Acta Univ. Sapientiae, Mathematica, **12**, 2 (2020) 260–271

DOI: 10.2478/ausm-2020-0017

# On some properties of split Horadam quaternions

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**Abstract.** In this paper we introduce and study the split Horadam quaternions. We give some identities, among others Binet's formula, Catalan's, Cassini's and d'Ocagne's identities for these numbers.

## 1 Introduction

Let $\mathbb{C}$ be the field of complex numbers. A quaternion $x$ is a hyper-complex number represented by

$$\mathbb{H} = \{x = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} : a_s \in \mathbb{R},\ s = 0, 1, 2, 3\},$$

where $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an orthonormal basis in $\mathbb{R}^4$, which satisfies the quaternion multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji},\ \mathbf{jk} = \mathbf{i} = -\mathbf{kj},\ \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

The quaternions were introduced by W. R. Hamilton in 1843.

**2010 Mathematics Subject Classification:** 11B37, 11R52
**Key words and phrases:** Horadam numbers, quaternion, split quaternion, split Horadam quaternion

Another extension of the complex numbers is the algebra of split quaternions. The split quaternions were introduced by J. Cockle in 1849 [2]. The set of split (or coquaternions) can be represented as

$$\hat{\mathbb{H}} = \{y = b_0 + b_1 i + b_2 j + b_3 k \colon b_s \in \mathbb{R},\ s = 0, 1, 2, 3\},$$

where $\{1, i, j, k\}$ is the basis of $\hat{\mathbb{H}}$ satisfying the following equalities

$$i^2 = -j^2 = -k^2 = -1, \tag{1}$$

$$ij = k = -ji,\ jk = -i = -kj,\ ki = j = -ik. \tag{2}$$

The split quaternion can be rewritten as

$$y = (b_0 + b_1 i) + (b_2 + b_3 i)j = z_1 + z_2 j,\ z_1, z_2 \in \mathbb{C}.$$

The split quaternions contain nontrivial zero divisors, nilpotent elements and idempotents. The conjugate of a split quaternion $y = b_0 + b_1 i + b_2 j + b_3 k$, denoted by $\overline{y}$, is given by $\overline{y} = b_0 - b_1 i - b_2 j - b_3 k$. The norm of $y$ is defined as

$$N(y) = y\overline{y} = b_0^2 + b_1^2 - b_2^2 - b_3^2. \tag{3}$$

Let $y_1, y_2 \in \hat{\mathbb{H}}$, $y_1 = a_1 + b_1 i + c_1 j + d_1 k, y_2 = a_2 + b_2 i + c_2 j + d_2 k$. Then addition and subtraction of the split quaternions is defined as follows

$$y_1 \pm y_2 = (a_1 \pm a_2) + (b_1 \pm b_2)i + (c_1 \pm c_2)j + (d_1 \pm d_2)k.$$

Multiplication of the split quaternions is defined by

$$\begin{aligned} y_1 \cdot y_2 = {} & a_1 a_2 - b_1 b_2 + c_1 c_2 + d_1 d_2 + (a_1 b_2 + b_1 a_2 - c_1 d_2 + d_1 c_2)i \\ & + (a_1 c_2 + c_1 a_2 - b_1 d_2 + d_1 b_2)j + (a_1 d_2 + d_1 a_2 + b_1 c_2 - c_1 b_2)k. \end{aligned} \tag{4}$$

For the basics on split quaternions theory, see [5].

## 2 The Horadam numbers

In [3] Horadam introduced a sequence $\{W_n\}$ defined by the following relation

$$W_0 = a, W_1 = b,\ W_n = pW_{n-1} + qW_{n-2} \text{ for } n \geq 2 \tag{5}$$

for arbitrary $a, b, p, q \in \mathbb{Z}$. This sequence is a certain generalization of famous sequences such as Fibonacci sequence $\{F_n\}$ $(a = 0, b = 1,\ p = q = 1)$, Lucas sequence $\{L_n\}$ ($a = 2, b = 1$, $p = q = 1$), Jacobsthal sequence $\{J_n\}$ ($a = 0, b = 1$, $p = 1, q = 2$), Pell sequence $\{P_n\}$ ($a = 0, b = 1$, $p = 2, q = 1$), Pell-Lucas sequence $\{PL_n\}$ ($a = b = 1$, $p = 2, q = 1$). The sequences defined by (5) are called sequences of the Fibonacci type.

