Acta Univ. Sapientiae, Mathematica, **1**, 2 (2009) 183–191

# On perfect numbers connected with the composition of arithmetic functions

József Sándor
Babeş-Bolyai University
Department of Mathematics
Cluj-Napoca, Romania
email: jjsandor@hotmail.com

Lehel István Kovács
Sapientia University,
Department of Mathematics and Informatics,
Târgu Mureş, Romania
email: klehel@ms.sapientia.ro

**Abstract.** We study two extensions of notions related to perfect numbers. One is the extension of "superperfect" numbers, the other one is a new notion called "aperfect" numbers. As particular cases, many results involving the arithmetical functions $\sigma$, $\sigma^*$, $\sigma^{**}$, $\varphi$, $\varphi^*$, $\psi$ and their compositions are presented in a unitary way.

## 1 Introduction

Let $\sigma(n)$ denote the sum of distinct divisors of the positive integer $n$. It is well-known that $n$ is called perfect if $\sigma(n) = 2n$. Euclid and Euler have determined all even perfect numbers (see [8] for history of this theorem) by showing that they are of the form $n = 2^k \cdot q$, where $q = 2^{k+1} - 1$ is a prime ($k \geq 1$). Prime numbers of the form $2^a - 1$ are called Mersenne primes, and it is one of the most difficult open problems of mathematics the proof of the infinitude of such primes. Up to now, only 46 Mersenne primes are known (see e.g. http://www.mersenne.org/). On the other hand, no odd perfect number is known ([3]). In 1969 D. Suryanarayana [10] defined the so-called superperfect numbers $n$, having the property $\sigma(\sigma(n)) = 2n$; and he and H. J. Kanold [4] obtained the general form of even superperfect numbers. All odd superperfect numbers must be perfect squares, but we do not know if there

**AMS 2000 subject classifications:** 11A25, 11A41, 11A99
**Key words and phrases:** arithmetic functions, perfect numbers, primes

exists at least one such number.
In what follows we denote by $\mathbb{N}$ the non-zero positive integers: $\mathbb{N} = \{1, 2, \ldots\}$.
We call a $g$ function *multiplicative*, if $g(ab) = g(a)g(b)$ for all $a, b \geq 1$, with $(a, b) = 1$.
In what follows, we denote by $\sigma^*(n)$ the sum of unitary divisors of $n$, i.e. those divisors $d|n$, with the property $(d, n/d) = 1$. A divisor $d$ of $n$ is called *bi-unitary* if the greatest common unitary divisor of $d$ and $n/d$ is 1. It is well-known that (see e.g. [2], [8]) $\sigma^*$ and $\sigma^{**}$ are multiplicative functions, and

$$\sigma^*(p^\alpha) = p^\alpha + 1, \tag{1}$$

$$\sigma^{**}(p^\beta) = \begin{cases} 1 + p + \ldots + p^{2\alpha} - p^\alpha, & \text{if } \beta = 2\alpha \\ 1 + p + \ldots + p^{2\alpha+1} = \sigma(p^\alpha), & \text{if } \beta = 2\alpha + 1 \end{cases}, \tag{2}$$

where $p$ is an arbitrary prime and $\alpha \geq 1$ is a positive integer.
Clearly, $\sigma$ is also a multiplicative function and

$$\sigma(p^\alpha) = 1 + p + \ldots + p^\alpha, \tag{3}$$

for any prime $p$ and $\alpha \geq 1$.
The Euler's totient function is a multiplicative function with

$$\varphi(p^\alpha) = p^{\alpha-1} \cdot (p - 1), \tag{4}$$

while its unitary analogue is a multiplicative function with

$$\varphi^*(p^\alpha) = p^\alpha - 1, \tag{5}$$

(see e.g. [2], [9]).
Finally, Dedekind's arithmetical function $\psi$ is a multiplicative function with the property

