_{k=1}^{i} M_k\left(\frac{\overline{d}(\Lambda_k\Delta^n_m X_k,\overline{0})}{\rho}\right)^{p_k}<\infty, \text{for some } \rho>0\right\}.$$

Lemma 2 [21] *Let* $1\le p<\infty$. *Then,*
(i) *The space* $C^F_p(M)$ *is a complete metric space with the metric,*

$$\eta_1(X,Y)=\inf\left\{\rho>0:\left(\sum_{i=1}^{\infty}\frac{1}{i}\sum_{k=1}^{i}\left(M\left(\frac{\overline{d}(X_k,Y_k)}{\rho}\right)\right)^p\right)^{\frac{1}{p}}\le 1\right\}.$$

(ii) *The space* $C^F_\infty(M)$ *is a complete metric space with the metric,*

$$\eta_2(X,Y)=\inf\left\{\rho>0:\sup_i\frac{1}{i}\sum_{k=1}^{i}\left(M\left(\frac{\overline{d}(X_k,Y_k)}{\rho}\right)\le 1\right\}.$$

(iii) *The space* $\ell^F_p(M)$ *is a complete metric space with the metric,*

$$\eta_3(X,Y)=\inf\left\{\rho>0:\left(\sum_{k=1}^{\infty}\left(M\left(\frac{\overline{d}(X_k,Y_k)}{\rho}\right)\right)^p\right)^{\frac{1}{p}}\le 1\right\}.$$

(iv) *The space* $O^F_p(M)$ *is a complete metric space with the metric,*

$$\eta_4(X,Y)=\inf\left\{\rho>0:\left(\sum_{i=1}^{\infty}\frac{1}{i}\sum_{k=1}^{i}\left(M\left(\frac{\overline{d}(X_k,Y_k)}{\rho}\right)\right)^p\right)^{\frac{1}{p}}\le 1\right\}.$$

(v) *The space* $O^F_\infty(M)$ *is a complete metric space with the metric,*

$$\eta_5(X,Y) = \inf\left\{\rho > 0 : \sup_i \frac{1}{i}\sum_{k=1}^{i}\left(M\left(\frac{\bar{d}(X_k,Y_k)}{\rho}\right) \le 1\right\}.$$

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \le \sup_k p_k = H$ and let $K = \max\{1, 2^{H-1}\}$. Then, for the factorable sequences (a_k) and (b_k) in the complex plane, we have

$$|a_k + b_k|^{p_k} \le K(|a_k|^{p_k} + |b_k|^{p_k}). \tag{2}$$

Also $|a_k|^{p_k} \le \max\{1, |a|^H\}$ for all $a \in \mathbb{C}$.

The main aim of this paper is to study some topological properties and prove some inclusion relations between above defined sequence spaces.

2 Main results

Theorem 1 *Let* $\mathcal{M} = (M_k)$ *be a Musielak-Orlicz function and* $p = (p_k)$ *be a bounded sequence of positive real numbers. Then the classes of sequences* $C^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$, $C^F_\infty(\mathcal{M}, \Lambda, \Delta^n_m, p)$, $\ell^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$, $O^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$ *and* $O^F_\infty(\mathcal{M}, \Lambda, \Delta^n_m, p)$ *are linear spaces over the field* $\mathbb{R}$ *of real numbers.*

Proof. We shall prove the result for the space $C^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$ and for other spaces, it will follow on applying similar arguments. Suppose $X = (X_k)$, $Y = (Y_k) \in C^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$ and $\alpha, \beta \in \mathbb{R}$. Then there exit positive real numbers ρ_1, ρ_2 such that

$$\sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{k=1}^{i}\left(M_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, \bar{0})}{\rho_1}\right)\right)\right)^{p_k} < \infty, \text{ for some } \rho_1 > 0$$

and

$$\sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{k=1}^{i}\left(M_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m Y_k, \bar{0})}{\rho_2}\right)\right)\right)^{p_k} < \infty, \text{ for some } \rho_2 > 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is a non-decreasing and convex so by using inequality (2), we have

$$\sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{k=1}^{i}\left(M_k\left(\frac{\bar{d}(\alpha\Lambda_k\Delta_m^n X_k+\beta\Lambda_k\Delta_m^n Y_k,\bar{0})}{\rho_3}\right)\right)\right)^{p_k}$$