The characteristic equation associated with the recurrence (5) is

$$r^2 - pr - q = 0.$$

Assuming that $p^2 + 4q > 0$, the equation has the following roots

$$r_1 = \frac{p + \sqrt{p^2 + 4q}}{2}, \quad r_2 = \frac{p - \sqrt{p^2 + 4q}}{2}. \tag{6}$$

Note that

$$r_1 + r_2 = p, \tag{7}$$

$$r_1 - r_2 = \sqrt{p^2 + 4q}, \tag{8}$$

$$r_1 r_2 = -q. \tag{9}$$

The Binet's formula for the sequence $\{W_n\}$ has the following form

$$W_n = \frac{(b - ar_2)r_1^n - (b - ar_1)r_2^n}{r_1 - r_2}.$$

Let

$$\alpha = \frac{b - ar_2}{r_1 - r_2}, \quad \beta = \frac{b - ar_1}{r_1 - r_2}. \tag{10}$$

Then

$$W_n = \alpha r_1^n - \beta r_2^n. \tag{11}$$

In the next section we will use the following result.

**Theorem 1** *Let* $n, p, q$ *be integers such that* $n \geq 0$, $p^2 + 4q > 0$. *Then*

$$\sum_{l=0}^{n-1} W_l = \frac{W_n + qW_{n-1} + a(p-1) - b}{p + q - 1}. \tag{12}$$

**Proof.** Using formula (11), (7) and (9), we get

$$\sum_{l=0}^{n-1} W_l = \sum_{l=0}^{n-1} (\alpha r_1^l - \beta r_2^l) = \alpha \frac{1 - r_1^n}{1 - r_1} - \beta \frac{1 - r_2^n}{1 - r_2}$$

$$= \frac{\alpha - \beta - (\alpha r_2 - \beta r_1) - (\alpha r_1^n - \beta r_2^n) + r_1 r_2(\alpha r_1^{n-1} - \beta r_2^{n-1})}{1 - (r_1 + r_2) + r_1 r_2}$$
$$= \frac{\alpha - \beta - (\alpha r_2 - \beta r_1) - W_n - qW_{n-1}}{1 - p - q}.$$

By simple calculations we have $\alpha - \beta = a$, $\alpha r_2 - \beta r_1 = ap - b$. Hence

$$\sum_{l=0}^{n-1} W_l = \frac{W_n + qW_{n-1} + a(p-1) - b}{p + q - 1}.$$

□

Numbers of the Fibonacci type appear in many subjects of mathematics. In [4] Horadam defined the Fibonacci and Lucas quaternions. In [1] the split Fibonacci quaternions $Q_n$ and split Lucas quaternions $T_n$ were introduced by the following relations

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},$$
$$T_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3},$$

where $F_n$, $L_n$ is $n$th Fibonacci and Lucas number, resp. and $\{i, j, k\}$ is the standard basis of split quaternions. In the literature there are many generalizations of the Fibonacci and Lucas sequences, among others $k$-Fibonacci sequence $\{F_{k,n}\}$, $k$-Lucas sequence $\{L_{k,n}\}$, defined for $k \in \mathbb{N}$ in the following way

$$F_{k,0} = 0, F_{k,1} = 1, \;\; F_{k,n} = kF_{k,n-1} + F_{k,n-2} \text{ for } n \geq 2,$$

$$L_{k,0} = 2, L_{k,1} = k, \;\; L_{k,n} = kL_{k,n-1} + L_{k,n-2} \text{ for } n \geq 2.$$

Some interesting results for the split $k$-Fibonacci and split $k$-Lucas quaternions can be found in [6]. In [7] the authors studied split Pell quaternions $SP_n$ and split Pell-Lucas quaternions $SPL_n$ defined by

$$SP_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3},$$
$$SPL_n = PL_n + iPL_{n+1} + jPL_{n+2} + kPL_{n+3},$$

where $P_n$ and $PL_n$ is $n$th Pell and Pell-Lucas number, resp.