$$\psi(p^\alpha) = p^{\alpha-1} \cdot (p + 1), \tag{6}$$

(see e.g. [3], [7]).
In what follows, we shall call a number $n$ "f-perfect", if

$$f(n) = 2n \tag{7}$$

Thus the classical perfect numbers are the $\sigma$-perfect numbers, while the superperfect numbers are in fact $\sigma \circ \sigma$-perfect numbers.
In 1989 the first author [6] determined all even $\psi \circ \sigma$-perfect numbers. In fact, he proved that for all even $n$ one has

$$\psi\big(\sigma(n)\big) \geq 2n, \tag{8}$$

with equality only if $n = 2^k$, where $2^{k+1} - 1$ is a Mersenne prime. Since $\sigma(m) \geq \psi(m)$ for all $m$, from (8) we get:

$$\sigma\big(\sigma(n)\big) \geq \psi\big(\sigma(n)\big) \geq 2n \text{ for } n = \text{even}, \tag{9}$$

an inequality, which refines in fact the Kanold-Suryanarayana theorem.
We note the contrary to the $\sigma \circ \sigma$-perfect numbers; at least one odd solution to $\psi \circ \sigma$-perfect numbers is known, namely $n = 3$.

## 2 Extensions of even superperfect numbers

The main result of this section is contained in the following.

**Theorem 1** *Let* $f, g : N \to N$ *be two arithmetic functions having the following properties:*

1. $g$ *is multiplicative*
2. $f(ab) \geq af(b)$ *for all* $a, b \geq 1$
3. $g(m) \geq m$*, with equality only for* $m = 1$
4. $f\big(g(2^k)\big) \geq 2^{k+1}$*, with equality only if* $2^{k+1} - 1 \in A$*, where* $A$ *is a set of positive integers*

*Then for all even* $n$ *one has*

$$f\big(g(n)\big) \geq 2n, \tag{10}$$

*and all even* $f \circ g$*-perfect numbers are of the form* $2^k$*, where* $2^{k+1} - 1 \in A$*.*

**Proof.** Let $n = 2^k \cdot m$ with $m =$ odd, be an even integer. By condition *1.* one has $g(n) = g(2^k)g(m)$, so by *2.* we can write that $f\big(g(n)\big) = f\big(g(2^k)g(m)\big) \geq g(m)f\big(g(2^k)\big)$. Since $g(m) \geq m$ (by *3.*) and $f\big(g(2^k)\big) \geq 2^{k+1}$ (by *4.*), we get that $f\big(g(m)\big) \geq 2n$, so (10) follows. For equality we must have $g(m) = 1$ and $f\big(g(2^k)\big) = 2^{k+1}$, so $m = 1$ and $2^{k+1} - 1 \in A$. This finishes the proof of Theorem 1. ■

**Remark 1** *If at least one of the inequalities 2.–4. is strict, then in (10) one has strict inequality. As a consequence,* $n$ *cannot be an even* $f \circ g$*-perfect number.*

**Corollary 1** *(Sándor [6])*
*All even* $\psi \circ \sigma$*-perfect numbers* $n$ *have the form* $n = 2^k$*, where* $2^{k+1} - 1$ *is prime.*

*– The first* $\psi \circ \sigma$*-perfect numbers are:* $2 = 2^1$, $3$, $4 = 2^2$, $16 = 2^4$, $64 = 2^6$, $4096 = 2^{12}$, $65536 = 2^{16}$, $262144 = 2^{18}$, *where* $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$, $2^7 - 1 = 127$, $2^{13} - 1 = 8191$, $2^{17} - 1 = 131071$, $2^{19} - 1 = 524287$ *are Mersenne primes.*

*– Put* $f(n) = \psi(n)$ *and* $g(n) = \sigma(n)$ *in Theorem 1. Then property 2. is known (see e.g. [7]), while 3. and 4. are well known. Since for* $t > 1$ *one has* $\psi(t) \geq t + 1$*, with equality only for* $t =$ *prime, by* $\sigma(2^k) = 2^{k+1} - 1$*, we get* $A =$ *set of primes of the form* $2^{k+1} - 1$*.*