$$\begin{aligned}
&= \sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{k=1}^{i}\left(M_k\left(\frac{\bar{d}(\alpha\Lambda_k\Delta_m^n X_k,\bar{0})}{\rho_3}+\frac{\bar{d}(\beta\Lambda_k\Delta_m^n Y_k,\bar{0})}{\rho_3}\right)\right)\right)^{p_k}\\
&\leq \sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{k=1}^{i}\frac{1}{2^{p_k}}\left(M_k\left(\frac{\bar{d}(\alpha\Lambda_k\Delta_m^n X_k,\bar{0})}{\rho_1}\right)+M_k\left(\frac{\bar{d}(\beta\Lambda_k\Delta_m^n Y_k,\bar{0})}{\rho_2}\right)\right)\right)^{p_k}\\
&\leq K\sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{k=1}^{i}\left(M_k\left(\frac{\bar{d}(\Lambda_k\Delta_m^n X_k,\bar{0})}{\rho_1}\right)\right)\right)^{p_k}\\
&+ K\sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{k=1}^{i}\left(M_k\left(\frac{\bar{d}(\Lambda_k\Delta_m^n Y_k,\bar{0})}{\rho_2}\right)\right)\right)^{p_k}\\
&< \infty.
\end{aligned}$$

Thus, $\alpha X+\beta Y\in C^F(\mathcal{M},\Lambda,\Delta_m^n,p)$. This proves that $C^F(\mathcal{M},\Lambda,\Delta_m^n,p)$ is a linear space. □

Proposition 1 *The classes of sequences* $C^F(\mathcal{M},\Lambda,\Delta_m^n,p)$, $C_\infty^F(\mathcal{M},\Lambda,\Delta_m^n,p)$, $\ell^F(\mathcal{M},\Lambda,\Delta_m^n,p)$, $O^F(\mathcal{M},\Lambda,\Delta_m^n,p)$ *and* $O_\infty^F(\mathcal{M},\Lambda,\Delta_m^n,p)$ *are metric spaces with respect to the metric,*

$$f(X,Y)=\sum_{k=1}^{mn}\bar{d}(X_k,\bar{0})+\eta(\Lambda_k\Delta_m^n X_k,\Lambda_k\Delta_m^n Y_k),$$

where $Z=C^F,C_\infty^F,O^F,O_\infty^F,\ell^F$.

Proof. The proof of the proposition is direct consequence of the Proposition 3.1 [21]. □

Theorem 2 *Let* $Z(\mathcal{M})$ *be a complete metric space with respect to the metric* η, *the space* $Z(\mathcal{M},\Lambda,\Delta_m^n,p)$ *is a complete metric space with respect to the metric,*

$$f(X,Y)=\sum_{k=1}^{mn}\bar{d}(X_k,\bar{0})+\eta(\Lambda_k\Delta_m^n X_k,\Lambda_k\Delta_m^n Y_k),$$

where $Z=C^F,C_\infty^F,O^F,O_\infty^F,\ell^F$.

Proof. Let $(X^{(u)})$ be a Cauchy sequence in $Z(\mathcal{M}, \Lambda, \Delta_m^n, p)$ such that $(X^{(u)}) = (X_n^{(u)})_{n=1}^{\infty}$. Then for $\epsilon > 0$, there exists a positive integer $n_0 = n_0(\epsilon)$ such that

$$f(X^{(u)}, X^{(v)}) < \epsilon \text{ for all } u, v \geq n_0.$$

By the definition of f, we get

$$\sum_{r=1}^{mn} \bar{d}(X_r^{(u)}, X_r^{(v)}) + \eta(\Lambda_k \Delta_m^n X_k^{(u)}, \Lambda_k \Delta_m^n X_k^{(v)}) < \epsilon, \text{ for all } u, v \geq n_0 \quad (3)$$

$$\implies \sum_{r=1}^{mn} \bar{d}(X_r^{(u)}, X_r^{(v)}) < \epsilon \ \ \forall\, u, v \geq n_0$$

$$\implies \bar{d}(X_r^{(u)}, X_r^{(v)}) < \epsilon \ \ \forall\, u, v \geq n_0, \ r = 1, 2, 3, ..., mn.$$

Hence, $(X_r^{(u)})$ is a Cauchy sequence in $R(I)$, so it is convergent in $R(I)$ by the completeness property of $R(I)$, for $r = 1, 2, 3, ..., mn$.
Let

$$\lim_{u \to \infty} X_r^{(u)} = X_r, \text{ for } r = 1, 2, 3, ..., mn. \quad (4)$$