We will focus on split Horadam quaternions. We will present some identities for the split Horadam quaternions, which generalize the results for the split Fibonacci quaternions, the split Lucas quaternions, the split Pell quaternions and the split Pell-Lucas quaternions.

## 3 The split Horadam quaternions

For $n \geq 0$ define the split Horadam quaternion $H_n$ by

$$H_n = W_n + iW_{n+1} + jW_{n+2} + kW_{n+3}, \tag{13}$$

where $W_n$ is the $n$th Horadam number and $i, j, k$ are split quaternionic units which satisfy the multiplication rules given by (1) and (2).

By (5) and (13) we obtain

$$\begin{aligned} H_0 &= a + bi + j(pb + qa) + k(p^2b + pqa + qb) \\ H_1 &= b + i(pb + qa) + j(p^2b + pqa + qb) + k(p^3b + p^2qa + 2pqb + q^2a) \\ H_2 &= pb + qa + i(p^2b + pqa + qb) + j(p^3b + p^2qa + 2pqb + q^2a) \\ &\quad + k(p^4b + p^3qa + 2pq(pb + qa) + p^2qb + q^2b). \end{aligned} \tag{14}$$

For any $n \geq 0$ we obtain the norm of $H_n$.

**Proposition 1** *Let $n, p, q$ be integers such that $n \geq 0$, $p^2 + 4q > 0$. Then*

$$\begin{aligned} N(H_n) = (1 - q^2 - p^2q^2)W_n^2 + (1 - p^2 - (p^2 + q^2)^2)W_{n+1}^2 \\ - 2pq(1 + p^2 + q)W_nW_{n+1}. \end{aligned}$$

**Proof.** Using formula (3) and (13), we get

$$\begin{aligned} N(H_n) &= W_n^2 + W_{n+1}^2 - W_{n+2}^2 - W_{n+3}^2 \\ &= W_n^2 + W_{n+1}^2 - (pW_{n+1} + qW_n)^2 - \Big((p^2 + q)W_{n+1} + pqW_n\Big)^2 \\ &= W_n^2 + W_{n+1}^2 - (p^2W_{n+1}^2 + 2pqW_nW_{n+1} + q^2W_n^2) \\ &\quad - ((p^2 + q)^2W_{n+1}^2 + 2pq(p^2 + q)W_nW_{n+1} + p^2q^2W_n^2). \end{aligned}$$

By simple calculations we get the result. □

By (13) we get a recurrence relation for the split Horadam quaternions.

**Proposition 2** *Let $n, p, q$ be integers such that $n \geq 2$, $p^2 + 4q > 0$. Then*

$$H_n = pH_{n-1} + qH_{n-2},$$

*where $H_0$, $H_1$ are given by (14).*

**Proof.** By formula (13) and (5) we get

$$\begin{aligned} pH_{n-1} + qH_{n-2} &= p(W_{n-1} + iW_n + jW_{n+1} + kW_{n+2}) \\ &\quad + q(W_{n-2} + iW_{n-1} + jW_n + kW_{n+1}) \\ &= pW_{n-1} + qW_{n-2} + i(pW_n + qW_{n-1}) \\ &\quad + j(pW_{n+1} + qW_n) + k(pW_{n+2} + qW_{n+1}) \\ &= W_n + iW_{n+1} + jW_{n+2} + kW_{n+3} = H_n, \end{aligned}$$

which ends the proof. □

**Theorem 2** *Let* $n, p, q$ *be integers such that* $n \geq 0$, $p^2 + 4q > 0$. *Then*

(i) $H_n + \overline{H_n} = 2W_n$,

(ii) $N(H_n) = 2W_n H_n - H_n^2$.

