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On some $L(2,1)$-coloring parameters of certain graph classes

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Abstract. Graph coloring can be considered as a random experiment with the color of a randomly selected vertex as the random variable. In this paper, we consider the $L(2,1)$-coloring of G as the random experiment and we discuss the concept of two fundamental statistical parameters – mean and variance – with respect to the $L(2,1)$-coloring of certain fundamental graph classes.

1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [1, 5, 13, 14]. Moreover, for notions and norms in graph colouring, see [2, 6, 8]. Unless mentioned otherwise, all graphs considered here are undirected, simple, finite and connected.

Graph coloring is an assignment of colors or labels or weights to the elements of the graph. A *vertex coloring* of a graph is a function $c : V(G) \to$

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$\mathcal{C} = \{c_1, c_2, c_3, ...c_l\}$, where $\mathcal{C}$ is a set of l distinct colors. Unless mentioned otherwise, by graph coloring, we mean a vertex coloring of G.

A *proper coloring* of a graph G is a coloring such that no two adjacent vertices receive the same color. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colors required in a proper vertex coloring of the graph G.

Note that the color set $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_l\}$ can also be written as $\mathcal{C} = \{1, 2, 3, \ldots, \}$. Invoking this representation, we have

Definition 1 [4] The $L(2,1)$-*coloring* of a graph G is a vertex coloring which assigns colors to the vertices of graph G satisfying the following two conditions:

$$\begin{aligned} |c(u) - c(v)| &\geq 2 \text{ if } d(u,v) = 1 \\ |c(u) - c(v)| &\geq 1 \text{ if } d(u,v) = 2 \end{aligned}$$

where u and v are vertices of G.

The *span* of a $L(2,1)$-coloring is its maximum label. The minimum span of a $L(2,1)$-coloring of a graph G is called the $L(2,1)$-*chromatic number* of G. This coloring scheme has significant applications in channel assignment problem and many other fields.

A proper k-coloring of graph G be given by c: $V(G) \to \mathcal{C} = \{c_1, c_2, ...c_k\}$. We denote number of vertices of G receiving the color c_i by $\theta(c_i)$ which is called the *color strength* or *color weight* of the color c_i. The *coloring sum* with respect to a given color set $\mathcal{C}$ of G is defined as $\omega_{\mathcal{C}}(G) = \sum_{i=1}^{k} i\theta(c_i)$ (see [7]).

Recently, some studies have been done by treating graph coloring as a random experiment (see [12, 3, 11, 10, 9]) and the color of an arbitrarily chosen vertex of G as the corresponding discrete random variable X. Then, the *probability mass function* (p.m.f) of this discrete random variable X is defined as

$$f(i) = \begin{cases} \frac{\theta(c_i)}{|V(G)|} & \text{if } i = 1, 2, ...k, \\ 0 & elsewhere. \end{cases}$$

where $\theta(c_i)$ is the cardinality of the color class of G with respect to the color c_i (c.f. [12]). If the context is clear, this p.m.f is referred as the p.m.f of G.

For mean and variance, we use the standard notation μ and σ. Therefore, for a graph G with color set $\mathcal{C}$, the *coloring mean* is defined as

$$\mu_{\mathcal{C}}(G) = \frac{\sum_{i=1}^{k} i\theta(c_i)}{\sum_{i=1}^{k} \theta(c_i)}$$

and the *coloring variance* is defined as

$$\sigma^2_{\mathcal{C}}(G) = \frac{\sum_{i=1}^{k} i^2\theta(c_i)}{\sum_{i=1}^{k} \theta(c_i)} - \left(\frac{\sum_{i=1}^{k} i\theta(c_i)}{\sum_{i=1}^{k} \theta(c_i)}\right)^2$$

In general, the r-th moment is given by

$$\mu_{\mathcal{C}^r}(G) = \frac{\sum_{i=1}^{k} i^r\theta(c_i)}{\sum_{i=1}^{k} \theta(c_i)}$$

where r is any positive integer. If context is clear, we say that $\mu_{\mathcal{C}}(G)$ and $\sigma^2_{\mathcal{C}}(G)$ are the chromatic mean and variance of G.

Motivated by the above studies, in this paper, we extend the notions of chromatic mean and variance to $L(2,1)$-coloring of graphs.

2 Discussion and new results

Throughout this discussion, we denote the $L(2,1)$-color set of G with the minimum possible color by $\mathcal{C}(G)$. In view of this convention, we have the following definitions:

Definition 2 Let $\mathcal{C} = \{c_1, c_2, ...c_l\}$ be the color set corresponding to an $L(2,1)$-coloring c of a given graph G. The coloring mean corresponding to the $L(2,1)$- coloring having minimum chromatic sum is called L_1^--*chromatic mean* of G and is denoted by $\mu_{\mathcal{C}_-}(G)$.

Definition 3 Let $\mathcal{C} = \{c_1, c_2, ...c_l\}$ be the color set corresponding to an $L(2,1)$- coloring c of a given graph G. The coloring variance corresponding to the $L(2,1)$- coloring having minimum chromatic sum is called L_1^--*chromatic variance* of G and is denoted by $\sigma^2_{\mathcal{C}_-}(G)$.

Definition 4 Let $\mathcal{C} = \{c_1, c_2, ...c_l\}$ be the color set corresponding to an $L(2,1)$- coloring c of a given graph G. The coloring mean corresponding to the $L(2,1)$- coloring having maximum chromatic sum is called L_1^+-*chromatic mean* of G and is denoted by $\mu_{\mathcal{C}_+}(G)$.

Definition 5 Let $\mathcal{C} = \{c_1, c_2, ...c_l\}$ be the color set corresponding to an $L(2,1)$- coloring c of a given graph G. The coloring variance corresponding to the $L(2,1)$- coloring having maximum chromatic sum is called L_1^+-*chromatic variance* of G and is denoted by $\sigma^2_{\mathcal{C}_+}(G)$.