**Corollary 2** *(Sándor [6])*
*The only even* $\sigma \circ \psi$*-perfect number* $n$ *is* $n = 2$*.*

*– Put* $f(n) = \sigma(n)$ *and* $g(n) = \psi(n)$ *in Theorem 1. Then properties 1.–3. are well-known; for 4. one has* $\psi(2^k) = 2^{k-1} \cdot 3$*; so* $\sigma(2^{k-1} \cdot 3) = ((2^k - 1) \cdot 4) \geq 2^{k+1} \Leftrightarrow 2^k \geq 2$*. Thus* $k = 1$ *and* $A = \{3\}$*.*

**Corollary 3** *(Kanold-Suryanarayana [4])*
*All even* $\sigma \circ \sigma$*-perfect numbers* $n$ *have the form* $n = 2^k$*, where* $2^{k+1} - 1$ *is prime.*

*– The first* $\sigma \circ \sigma$*-perfect numbers are:*
$2 = 2^1$, $4 = 2^2$, $16 = 2^4$, $64 = 2^6$, $4096 = 2^{12}$, $65536 = 2^{16}$, $262144 = 2^{18}$, $1073741824 = 2^{30}$, $1152921504606846976 = 2^{60}$, *where* $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$, $2^7 - 1 = 127$, $2^{13} - 1 = 8191$, $2^{17} - 1 = 131071$, $2^{19} - 1 = 524287$, $2^{31} - 1 = 2147483647$, $2^{61} - 1 = 2305843009213693951$ *are Mersenne primes.*

*– This also follows from inequality (10) for* $f(n) = \sigma(n)$ *and* $g(n) = \psi(n)$*, but a direct proof applies for* $f(n) = g(n) = \sigma(n)$*.*

**Corollary 4** *(Sándor [6])*
*There is no even* $\psi \circ \psi$*-perfect number.*

*– Put* $f(n) = g(n) = \psi(n)$*. Since inequality 4. will be strict, inequality (10) holds true also with strict inequality.*

**Remark 2** *In [6] it is proved also that the only odd* $\psi \circ \psi$*-perfect number is* $n = 3$.

**Corollary 5** *The only even* $\sigma \circ \sigma^{**}$*-perfect number* $n$ *is* $n = 2$*.*

*– Let* $f(n) = \sigma(n)$ *and* $g(n) = \sigma^{**}(n)$ *in Theorem 1. Clearly 3. holds true, as more generally it is known that (see e.g. [1], [8]):*

$$\sigma^{**}(m) \geq m + 1 \text{ for } m > 1, \tag{11}$$

*with equality only for* $m = p$ *or* $m = p^2$ *(*$p$ *= prime).*

*Now, let* $k$ *be odd. Then* $\sigma^{**}(2^k) = \sigma(2^k) = 2^{k+1} - 1$ *and* $\sigma(\sigma^{**}(2^k)) = \sigma(2^{k+1} - 1) \geq 2^{k+1}$*, with equality only if* $2^{k+1} - 1$ *= prime. For* $k \geq 3$*, as* $k$ *is odd, clearly* $k + 1$ *is even, so it is immediate that* $2^{k+1} - 1 \equiv 0 \pmod 3$*. Thus we must have* $k = 1$*, i.e.* $n = 2$ *is a solution.*

*When* $k$ *is even, put* $k = 2a$*. Then* $\sigma^{**}(2^k) = \sigma^{**}(2^{2a}) = 1 + 2 + \ldots + 2^{a-1} + \underbrace{2^{a+1} + \ldots + 2^{2a}}_{2^{a+1}\cdot(1+2+\ldots+2^{a-1})} = (1 + 2 + \cdots + 2^{a-1}) \cdot (1 + 2^{a+1}) = (2^a - 1)(2^{a+1} + 1)$*. Thus,* $\sigma(\sigma^*(2^k)) = \sigma((2^a - 1) \cdot (2^{a+1} + 1)) \geq (2^a - 1)\sigma(2^a - 1) \geq (2^{a+1} + 1) \cdot 2^a > 2^{2a+1} = 2^{k+1}$*, so inequality 4) is strict for* $k$ *even number.*