**Proof.** (i) Using the definition of the conjugate of a split quaternion we obtain the result.
(ii) By formula (13) we have

$$\begin{aligned} H_n^2 &= W_n^2 - W_{n+1}^2 + W_{n+2}^2 + W_{n+3}^2 \\ &\quad + 2iW_nW_{n+1} + 2jW_nW_{n+2} + 2kW_nW_{n+3} \\ &= -W_n^2 - W_{n+1}^2 + W_{n+2}^2 + W_{n+3}^2 \\ &\quad + 2(W_n^2 + iW_nW_{n+1} + jW_nW_{n+2} + kW_nW_{n+3}) \\ &= 2W_n(W_n + iW_{n+1} + jW_{n+2} + kW_{n+3}) \\ &\quad - W_n^2 - W_{n+1}^2 + W_{n+2}^2 + W_{n+3}^2 \\ &= 2W_nH_n - N(H_n). \end{aligned}$$

Hence we get the result. □

The next theorem presents the Binet's formula for the split Horadam quaternions.

**Theorem 3** (*Binet's formula*) *Let* $n, p, q$ *be integers such that* $n \geq 0$, $p^2 + 4q > 0$. *Then*

$$H_n = \alpha \hat{r_1} r_1^n - \beta \hat{r_2} r_2^n, \tag{15}$$

*where* $r_1, r_2, \alpha, \beta$ *are given by* (6), (10), *resp. and* $\hat{r_1} = 1 + ir_1 + jr_1^2 + kr_1^3$, $\hat{r_2} = 1 + ir_2 + jr_2^2 + kr_2^3$.

**Proof.** By (11) we have

$$\begin{aligned} H_n &= W_n + iW_{n+1} + jW_{n+2} + kW_{n+3} \\ &= \alpha r_1^n - \beta r_2^n + i(\alpha r_1^{n+1} - \beta r_2^{n+1}) + j(\alpha r_1^{n+2} - \beta r_2^{n+2}) \\ &\quad + k(\alpha r_1^{n+3} - \beta r^{n+3}) \\ &= \alpha r_1^n \left(1 + ir_1 + jr_1^2 + kr_1^3\right) - \beta r_2^n \left(1 + ir_2 + jr_2^2 + kr_2^3\right) \\ &= \alpha \hat{r}_1 r_1^n - \beta \hat{r}_2 r_2^n. \end{aligned}$$

□

Using the Binet's formula (15), we can obtain some new identities for the split Horadam quaternions. We will use the following lemma.

**Lemma 1** *Let* $\hat{r}_1 = 1 + ir_1 + jr_1^2 + kr_1^3$, $\hat{r}_2 = 1 + ir_2 + jr_2^2 + kr_2^3$, *where* $r_1, r_2$ *are given by* (6). *Then*

$$\begin{aligned} \hat{r}_1\hat{r}_2 =& 1 + q + q^2 - q^3 + i(p + q^2\sqrt{p^2+4q}) \\ &+ j(p^2 + 2q - pq\sqrt{p^2+4q}) + k(p^3 + 3pq + q\sqrt{p^2+4q}), \end{aligned} \tag{16}$$

$$\begin{aligned} \hat{r}_2\hat{r}_1 =& 1 + q + q^2 - q^3 + i(p - q^2\sqrt{p^2+4q}) \\ &+ j(p^2 + 2q + pq\sqrt{p^2+4q}) + k(p^3 + 3pq - q\sqrt{p^2+4q}). \end{aligned} \tag{17}$$

**Proof.** Using formula (4), we have

$$\begin{aligned} \hat{r}_1\hat{r}_2 &= 1 - r_1r_2 + (r_1r_2)^2 + (r_1r_2)^3 + i(r_1 + r_2 + (r_1r_2)^2(r_1 - r_2)) \\ &\quad + j(r_1^2 + r_2^2 + r_1r_2(r_1^2 - r_2^2)) + k(r_1^3 + r_2^3 - r_1r_2(r_1 - r_2)), \\ \hat{r}_2\hat{r}_1 &= 1 - r_1r_2 + (r_1r_2)^2 + (r_1r_2)^3 + i(r_1 + r_2 - (r_1r_2)^2(r_1 - r_2)) \\ &\quad + j(r_1^2 + r_2^2 - r_1r_2(r_1^2 - r_2^2)) + k(r_1^3 + r_2^3 + r_1r_2(r_1 - r_2)). \end{aligned}$$