In view of the above notions, the chromatic mean and variance corresponding to L_1^- and L_1^+ coloring of complete graphs is discussed below:

Theorem 6 *For a complete graph K_n, the coloring parameters, L_1^--chromatic mean and variance and L_1^+-chromatic mean and variance are given by*

$$\mu_{\mathcal{C}_-}(K_n) = \mu_{\mathcal{C}_+}(K_n) = n$$

$$\sigma^2_{\mathcal{C}_-}(K_n) = \sigma^2_{\mathcal{C}_+}(K_n) = \frac{n^2-1}{3}$$

Proof. In a complete graph, each vertex receives distinct color and color difference between any two vertices is at least 2. Therefore, we need at least $(2n-1)$ colors say, $\{c_1, c_3, c_5, ...c_{2n-1}\}$, for coloring the vertices of K_n. We cannot use the colors $\{c_2, c_4, ...c_{2n}\}$ by the protocol of $L(2,1)$- coloring. For illustration, see Figure 1. Hence, the corresponding $p.m.f$ is given by

$$f(i) = \begin{cases} \frac{1}{n} & \text{for } i = 1, 3, 5, ...(2n-1), \\ 0 & \text{elsewhere.} \end{cases}$$

Here, we observe that the minimum and maximum coloring sum remains the same. Therefore,

$$\begin{aligned} \mu_{\mathcal{C}_-}(K_n) = \mu_{\mathcal{C}_+}(K_n) &= \frac{1+3+5+....+(2n-1)}{n} \\ &= \frac{1}{n}(n^2) \\ &= n \end{aligned}$$

$$\begin{aligned}\sigma^2_{\mathcal{C}_-}(K_n) = \sigma^2_{\mathcal{C}_+}(K_n) &= \frac{1^2+3^2+5^2+....+(2n-1)^2}{n} - n^2\\ &= \frac{1}{n}\frac{n(2n-1)(2n+1)}{3} - n^2\\ &= \frac{n^2-1}{3}\end{aligned}$$

□

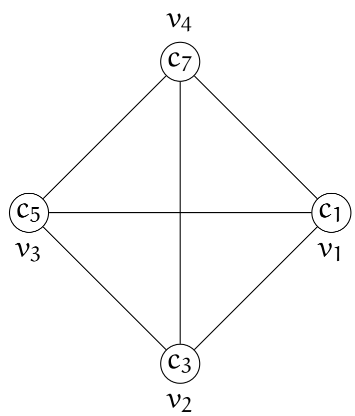


Figure 1

Theorem 7 *For path P_n of length $n \equiv 1, 2 \pmod 3$, The L_1^--chromatic mean is*

$$\mu_{\mathcal{C}_-}(P_n) = \frac{3n-2}{n}$$

and their L_1^--chromatic variance is given by

$$\sigma^2_{\mathcal{C}_-}(P_n) = \begin{cases} \dfrac{8n^2+4n-12}{3n^2} & \text{if } n \equiv 1 \pmod 3\\ \dfrac{8n^2-4n-12}{3n^2} & \text{if } n \equiv 2 \pmod 3\end{cases}$$

Also, for $n \equiv 0 \pmod 3$, the L_1^--chromatic mean for path P_n is

$$\mu_{\mathcal{C}_-}(P_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 5,\\ \dfrac{3n-2}{n} & \text{if } n \equiv 1,2,3 \pmod 5\\ \dfrac{3n-1}{n} & \text{if } n \equiv 4 \pmod 5\end{cases}$$

and it's L_1^-*-chromatic variance is given by*

$$\sigma^2_{\mathcal{C}_-}(P_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 5 \\ \dfrac{2n^2+2n-4}{n^2} & \text{if } n \equiv 1 \pmod 5 \\ \dfrac{2n^2-4}{n^2} & \text{if } n \equiv 2,3 \pmod 5 \\ \dfrac{2n^2+n-1}{n^2} & \text{if } n \equiv 4 \pmod 5 \end{cases}$$

Proof. Note that according to the L(2, 1)- coloring protocol, any three consecutive vertices of P_n must receive distinct colors. L(2, 1)- chromatic number of P_n is 3, thus we have the color set as $\{c_1, c_2, c_3, c_4, c_5\}$. Now let us consider each case separately.
Case 1: When $n \equiv 1 \pmod 3$, we observe that $(\frac{n+2}{3})$ vertices receive the color c_1, $(\frac{n-1}{3})$ vertices each receive color c_3 and c_5. In accordance with L(2, 1)-coloring protocol, c_2 and c_4 cannot be assigned to any vertex. Then, the corresponding p.m.f is given by

$$f(i) = \begin{cases} \dfrac{n+2}{3n} & \text{if } i = 1, \\ \dfrac{n-1}{3n} & \text{if } i = 3, 5, \\ 0 & \text{elsewhere.} \end{cases}$$

Therefore, the L_1^--chromatic mean $= (1)\frac{n+2}{3n} + (3+5)\frac{n-1}{3n} = \frac{3n-2}{n}$ and variance $= (1^2)\frac{n+2}{3n} + (3^2+5^2)\frac{n-1}{3n} - (\frac{3n-2}{n})^2 = \frac{8n^2+4n-12}{3n^2}$ (refer to Figure 2).

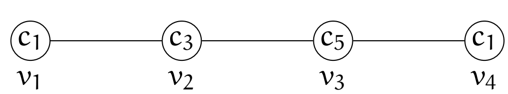


Figure 2

Case 2: When $n \equiv 2 \pmod 3$, we observe that $(\frac{n+1}{3})$ vertices each receive the color c_1 and c_3, $(\frac{n-2}{3})$ vertices receive color c_5. In accordance with L(2, 1)-coloring protocol, c_2 and c_4 cannot be assigned to any vertex. Then, the corresponding $p.m.f$ is given by

$$f(i) = \begin{cases} \frac{n+1}{3n} & \text{if } i = 1, 3, \\ \frac{n-2}{3n} & \text{if } i = 5, \\ 0 & \text{elsewhere.} \end{cases}$$

Then, the L_1^--chromatic mean $= (1+3)\frac{n+1}{3n} + (5)\frac{n-2}{3n} = \frac{3n-2}{n}$ and variance $= (1^2+3^2)\frac{n+1}{3n} + (5^2)\frac{n-2}{3n} - (\frac{3n-2}{n})^2 = \frac{8n^2-4n-12}{3n^2}$ (refer to Figure 3).