Next, we have

$$\eta(\Lambda_k \Delta_m^n X_k^{(u)}, \Lambda_k \Delta_m^n X_k^{(v)}) < \epsilon \text{ for all } u, v \geq n_0$$

which implies that $(\Lambda_k \Delta_m^n X_k^{(u)})$ is a Cauchy sequence in $Z(\mathcal{M})$, Since $\mathcal{M} = (M_k)$ is continuous function and so it is convergent in $Z(\mathcal{M})$ by the completeness property of $Z(\mathcal{M})$.
Let $\lim_u \Lambda_k \Delta_m^n X_k^{(u)} = Y_k$ (say), in $Z(\mathcal{M})$, for each $k \in N$. We have to prove $\lim_u X^{(u)} = X$ and $X \in Z(\mathcal{M}, \Lambda, \Delta_m^n, p)$.
For $k = 1$, we have from equation (1) and (4),

$$\lim_u X_{mn+1}^{(u)} + X_{mn+1}, \text{ for } m \geq 1, n \geq 1.$$

Proceeding in this way of induction, we get

$$\lim_u X_k^{(u)} + X_k, \text{ for each } \ k \in N.$$

Also, $\lim_u \Lambda_k \Delta_m^n X_k^{(u)} = \Lambda_k \Delta_m^n X_k$ for each $k \in N$. Now, taking $v \to \infty$ and fixing u, it follows from (3),

$$\sum_{r=1}^{mn} \bar{d}(X_r^{(u)}, X_r) + \eta(\Lambda_k \Delta_m^n X_k^{(u)}, \Lambda_k \Delta_m^n X_k) < \epsilon, \text{ for all } u, v \geq n_0.$$

$$\Longrightarrow f(X^{(u)}, X) < \epsilon, \text{ for all } u \geq n_0.$$

Therefore, we have $\lim_u X^{(u)} = X$.
Now, we show that $X \in Z(\mathcal{M}, \Lambda, \Delta_m^n, p)$. Since

$$f(\Lambda_k \Delta_m^n X_k, \bar{0}) \leq f(\Lambda_k \Delta_m^n X_k^{(i)},\ \Lambda_k \Delta_m^n X_k) + f(\Lambda_k \Delta_m^n X_k^{(i)}, \bar{0}) < \infty.$$

$\Longrightarrow X \in Z(\mathcal{M}, \Lambda, \Delta_m^n, p)$. Hence, $Z(\mathcal{M}, \Lambda, \Delta_m^n, p)$ is a complete metric space. □

Proposition 2 *Let* $1 \leq p = \sup_k p_k < \infty$. *Then,*
(i) *The space* $C^F(\mathcal{M}, \Lambda, \Delta_m^n, p)$ *is a complete metric space with the metric,*

$f_1(X, Y) =$

$$\sum_{r=1}^{mn} \bar{d}(X_r, Y_r) + \inf\left\{\rho > 0 : \left(\sum_{i=1}^{\infty} \frac{1}{i} \sum_{k=1}^{i} \left(M_k\left(\frac{\bar{d}(\Lambda_k \Delta_m^n X_k, \Lambda_k \Delta_m^n Y_k)}{\rho}\right)\right)^p\right)^{\frac{1}{p}} \leq 1\right\}.$$

(ii) *The space* $C_\infty^F(\mathcal{M}, \Lambda, \Delta_m^n, p)$ *is a complete metric space with the metric,*

$f_2(X, Y) =$

$$\sum_{r=1}^{mn} \bar{d}(X_r, Y_r) + \inf\left\{\rho > 0 : \sup_i \frac{1}{i} \sum_{k=1}^{i} \left(M_k\left(\frac{\bar{d}(\Lambda_k \Delta_m^n X_k, \Lambda_k \Delta_m^n Y_k)}{\rho}\right)\right)^{p_k} \leq 1\right\}.$$

(iii) *The space* $\ell^F(\mathcal{M}, \Lambda, \Delta_m^n, p)$ *is a complete metric space with the metric,*

$f_3(X, Y) =$

$$\sum_{r=1}^{mn} \bar{d}(X_r, Y_r) + \inf\left\{\rho > 0 : \left(\sum_{k=1}^{\infty} \left(M_k\left(\frac{\bar{d}(\Lambda_k \Delta_m^n X_k, \Lambda_k \Delta_m^n Y_k)}{\rho}\right)\right)^p\right)^{\frac{1}{p}} \leq 1\right\}.$$

(iv) *The space* $O^F(\mathcal{M}, \Lambda, \Delta_m^n, p)$ *is a complete metric space with the metric,*