By (7) and (9) we get

$$\begin{aligned} r_1^2 + r_2^2 &= (r_1 + r_2)^2 - 2r_1r_2 = p^2 + 2q, \\ r_1^3 + r_2^3 &= (r_1 + r_2)^3 - 3r_1r_2(r_1 + r_2) = p^3 + 3pq. \end{aligned}$$

Hence

$$\begin{aligned} \hat{r}_1\hat{r}_2 &= 1 + q + q^2 - q^3 + i(p + q^2\sqrt{p^2+4q}) \\ &\quad + j(p^2 + 2q - pq\sqrt{p^2+4q}) + k(p^3 + 3pq + q\sqrt{p^2+4q}), \end{aligned}$$

$$\hat{r_2}\hat{r_1} = 1 + q + q^2 - q^3 + i(p - q^2\sqrt{p^2+4q}) \\ + j(p^2 + 2q + pq\sqrt{p^2+4q}) + k(p^3 + 3pq - q\sqrt{p^2+4q}).$$

□

**Corollary 1**

$$\hat{r_1}\hat{r_2} + \hat{r_2}\hat{r_1} = 2(1 + q + q^2 - q^3 + pi + j(p^2 + 2q) + k(p^3 + 3pq)). \tag{18}$$

**Theorem 4** (*Catalan's identity*) *Let* $n, m, p, q$ *be integers such that* $n \geq m$, $p^2 + 4q > 0$. *Then*

$$H_{n-m}H_{n+m} - H_n^2 = \alpha\beta(-q)^{n-m}[(-q)^m(\hat{r_1}\hat{r_2} + \hat{r_2}\hat{r_1}) - r_2^{2m}\hat{r_1}\hat{r_2} - r_1^{2m}\hat{r_2}\hat{r_1}],$$

*where* $\alpha, \beta$, $\hat{r_1}\hat{r_2} + \hat{r_2}\hat{r_1}$, $\hat{r_1}\hat{r_2}$, $\hat{r_2}\hat{r_1}$ *are given by* (10), (18), (16), (17), *resp.*

**Proof.** By (15) we get

$$\begin{aligned} H_{n-m}H_{n+m} - H_n^2 =& (\alpha\hat{r_1}r_1^{n-m} - \beta\hat{r_2}r_2^{n-m})(\alpha\hat{r_1}r_1^{n+m} - \beta\hat{r_2}r_2^{n+m}) \\ & - (\alpha\hat{r_1}r_1^{n} - \beta\hat{r_2}r_2^{n})(\alpha\hat{r_1}r_1^{n} - \beta\hat{r_2}r_2^{n}) \\ =& \alpha\beta(r_1r_2)^{n-m}[(r_1r_2)^m(\hat{r_1}\hat{r_2} + \hat{r_2}\hat{r_1}) \\ & - r_2^{2m}\hat{r_1}\hat{r_2} - r_1^{2m}\hat{r_2}\hat{r_1}]. \end{aligned}$$

Using formula (9), we obtain

$$H_{n-m}H_{n+m} - H_n^2 = \alpha\beta(-q)^{n-m}\left((-q)^m(\hat{r_1}\hat{r_2} + \hat{r_2}\hat{r_1}) - r_2^{2m}\hat{r_1}\hat{r_2} - r_1^{2m}\hat{r_2}\hat{r_1}\right).$$

□

**Corollary 2** (*Cassini's identity*) *Let* $n, p, q$ *be integers such that* $n \geq 0$, $p^2 + 4q > 0$. *Then*

$$H_{n-1}H_{n+1} - H_n^2 = -\alpha\beta(-q)^{n-1}\left(q(\hat{r_1}\hat{r_2} + \hat{r_2}\hat{r_1}) + r_2^2\hat{r_1}\hat{r_2} + r_1^2\hat{r_2}\hat{r_1}\right).$$

Note that for $p = q = 1$ we get the Cassini's identity for the split Fibonacci quaternions $Q_n$ and the split Lucas quaternions $T_n$ ([1]).