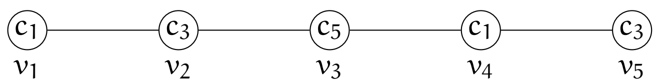


Figure 3

Case 3: When $n \equiv 0 \pmod 5$, each color c_1, c_2, c_3, c_4 and c_5 is given to $(\frac{n}{5})$ vertices. Then, the corresponding $p.m.f$ is given by

$$f(i) = \begin{cases} \frac{1}{5} & \text{if } i = 1, 2, 3, 4, 5, \\ 0 & \text{elsewhere.} \end{cases}$$

The L_1^--chromatic mean $= \sum_{i=1}^{5}(i)\frac{1}{5} = \frac{15}{5} = 3$ and

variance $= \sum_{i=1}^{5}(i^2)\frac{1}{5} - (3^2) = 11 - 9 = 2$ (refer to Figure 4).

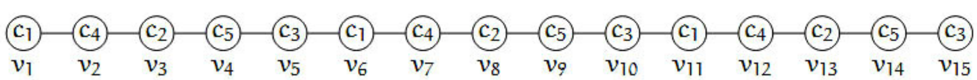


Figure 4

Case 4: When $n \equiv 1 \pmod 5$, we shall see that $(\frac{n+4}{5})$ vertices receive color c_1, and $(\frac{n-1}{5})$ vertices each receive color c_2, c_3, c_4 and c_5. Then, the $p.m.f$ is given by

$$f(i) = \begin{cases} \frac{n+4}{5n} & \text{if } i = 1 \\ \frac{n-1}{5n} & \text{if } i = 2, 3, 4, 5, \\ 0 & \text{elsewhere.} \end{cases}$$

The L_1^--chromatic mean $= (1)\frac{n+4}{5n} + (2+3+4+5)\frac{n-1}{5n} = \frac{15n-10}{5n} = \frac{3n-2}{n}$ and variance $= (1^2)\frac{n+4}{5n} + (2^2+3^2+4^2+5^2)\frac{n-1}{5n} - (\frac{3n-2}{n})^2 = \frac{2n^2+2n-4}{n^2}$ (refer to Figure 5).

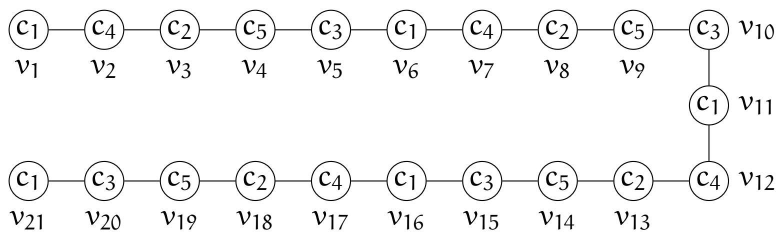


Figure 5

Case 5: When $n \equiv 2 \pmod 5$, we shall give $(\frac{n+3}{5})$ vertices each color c_1 and c_3; $(\frac{n-2}{5})$ vertices each color c_2, c_4 and c_5. Then, the $p.m.f$ is given by

$$f(i) = \begin{cases} \dfrac{n+3}{5n} & \text{if } i = 1,3, \\ \dfrac{n-2}{5n} & \text{if } i = 2,4,5 \\ 0 & \text{elsewhere.} \end{cases}$$

The L_1^--chromatic mean $= (1+3)\frac{n+3}{5n} + (2+4+5)\frac{n-2}{5n} = \frac{15n-10}{5n} = \frac{3n-2}{n}$ and variance $= (1^2+3^2)\frac{n+3}{5n} + (2^2+4^2+5^2)\frac{n-2}{5n} - (\frac{3n-1}{n})^2 = \frac{2n^2-4}{n^2}$ (refer to Figure 6).

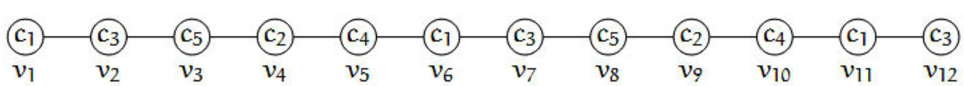


Figure 6

Case 6: When $n \equiv 3 \pmod 5$, we observe that each $(\frac{n+2}{5})$ vertices receive the color c_1, c_4 and c_2; and each $(\frac{n-3}{5})$ vertices receive the color c_3 and c_5.

Then, the p.m.f is given by

$$f(i) = \begin{cases} \dfrac{n+2}{5n} & \text{if } i = 1,4,2 \\ \dfrac{n-3}{5n} & \text{if } i = 3,5, \\ 0 & \text{elsewhere.} \end{cases}$$

The L_1^--chromatic mean $= (1+2+4)\frac{n+2}{5n} + (3+5)\frac{n-3}{5n} = \frac{15n-10}{5n} = \frac{3n-2}{n}$ and variance $= (1^2+2^2+4^2)\frac{n+2}{5n}+(3^2++5^2)\frac{n-3}{5n}-(\frac{3n-2}{n})^2 = \frac{2n^2-4}{n^2}$ (refer to Figure 7).

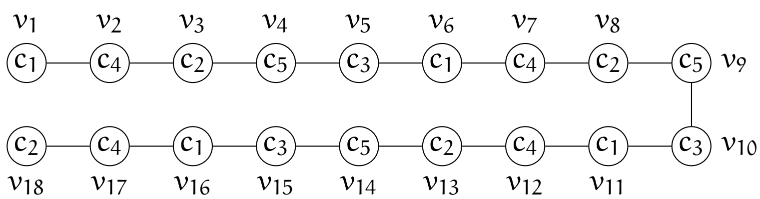


Figure 7

Case 7: When $n \equiv 4 \pmod 5$, we observe that $(\frac{n-4}{5})$vertices receive color c_4, $(\frac{n+1}{5})$ vertices each receives color c_1, c_2, c_3 and c_5. Then, the p.m.f is given by

$$f(i) = \begin{cases} \dfrac{n+1}{5n} & \text{if } i = 1,2,3,5 \\ \dfrac{n-4}{5n} & \text{if } i = 4, \\ 0 & \text{elsewhere.} \end{cases}$$

The L_1^--chromatic mean $= (1+2+3+5)\frac{n+1}{5n} + (4)\frac{n-4}{5n} = \frac{15n-5}{5n} = \frac{3n-1}{n}$ and variance $= (1^2+2^2+3^2+5^2)\frac{n+1}{5n}+(4^2)\frac{n-4}{5n}-(\frac{3n-1}{5n})^2 = \frac{2n^2+n-1}{n^2}$ (refer to Figure 8).