$f_4(X, Y) =$

$$\sum_{r=1}^{mn} \bar{d}(X_r, Y_r) + \inf\left\{\rho > 0 : \left(\sum_{i=1}^{\infty} \frac{1}{i} \sum_{k=1}^{i} \left(M_k\left(\frac{\bar{d}(\Lambda_k \Delta_m^n X_k, \Lambda_k \Delta_m^n Y_k)}{\rho}\right)\right)^p\right)^{\frac{1}{p}} \leq 1\right\}.$$

(v) *The space* $O^F_\infty(\mathcal{M},\Lambda,\Delta^n_m,p)$ *is a complete metric space with the metric,*

$f_5(X,Y) =$

$$\sum_{r=1}^{mn}\overline{d}(X_r,Y_r)+\inf\left\{\rho>0:\sup_i\frac{1}{i}\sum_{k=1}^{i}\left(M_k\left(\frac{\overline{d}(\Lambda_k\Delta^n_mX_k,\Lambda_k\Delta^n_mY_k)}{\rho}\right)\right)^{p_k}\leq 1\right\}.$$

Proof. The proof directly comes from ([21], Proposition 3.2). □

Theorem 3 (a) $\ell^F(\mathcal{M},\Lambda,\Delta^n_m,p)\subset O^F(\mathcal{M},\Lambda,\Delta^n_m,p)\subset C^F_\infty(\mathcal{M},\Lambda,\Delta^n_m,p)$ *and the inclusions are strict.*
(b) $Z(\mathcal{M},\Lambda,\Delta^{n-1}_m,p)\subset Z(\mathcal{M},\Lambda,\Delta^n_m,p)$ *(in general* $Z(\mathcal{M},\Lambda,\Delta^i_m,p)\subset Z(\mathcal{M},\Lambda,\Delta^n_m,p)$ *for* $i=1,2,3...,n-1$*), for* $Z=C^F,C^F_\infty,O^F,O^F_\infty,\ell^F$.
(c) $O^F_\infty(\mathcal{M},\Lambda,\Delta^n_m,p)\subset C^F_\infty(\mathcal{M},\Lambda,\Delta^n_m,p)$ *and the inclusion is strict.*

Proof. We shall prove the result for the space $Z=C_\infty$ only and others can be proved in the similar way. Let $(X_k)\in C^F_\infty(\mathcal{M},\Lambda,\Delta^{n-1}_m,p)$. Then, we have

$$\sup_i\frac{1}{i}\left(\sum_{k=1}^{i}M_k\left(\frac{\overline{d}(\Lambda_k\Delta^{n-1}_mX_k,\overline{0})}{\rho}\right)\right)^{p_k}<\infty,\text{ for some }\rho>0.$$

Now, we have

$$\begin{aligned}
&\sup_i\frac{1}{i}\left(\sum_{k=1}^{i}M_k\left(\frac{\overline{d}(\Lambda_k\Delta^n_mX_k,\overline{0})}{2\rho}\right)\right)^{p_k}\\
&= \sup_i\frac{1}{i}\left(\sum_{k=1}^{i}M_k\left(\frac{\overline{d}(\Lambda_k\Delta^{n-1}_mX_k-\Lambda_k\Delta^{n-1}_mX_{k+1},\overline{0})}{2\rho}\right)\right)^{p_k}\\
&\leq \sup_i\frac{1}{2}\left(\frac{1}{i}\left(\sum_{k=1}^{i}M_k\left(\frac{\overline{d}(\Lambda_k\Delta^{n-1}_mX_k,\overline{0})}{2\rho}\right)\right)\right)^{p_k}\\
&+ \sup_i\frac{1}{2}\left(\frac{1}{i}\left(\sum_{k=1}^{i}M_k\left(\frac{\overline{d}(\Lambda_k\Delta^{n-1}_mX_{k+1},\overline{0})}{2\rho}\right)\right)\right)^{p_k}\\
&< \infty.
\end{aligned}$$

Proceeding in this way, we have $Z(\mathcal{M},\Lambda,\Delta^i_m,p)\subset Z(\mathcal{M},\Lambda,\Delta^n_m,p)$, for $0\leq i<n$, for $Z=C^F,C^F_\infty,O^F,O^F_\infty,\ell^F$. □

Theorem 4 (a) *If* $1 \leq p < q < \infty$*, then*
(i) $C^F(\mathcal{M}, \Lambda, \Delta^n_m, p) \subset C^F(\mathcal{M}, \Lambda, \Delta^n_m, q)$;
(ii) $\ell^F(\mathcal{M}, \Lambda, \Delta^n_m, p) \subset \ell^F(\mathcal{M}, \Lambda, \Delta^n_m, q)$;
(b) $C^F(\mathcal{M}, \Lambda, p) \subset C^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$ *for all* $m \geq 1$ *and* $n \geq 1$.