**Corollary 3** *Let* $n \geq 1$ *be an integer. Then*

(i) $Q_{n-1}Q_{n+1} - Q_n^2 = (-1)^n(2Q_1 - 2i - 3k)$,

(ii) $T_{n-1}T_{n+1} - T_n^2 = 5(-1)^{n+1}(2Q_1 - 2i - 3k)$.

**Proof.** (i) Using Lemma 1, for $p = q = 1$ we get

$$\begin{aligned} \hat{r_1}\hat{r_2} &= 2 + (1+\sqrt{5})i + (3-\sqrt{5})j + (4+\sqrt{5})k, \\ \hat{r_2}\hat{r_1} &= 2 + (1-\sqrt{5})i + (3+\sqrt{5})j + (4-\sqrt{5})k, \\ \hat{r_1}\hat{r_2} + \hat{r_2}\hat{r_1} &= 4 + 2i + 6j + 8k. \end{aligned}$$

Hence and by Corollary 2 we have

$$\begin{aligned} Q_{n-1}Q_{n+1} - Q_n^2 = &-\frac{1}{5}(-1)^{n-1}[4 + 2i + 6j + 8k \\ &+ \frac{3-\sqrt{5}}{2}(2 + (1+\sqrt{5})i + (3-\sqrt{5})j + (4+\sqrt{5})k) \\ &+ \frac{3+\sqrt{5}}{2}(2 + (1-\sqrt{5})i + (3+\sqrt{5})j + (4-\sqrt{5})k)] \\ = &\, (-1)^n(2 + 4j + 3k) = (-1)^n(2Q_1 - 2i - 3k). \end{aligned}$$

We omit the proof of (ii). □

**Proposition 3** *Let $n, p, q$ be integers such that $n \geq 0$, $p^2 + 4q > 0$. Then*

$$H_{n+1}H_{n-1} - H_n^2 = -\alpha\beta(-q)^{n-1}\left(q(\hat{r_1}\hat{r_2} + \hat{r_2}\hat{r_1}) + r_1^2\hat{r_1}\hat{r_2} + r_2^2\hat{r_2}\hat{r_1}\right).$$

For $p = 2$ and $q = 1$ we get the Cassini's identity for the split Pell quaternions $SP_n$ and the split Pell-Lucas quaternions $SPL_n$ ([7]).

**Corollary 4** *Let $n \geq 1$ be an integer. Then*

$$\begin{aligned} SP_{n+1}SP_{n-1} - SP_n^2 &= (-1)^n(2 + 4i + 2j + 16k), \\ SPL_{n+1}SPL_{n-1} - SPL_n^2 &= (-1)^{n-1}(4 + 8i + 4j + 32k). \end{aligned}$$

**Theorem 5** (*d'Ocagne's identity*) *Let $m, n, p, q$ be integers such that $n \geq 0$, $p^2 + 4q > 0$. Then*

$$H_nH_{m+1} - H_{n+1}H_m = \frac{(-q)^m(b - ar_2)(b - ar_1)}{r_1 - r_2}\left(r_1^{n-m}\hat{r_1}\hat{r_2} - r_2^{n-m}\hat{r_2}\hat{r_1}\right),$$

*where $\hat{r_1}\hat{r_2}$, $\hat{r_2}\hat{r_1}$ are given by (16), (17), resp.*

**Proof.** By (15) we get

$$
\begin{aligned}
H_nH_{m+1} - H_{n+1}H_m &= (\alpha\hat{r}_1 r_1^n - \beta\hat{r}_2 r_2^n)(\alpha\hat{r}_1 r_1^{m+1} - \beta\hat{r}_2 r_2^{m+1}) \\
&\quad - (\alpha\hat{r}_1 r_1^{n+1} - \beta\hat{r}_2 r_2^{n+1})(\alpha\hat{r}_1 r_1^m - \beta\hat{r}_2 r_2^m) \\
&= \alpha\beta(r_1 - r_2)\left(r_1^n r_2^m \hat{r}_1\hat{r}_2 - r_1^m r_2^n \hat{r}_2\hat{r}_1\right) \\
&= \alpha\beta(r_1 - r_2)(r_1r_2)^m\left(r_1^{n-m}\hat{r}_1\hat{r}_2 - r_2^{n-m}\hat{r}_2\hat{r}_1\right) \\
&= \frac{(b - ar_2)(b - ar_1)(-q)^m}{r_1 - r_2}\left(r_1^{n-m}\hat{r}_1\hat{r}_2 - r_2^{n-m}\hat{r}_2\hat{r}_1\right).
\end{aligned}
$$