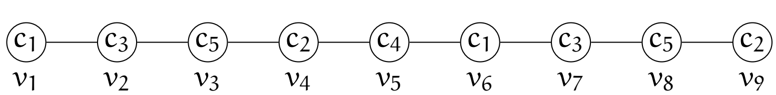


Figure 8

□

Theorem 8 *The* L_1^+*-chromatic mean of path* P_n *of length* $n \equiv 1, 2 \pmod 3$ *is*

$$\mu_{\mathcal{C}_+}(P_n) = \frac{3n+2}{n}$$

and their L_1^+*-chromatic variance is given by*

$$\sigma^2_{\mathcal{C}_+}(P_n) = \begin{cases} \dfrac{8n^2+4n-12}{3n^2} & \text{if } n \equiv 1 \pmod 3 \\ \dfrac{8n^2-4n-12}{3n^2} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Also, the L_1^+*-chromatic mean for path* P_n *of length* $n \equiv 0 \pmod 3$ *is*

$$\mu_{\mathcal{C}_+}(P_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 5 \\ \dfrac{3n+1}{n} & \text{if } n \equiv 1 \pmod 5 \\ \dfrac{3n+2}{n} & \text{if } n \equiv 2, 3, 4 \pmod 5 \\ 0 & \text{elsewhere.} \end{cases}$$

and it's L_1^+*-chromatic variance is given by*

$$\sigma^2_{\mathcal{C}_+}(P_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 5 \\ \dfrac{2n^2-n-1}{n^2} & \text{if } n \equiv 1 \pmod 5 \\ \dfrac{2n^2-4}{n^2} & \text{if } n \equiv 2, 3 \pmod 5 \\ \dfrac{2n^2-2n-4}{n^2} & \text{if } n \equiv 4 \pmod 5 \\ 0 & \text{elsewhere.} \end{cases}$$

Proof. In accordance with $L(2,1)$- coloring protocol, any three consecutive vertices of P_n must receive distinct colors. $L(2,1)$- chromatic number of P_n is 3 and the corresponding color set is $\{c_1, c_2, c_3, c_4, c_5\}$. Considering each case separately,

Case 1: When $n \equiv 1 \pmod 3$, we shall give color c_5 to $(\frac{n+2}{3})$ vertices and color c_1 and c_3 to $(\frac{n-1}{3})$ vertices. c_2 and c_4 cannot be assigned to any vertex according to the $L(2,1)$- coloring protocol. Then, the corresponding $p.m.f$ is given by

$$f(i) = \begin{cases} \dfrac{n-1}{3n} & \text{if } i = 1, 3, \\ \dfrac{n+2}{3n} & \text{if } i = 5, \\ 0 & \text{elsewhere.} \end{cases}$$

Then, the L_1^+-chromatic mean $= (1+3)\frac{n-1}{3n} + (5)\frac{n+2}{3n} = \frac{3n+2}{n}$ and variance $= (1^2+3^2)\frac{n-1}{3n} + (5^2)\frac{n+2}{3n} - (\frac{3n+2}{n})^2 = \frac{8n^2+4n-12}{3n^2}$ (refer to Figure 9).

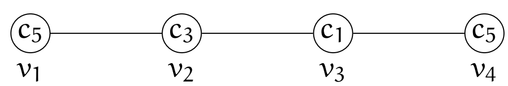


Figure 9

Case 2: When $n \equiv 2 \pmod 3$, we shall give $(\frac{n-2}{3})$ vertices color c_1 and $(\frac{n+1}{3}$ vertices each receive color c_3 and c_5. Then, the $p.m.f$ is given by

$$f(i) = \begin{cases} \frac{n-2}{3n} & \text{if } i = 1, \\ \frac{n+1}{3n} & \text{if } i = 3, 5, \\ 0 & \text{elsewhere.} \end{cases}$$

Then, the L_1^+-chromatic mean $= (1)\frac{n-2}{3n} + (3+5)\frac{n+1}{3n} = \frac{3n+2}{n}$ and variance $= (1^2)\frac{n-2}{3n} + (3^2+5^2)\frac{n+1}{3n} - (\frac{3n+2}{n})^2 = \frac{8n^2-4n-12}{3n^2}$ (refer to Figure 10).

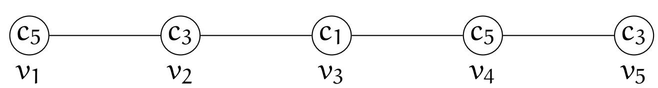


Figure 10

When n ≡ 0 (mod 3), the $p.m.f$ is given by

Case 3: When $n \equiv 0 \pmod 5$, each color c_1, c_2, c_3, c_4 and c_5 is given to $(\frac{n}{5})$ vertices. Then, the corresponding $p.m.f$ is given by

$$f(i) = \begin{cases} \dfrac{1}{5} & \text{if } i = 1, 2, 3, 4, 5, \\ 0 & \text{elsewhere.} \end{cases}$$

The L_1^+-chromatic mean $= \sum_{i=1}^{5}(i)\frac{1}{5} = \frac{15}{5} = 3$ and

variance $= \sum_{i=1}^{5}(i^2)\frac{1}{5} - (3^2) = 11 - 9 = 2$ (refer to Figure 11).