Proof. (i) We shall prove the result for the space $C^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$ and others can be proved in the similar way. Let $X \in C^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$. Then there exists $\rho > 0$ such that

$$\sum_{i=1}^{\infty} \left(\frac{1}{i} \sum_{k=1}^{i} \left(M_k \left(\frac{\bar{d}(\Lambda_k \Delta^n_m X_k, \bar{0})}{\rho} \right) \right) \right)^{p_k} < \infty.$$

This implies that

$$\frac{1}{i} \sum_{k=1}^{i} \left(M_k \left(\frac{\bar{d}(\Lambda_k \Delta^n_m X_k, \bar{0})}{\rho} \right) \right)^{p_k} < 1$$

for sufficiently large values of i. Since (M_k) is non-decreasing, we get

$$\sum_{i=1}^{\infty} \left(\frac{1}{i} \sum_{k=1}^{i} \left(M_k \left(\frac{\bar{d}(\Lambda_k \Delta^n_m X_k, \bar{0})}{\rho} \right) \right) \right)^{q_k}$$

$$\leq \sum_{i=1}^{\infty} \left(\frac{1}{i} \sum_{k=1}^{i} \left(M_k \left(\frac{\bar{d}(\Lambda_k \Delta^n_m X_k, \bar{0})}{\rho} \right) \right) \right)^{p_k} < \infty.$$

Thus, $X \in C^F(\mathcal{M}, \Lambda, \Delta^n_m, q)$. This completes the proof. □

Theorem 5 *Let* $\mathcal{M} = (M_k)$*,* $\mathcal{M}' = (M'_k)$ *and* $\mathcal{M}'' = (M''_k)$ *be Musielak-Orlicz functions satisfying* Δ_2-*condition. Then for* $Z = C^F, C^F_\infty, O^F, O^F_\infty, \ell^F$*, we have*
(i) $Z(\mathcal{M}', \Lambda, \Delta^n_m, p) \subseteq Z(\mathcal{M} \circ \mathcal{M}', \Lambda, \Delta^n_m, p)$.
(ii) $Z(\mathcal{M}', \Lambda, \Delta^n_m, p) \cap Z(\mathcal{M}'', \Lambda, \Delta^n_m, p) \subseteq Z(\mathcal{M}' + \mathcal{M}'', \Lambda, \Delta^n_m, p)$.

Proof. Let $(X_k) \in Z(\mathcal{M}', \Lambda, \Delta^n_m, p)$. For $\epsilon > 0$, there exists $\eta > 0$ such that $\epsilon = \mathcal{M}(\eta)$. Then,

$$M'_k \left(\frac{\bar{d}(\Lambda_k \Delta^n_m X_k, L)}{\rho} \right)^{p_k} < \eta, \text{ for some } \rho > 0, \ L \in R(I).$$

Let $Y_k = M'_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, L)}{\rho}\right)^{p_k}$, for some $\rho > 0$, $L \in R(I)$. Since $\mathcal{M} = (M_k)$ is continuous and non-decreasing, we get

$$M_k(Y_k) = M_k\left(M'_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, L)}{\rho}\right)^{p_k} < M_k(\eta) = \epsilon, \text{ for some } \rho > 0.$$

$\Longrightarrow (X_k) \in Z(\mathcal{M} \circ \mathcal{M}', \Lambda, \Delta^n_m, p)$.
(ii) Let $(X_k) \in Z(\mathcal{M}', \Lambda, \Delta^n_m, p) \cap Z(\mathcal{M}'', \Lambda, \Delta^n_m, p)$. Then,

$$M'_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, L)}{\rho}\right)^{p_k} < \epsilon, \text{ for some } \rho > 0, \ L \in R(I)$$

and

$$M''_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, L)}{\rho}\right)^{p_k} < \epsilon, \text{ for some } \rho > 0, \ L \in R(I).$$

The rest of the proof follows from the equality

$$\begin{aligned}(M'_k + M''_k)&\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, L)}{\rho}\right)^{p_k} \\ &= M'_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, L)}{\rho}\right)^{p_k} + M''_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, L)}{\rho}\right)^{p_k} \\ &< \epsilon + \epsilon = 2\epsilon, \text{ for some } \rho > 0\end{aligned}$$

which implies that $(X_k) \in Z(\mathcal{M}' + \mathcal{M}'', \Lambda, \Delta^n_m, p)$. This completes the proof. □

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