□

In the next theorem we give a summation formula for the split Horadam quaternions.

**Theorem 6** *Let* $n, p, q$ *be integers such that* $n \geq 0$, $p^2 + 4q > 0$. *Then*

$$
\begin{aligned}
\sum_{l=0}^{n} H_l &= \frac{H_{n+1} + qH_n + (ap - a - b)(1 + i + j + k)}{p + q - 1} \\
&\quad -ia - j(a + b) - k(a + b + pb + qa).
\end{aligned}
$$

**Proof.** By formula (12) we get

$$
\sum_{l=0}^{n} H_l = \sum_{l=0}^{n} H_l + i\sum_{l=0}^{n} H_{l+1} + j\sum_{l=0}^{n} H_{l+2} + k\sum_{l=0}^{n} H_{l+3}
$$

$$
\begin{aligned}
&= \frac{1}{p+q-1}[W_{n+1} + qW_n + a(p-1) - b + i(W_{n+2} + qW_{n+1} + a(p-1) - b) \\
&\quad + j(W_{n+3} + qW_{n+2} + a(p-1) - b) + k(W_{n+4} + qW_{n+3} + a(p-1) - b)] \\
&\quad - iW_0 - j(W_0 + W_1) - k(W_0 + W_1 + W_2).
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
\sum_{l=0}^{n} H_l &= \frac{1}{p+q-1}[W_{n+1} + iW_{n+2} + jW_{n+3} + kW_{n+4} \\
&\quad + q(W_n + iW_{n+1} + jW_{n+2} + kW_{n+3} + (ap - a - b)(1 + i + j + k)] \\
&\quad - ia - j(a + b) - k(a + b + pb + qa) \\
&= \frac{H_{n+1} + qH_n + (ap - a - b)(1 + i + j + k)}{p + q - 1} \\
&\quad - ia - j(a + b) - k(a + b + pb + qa).
\end{aligned}
$$

□

For $p = q = 1$ and $a = 0, b = 1$ we get the result for the split Fibonacci quaternions $Q_n$ ([1]).

**Corollary 5** $\sum_{l=1}^{n} Q_l = Q_{n+2} - Q_2$.

Now we will give the generating function of the split Horadam quaternions.

**Theorem 7** *The generating function of the split Horadam quaternions is*

$$f(x) = \frac{H_0 + (H_1 - pH_0)x}{1 - px - qx^2}.$$

**Proof.** Let $f(x) = H_0 + H_1x + H_2x^2 + \ldots + H_nx^n + \ldots$. Then

$$pxf(x) = pH_0x + pH_1x^2 + pH_2x^3 + \ldots + pH_{n-1}x^n + \ldots$$
$$qx^2f(x) = qH_0x^2 + qH_1x^3 + qH_2x^4 + \ldots + qH_{n-2}x^n + \ldots.$$

Hence, by Proposition 2, we get

$$\begin{aligned} &f(x) - pxf(x) - qx^2f(x) \\ &= H_0 + (H_1 - pH_0)x + (H_2 - pH_1 - qH_0)x^2 + \ldots \\ &= H_0 + (H_1 - pH_0)x. \end{aligned}$$

Thus

$$f(x) = \frac{H_0 + (H_1 - pH_0)x}{1 - px - qx^2}.$$

Moreover, by (14) we obtain

$$H_0 = a + ib + j(pb + qa) + k(p^2b + pqa + qb),$$
$$H_1 - pH_0 = b - pa + iqa + jqb + k(pqb + q^2a).$$

□

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*Received: August 21, 2019*