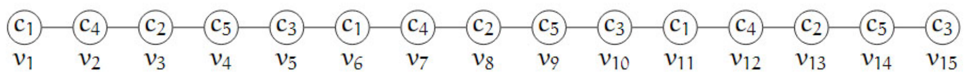


Figure 11

Case 4: When $n \equiv 1 \pmod 5$, we observe that $(\frac{n+4}{5})$ vertices receive the color c_4 and $(\frac{n-1}{5})$ vertices each receive c_1, c_2, c_4 and c_5. Then, the $p.m.f$ is given by

$$f(i) = \begin{cases} \dfrac{n+4}{5n} & \text{if } i = 4 \\ \dfrac{n-1}{5n} & \text{if } i = 1, 2, 3, 5, \\ 0 & \text{elsewhere.} \end{cases}$$

The L_1^+-chromatic mean $= (4)\frac{n+4}{5n} + (1+2+3+5)\frac{n-1}{5n} = \frac{15n+5}{5n} = \frac{3n+1}{n}$ and variance $= (4^2)\frac{n+4}{5n} + (1^2+2^2+3^2+5^2)\frac{n-1}{5n} - (3)^2 = \frac{2n^2-n-1}{n^2}$ (refer to Figure 12).

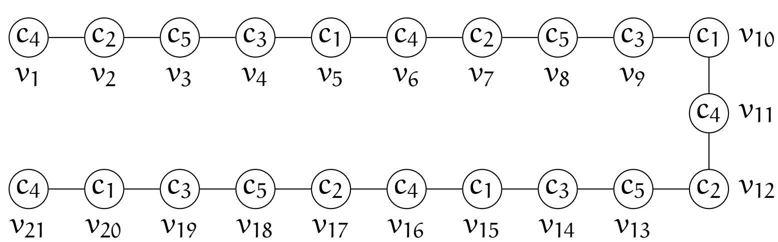


Figure 12

Case 5: When $n \equiv 2 \pmod 5$, we observe that $(\frac{n+3}{5})$ vertices each receive

c_3, c_5 and $(\frac{n-2}{5})$ vertices each receive c_1, c_2 and c_4. The $p.m.f$ is given by

$$f(i) = \begin{cases} \dfrac{n+3}{5n} & \text{if } i = 3,5 \\ \dfrac{n-2}{5n} & \text{if } i = 1,2,4, \\ 0 & \text{elsewhere.} \end{cases}$$

The L_1^+-chromatic mean $= (3+5)\frac{n+3}{5n} + (1+2+4)\frac{n-2}{5n} = \frac{15n+10}{5n} = \frac{3n+2}{n}$ and variance $= (3^2+5^2)\frac{n+3}{5n} + (1^2+2^2+4^2)\frac{n-2}{5n} - (\frac{3n+2}{n})^2 = \frac{2n^2-4}{n^2}$ (refer to Figure 13).

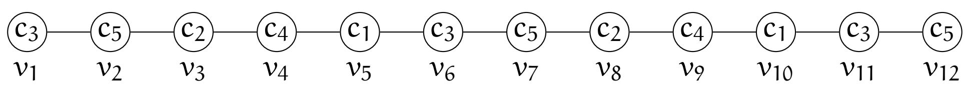


Figure 13

Case 6: When $n \equiv 3 \pmod 5$, we shall give color c_2, c_4 and c_5 to each $(\frac{n+2}{5})$ vertices and color c_1 and c_3 to each $(\frac{n-3}{5})$ vertices. Then, the corresponding $p.m.f$ is given by

$$f(i) = \begin{cases} \dfrac{n+2}{5n} & \text{if } i = 2,4,5 \\ \dfrac{n-3}{5n} & \text{if } i = 1,3, \\ 0 & \text{elsewhere.} \end{cases}$$

The L_1^+-chromatic mean $= (2+4+5)\frac{n+2}{5n} + (1+3)\frac{n-3}{5n} = \frac{15n+10}{5n} = \frac{3n+2}{n}$ and variance $= (2^2+4^2+5^2)\frac{n+2}{5n} + (1^2+3^2)\frac{n-3}{5n} - (\frac{3n+2}{n})^2 = \frac{2n^2-4}{n^2}$ (refer to Figure 14).

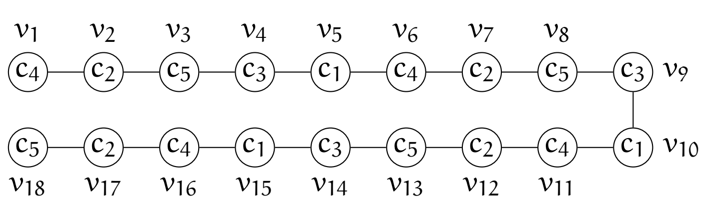


Figure 14

Case 7: When $n \equiv 4 \pmod 5$, we give c_1 to $(\frac{n-4}{5})$ vertices and each color c_2, c_3, c_4, c_5 to $(\frac{n+1}{5})$ vertices. Then, the corresponding $p.m.f$ is given by

$$f(i) = \begin{cases} \dfrac{n+1}{5n} & \text{if } i = 2,3,4,5 \\ \dfrac{n-4}{5n} & \text{if } i = 1, \\ 0 & \text{elsewhere.} \end{cases}$$

The L_1^+-chromatic mean $= (2+3+4+5)\frac{n+1}{5n} + (1)\frac{n-4}{5n} = \frac{15n+10}{5n} = \frac{3n+2}{n}$ and variance $= (2^2+3^2+4^2+5^2)\frac{n+1}{5n} + (1^2)\frac{n-4}{5n} - (\frac{3n+2}{n})^2 = \frac{2n^2-2n-4}{n^2}$ (refer to Figure 15).

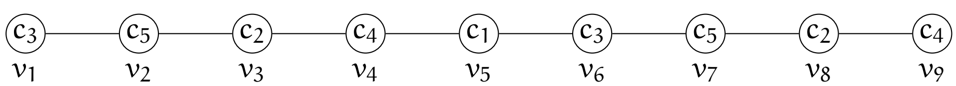


Figure 15

□

Next our aim is to find L_1^--chromatic mean of cycles. Consider C_3 and C_6 and their color set $\{c_1, c_3, c_5\}$, their $p.m.f$ is given by

$$f(i) = \begin{cases} \dfrac{1}{3} & \text{if } i = 1,3,5, \\ 0 & \text{elsewhere.} \end{cases}$$

The L_1^--chromatic mean $= (1+3+5)\frac{1}{3} = \frac{9}{3} = 3$ and
variance $= (1^2+3^2+5^2)\frac{1}{3} - (3^2) = \frac{35}{3} - 9 = \frac{8}{3}$.
Therefore, for C_3 and C_6 L_1^--chromatic mean is 3 and L_1^--chromatic variance is $\frac{8}{3}$.

Theorem 9 *The* L_1^-*-chromatic mean of cycle* C_n *where* $n \neq 3, 6$ *is*

$$\mu_{\mathcal{C}_-}(C_n) = 3$$

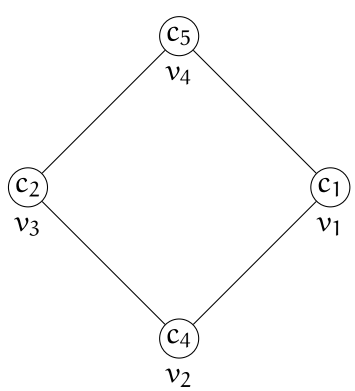
and L_1^-*-chromatic variance for* C_n *where* $n \neq 3, 6$ *is given by*

$$\sigma^2_{C\_}(C_n) = \begin{cases} \frac{5}{2} & \textit{if } n \equiv 0 \text{ mod } 4) \\ \frac{5n-5}{2n} & \textit{if } n \equiv 1 \text{ (mod 4)} \\ \frac{5n-10}{2n} & \textit{if } n \equiv 2 \text{ (mod 4)} \\ \frac{5n+1}{2n} & \textit{if } n \equiv 3 \text{ (mod 4)} \end{cases}$$

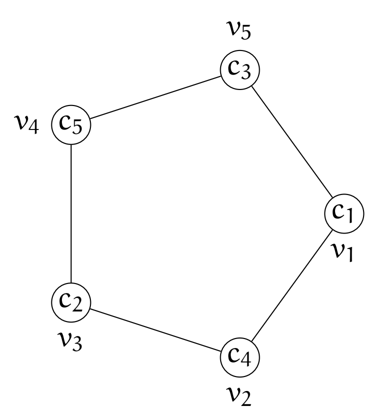
Proof. From the definition of $L(2,1)$- coloring, any three consecutive vertices of C_n must receive distinct colors. Chromatic number of C_n is 5 and the color classes used are c_1, c_2, c_3, c_4, c_5. Now let us consider each case separately.
Case 1: When $n \equiv 0$ (mod 4),each color c_1, c_2, c_4 and c_5 is received by $(\frac{n}{4})$ vertices. Hence, the $p.m.f$ is given by

$$f(i) = \begin{cases} \frac{1}{4} & \text{if } i = 1, 2, 4, 5, \\ 0 & \text{elsewhere.} \end{cases}$$

The L_1^--chromatic mean $= (1+2+4+5)\frac{1}{4} = \frac{12}{4} = 3$ and
variance $= (1^2+2^2+4^2+5^2)\frac{1}{4} - (3^2) = \frac{46}{4} - 9 = \frac{5}{2}$ (refer to Figure 16a).



(a)



(b)

Figure 16

Case 2: When $n \equiv 1 \pmod 4$, c_3 is given to the last vertex i.e. v_n and remaining $(\frac{n-1}{4})$ vertices each receive c_1, c_2, c_4 and c_5. Then, the $p.m.f$ is given by

$$f(i) = \begin{cases} \dfrac{n-1}{4n} & \text{if } i = 1,2,4,5 \\ \dfrac{1}{n} & \text{if } i = 3, \\ 0 & \text{elsewhere.} \end{cases}$$

The L_1^--chromatic mean $= (1+2+4+5)\frac{n-1}{4n} + (3)\frac{1}{n} = 3$ and variance $= (1^2+2^2+4^2+5^2)\frac{n-1}{4n} + (3^2)\frac{1}{n} - (3)^2 = \frac{5n-5}{2n}$ (refer to Figure 16b).
Case 3: When $n \equiv 2 \pmod 4$, we observe that two vertices receive c_3 and $(\frac{n-2}{4})$ vertices each receive c_1, c_2, c_4 and c_5. Then, the $p.m.f$ is given by

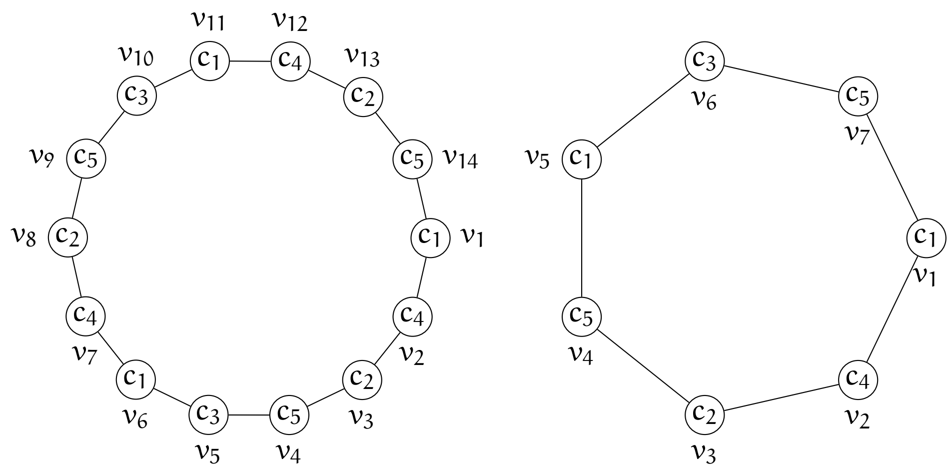
$$f(i) = \begin{cases} \dfrac{n-2}{4n} & \text{if } i = 1,2,4,5 \\ \dfrac{2}{n} & \text{if } i = 3, \\ 0 & \text{elsewhere.} \end{cases}$$

The L_1^--chromatic mean $= (1+2+4+5)\frac{n-2}{4n} + (3)\frac{2}{n} = 3$ and variance $= (1^2+2^2+4^2+5^2)\frac{n-2}{4n} + (3^2)\frac{2}{n} - (3)^2 = \frac{5n-10}{2n}$ (refer to Figure 17a).

Case 4: When $n \equiv 3 \pmod 4$,each set of $(\frac{n+1}{4})$ vertices receive c_1 and c_5; each set of $(\frac{n-3}{4})$ vertices receive c_2 and c_4; and one vertex receives c_3. Then, the corresponding $p.m.f$ is given by

$$f(i) = \begin{cases} \dfrac{n+1}{4n} & \text{if } i = 1,5 \\ \dfrac{n-3}{4n} & \text{if } i = 2,4 \\ \dfrac{1}{n} & \text{if } i = 3, \\ 0 & \text{elsewhere.} \end{cases}$$

The L_1^--chromatic mean $= (1+5)\frac{n+1}{4n} + (2+4)\frac{n-3}{4n} + (3)\frac{1}{n} = 3$ and variance $= (1^2+5^2)\frac{n+1}{4n} + (2^2+4^2)\frac{n-3}{4n} + (3^2)\frac{1}{n} - (3)^2 = \frac{5n+1}{2n}$ (refer to Figure 17b). □



(a) (b)

Figure 17

Theorem 10 *The* L_1^+*-chromatic mean of cycle of length* $n \equiv 0 \pmod 3$ *is*

$$\mu_{\mathcal{C}_+}(C_n) = 3$$

and L_1^+*-chromatic variance by*

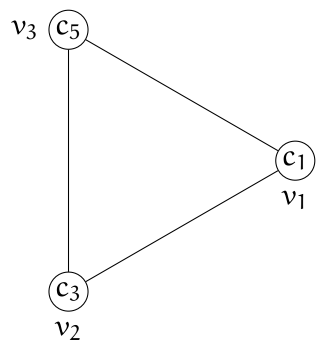
$$\sigma^2_{\mathcal{C}_+}(C_n) = \frac{8}{3}$$

Proof. In case of $n \equiv 1,2 \pmod 3$, L_1^- and L_1^+-chromatic mean are same and so is the case of L_1^- and L_1^+-chromatic variance. Therefore, we just consider the cycle of length $n \equiv 0 \pmod 3$. Here, $(\frac{n}{3})$ vertices each receive color c_1, c_3 and c_5.c_2 and c_4 are not received by any vertex of graph G. For illustration, see Figure 18a. The corresponding $p.m.f$ for L_1^+ coloring is given by

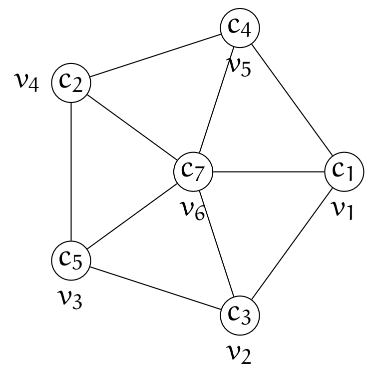
$$f(i) = \begin{cases} \frac{1}{3}, & \text{for i = 1, 3, 5} \\ 0 & \text{elsewhere} \end{cases}$$

The L_1^+-chromatic mean $= (1+3+5)\frac{1}{3} = 3$ and variance $= (1^2+3^2+5^2)\frac{1}{3} - (3)^2 = \frac{8}{3}$.

□



(a)



(b)

Figure 18

Theorem 11 *For wheel graphs having n vertices, where $n \geq 6$, mean and variance for L_1^- and L_1^+ coloring are given by*

$$\mu_{\mathcal{C}_-}(W_n) = \mu_{\mathcal{C}_+}(W_n) = \frac{n^2+n+2}{2n}$$

and

$$\sigma^2_{\mathcal{C}_-}(W_n) = \sigma^2_{\mathcal{C}_+}(W_n) = \frac{n^4+11n^2-12}{12n^2}$$

Proof. The diameter of wheel graph is 2. Also, the central vertex is adjacent to all the other vertices. Hence, we need (n+1) colors. We give the color c_{n+1} to the central vertex and remaining colors to the other vertices of G. Its $p.m.f$ is given by

$$f(i) = \begin{cases} \dfrac{1}{n}, & \text{if } i = 1, 2, ...(n-1), (n+1) \\ 0 & \text{elsewhere} \end{cases}$$

Therefore, L_1^- and L_1^+-chromatic mean $= (1+2+...n-1+n+1)\frac{1}{n} = \frac{n^2+n+2}{2n}$. L_1^- and L_1^+ chromatic variance $= (1^2+2^2+...(n-1)^2+(n+1)^2\frac{1}{n} - (\frac{n^2+n+2}{2n})^2 = \frac{n^4+11n^2-12}{12n^2}$ (refer to Figure 18b). □

Theorem 12 *For helm graphs having $2n+1$ vertices, where $n \geq 7$, L_1^--chromatic mean is given by*

$$\mu_{\mathcal{C}_-}(H_n) = \frac{n^2+5n+28}{4n+2}$$

and L_1^-*-chromatic variance is given by*

$$\sigma^2_{\mathcal{C}_-}(H_n) = \frac{2n^3+9n^2+31n+198}{6(2n+1)}$$

Proof. We need $n+2$ colors to color the vertices of helm graph. The wheel graph induced from the given helm graph is colored as discussed in the previous theorem. Among the remaining n vertices, $n-4$ vertices receive c_1, 2 vertices receive C_2, and c_3, c_4 is given to one vertex each. For illustration, see Figure 19a. The corresponding $p.m.f$ is given by:

$$f(i) = \begin{cases} \frac{n-3}{2n+1} & \text{if } i = 1 \\ \frac{3}{2n+1} & \text{if } i = 2 \\ \frac{2}{2n+1} & \text{if } i = 3, 4 \\ \frac{1}{2n+1} & \text{if } i = 5, 6, 7, ...n, (n+2) \\ 0 & \text{elsewhere.} \end{cases}$$

L_1^--chromatic mean $= 1\frac{n-3}{2n+1} + (2)\frac{3}{2n+1} + (3+4)\frac{2}{2n+1} + (5+6+...n+(n+2))\frac{1}{2n+1} = \frac{n^2+5n+28}{4n+2}$ and variance $= (1^2)\frac{n-3}{2n+1} + (2^2)\frac{3}{2n+1} + (3^2+4^2)\frac{2}{2n+1}(5^2+6^2+..n^2+(n+2)^2)\frac{1}{2n+1} - (\frac{n^2+5n+28}{4n+2})^2 = \frac{2n^3+9n^2+31n+198}{6(2n+1)}$

□

Theorem 13 *For helm graphs having* $2n+1$ *vertices, where* $n \geq 7$, L_1^+*-chromatic mean is given by*

$$\mu_{\mathcal{C}_+}(H_n) = \frac{n^2+2n+2}{2n+1}$$

and L_1^+*-chromatic variance is given by*

$$\sigma^2_{\mathcal{C}_+}(H_n) = \frac{n^4+2n^3+8n^2+13n}{3(2n+1)^2}$$

Proof. We need $n+2$ colors to color the vertices of helm graph. The wheel graph induced from the given helm graph is colored as discussed in Theorem

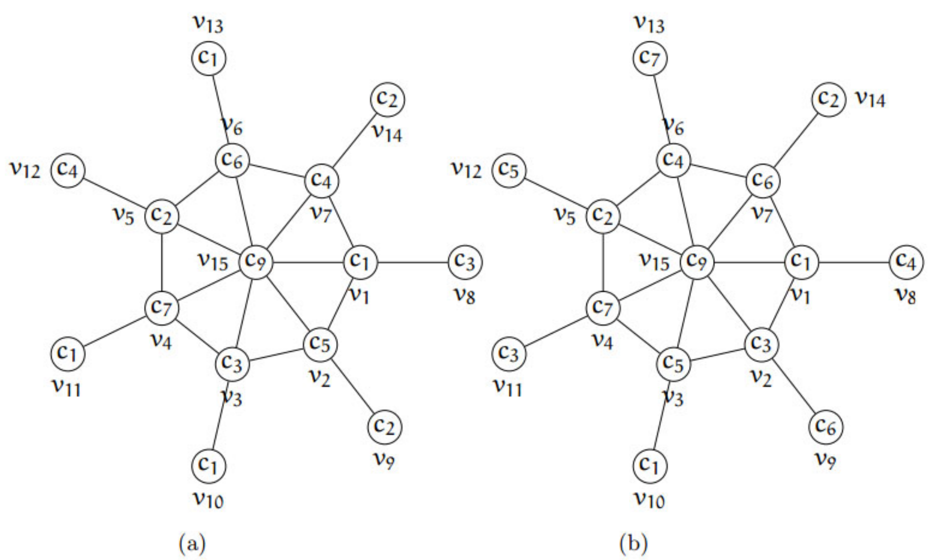


Figure 19

6. And each vertex in the remaining n vertices receive distinct color c_i (where $i = 1, 2, ..n$). The corresponding $p.m.f$ is given by:

$$f(i) = \begin{cases} \dfrac{2}{2n+1} & 1,\ 2,\ \ldots\ n \\ \dfrac{1}{2n+1} & n+2 \end{cases}$$

L_1^+-chromatic mean $= (1+2+..n)\frac{2}{2n+1} + (n+2)\frac{1}{2n+1} = \frac{n^2+2n+2}{2n+1}$ and variance $= (1^2+2^2+...n^2)\frac{2}{2n+1} + (n+2)^2\frac{1}{2n+1} - (\frac{n^2+2n+2}{2n+1})^2 = \frac{n^4+2n^3+8n^2+13n}{3(2n+1)^2}$ (refer to Figure 19b). □

Theorem 14 *For flower graph having* $n+1$ *vertices, where* $n \geq 6$, L_1^- *and* L_1^+*-chromatic mean and variance are given by*

$$\mu_{\mathcal{C}_-}(Fl_n) = \mu_{\mathcal{C}_+}(Fl_n) = \frac{n^2+3n+4}{2n+2}$$

and

$$\sigma^2_{\mathcal{C}_-}(Fl_n) = \sigma^2_{\mathcal{C}_+}(Fl_n) = \frac{n^4+4n^3+17n^2+26n}{12(n+1)^2}$$

Proof. The diameter of flower graph is 2. Thus, each vertex receives distinct color and central vertex is adjacent to all the other vertices. By definition, color difference between central vertex and any other vertex is 2. so we shall give the color c_{n+2} to the central vertex and other vertices receive distinct color c_i, (where $i = 1, 2, ...n$). For illustration, see Figure 20. The corresponding $p.m.f$ is given by

$$f(i) = \begin{cases} \dfrac{1}{n+1}, & \text{if } i = 1, 2, ...n, (n+2) \\ 0 & \text{elsewhere.} \end{cases}$$

Therefore, L_1^- and L_+^+ chromatic mean $= (1 + 2 + ...n + (n+2))\frac{1}{n+1} = \frac{n^2+3n+4}{2n+2}$ and variance $= (1^2 + 2^2 + ...n^2 + (n+2)^2)\frac{1}{n+1} - (\frac{n^2+2n+2}{n+1})^2 = \frac{n^4+4n^3+17n^2+26n}{12(n+1)^2}$

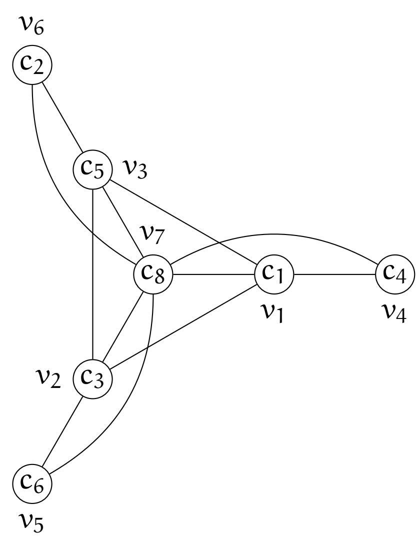


Figure 20

∎

3 Conclusion

In this paper, we have introduced the notions of certain coloring means and variances related to $L(2, 1)$-coloring and discussed these parameters in context of some fundamental graph classes. Further investigations are possible in this area, as the above-mentioned parameters can be discussed for many other classes of graphs, graph operations, graph products and known derived graphs. The coloring parameters play vital role in many areas such as network analysis, distribution problems, transportation problems, etc.

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