## 3 Aperfect numbers

The equality $f(n) = n + 2$, for $f(n) > n$ is a kind of additive analogue of $f(n) = n \cdot 2$, i.e. of classical perfect numbers. We shall call a number $n$ $f$-*plus aperfect* (aperfect = "additive perfect"), if

$$f(n) = n + 2. \tag{12}$$

This notion also extends the notion of perfect numbers. Put e.g. $f(n) = \sigma(n) - n + 2$. Then $\sigma(n) = 2n$, so we obtain again the perfect numbers.

Similary, for $f(n) < n$ we have a similar notion. We call $n$ $f$-*minus aperfect*, if

$$f(n) = n - 2. \tag{13}$$

We can state the following general result:

**Theorem 2** *Let* $f, g : N \to N$ *be two arithmetic functions such that* $g(n) \geq n + 1$ *for* $n > 1$*, with equality only for* $n = p^\alpha$ *(*$p$ *prime,* $\alpha \geq 1$ *integer) and* $f(m) \geq m + 1$*, for* $m > 1$*, with equality only for* $m = q^\beta$ *(*$q$ *prime,* $\beta \geq 1$ *integer). Then one has the inequality*

$$f\big(g(n)\big) \geq n + 2 \tag{14}$$

*for all* $n$*, and* $n$ *is* $f \circ g$*-plus aperfect only if the prime powers* $p^\alpha$ *and* $q^\beta$ *satisfy the equation*

$$g(p^\alpha) = q^\beta. \tag{15}$$

**Proof.** From the stated conditions, one can write $f\big(g(n)\big) \geq g(n) + 1 \geq (n + 1) + 1 = n + 2$. One has equality only if $n = p^\alpha$ and $g(n) = q^\beta$, i.e. $g(p^\alpha) = q^\beta$, which means equality (15). ■

**Corollary 6** *(Sándor [6])*
*All* $\sigma \circ \sigma^*$*-plus aperfect numbers* $n$ *have form* $n = 2^s$*, where* $2^s + 1$ *is a prime (i.e. Fermat prime,* $s = 2^a$ *).*

*– The first* $\sigma \circ \sigma^*$*-plus aperfect numbers are:* $2 = 2^1$*,* $4 = 2^2$*,* $16 = 2^4$*,* $256 = 2^8$*,* $65536 = 2^{16}$*, where* $2^1 + 1 = 3$*,* $2^2 + 1 = 5$*,* $2^4 + 1 = 17$*,* $2^8 + 1 = 257$*,* $2^{16} + 1 = 65537$ *are Fermat primes.*

*– Let* $f(n) = \sigma(n), g(n) = \sigma^*(n)$*. Then (15) may be written as* $\sigma^*(p^\alpha) = q^\beta$*. Since* $\sigma(m) = m + 1$ *only for* $m =$ *prime, we have* $\beta = 1$*, thus* $p^\alpha + 1 = q$*. For* $p \geq 3, p^\alpha + 1$ *is even number, so we must have* $p = 2$*, i.e.* $q = 2^\alpha + 1$*. Since* $n = p^\alpha = 2^\alpha$*, then result follows.*

**Corollary 7** *All* $\sigma^* \circ \sigma$*-plus aperfect numbers are* $n = 2$*, or have the form* $n = 2^k - 1$*, where* $2^k - 1$ *is a Mersenne prime.*

*– The first* $\sigma^* \circ \sigma$*-plus aperfect numbers are:* $2$*,* $2^2 - 1 = 3$*,* $2^3 - 1 = 7$*,* $2^5 - 1 = 31$*,* $2^7 - 1 = 127$*,* $2^{13} - 1 = 8191$*,* $2^{17} - 1 = 131071$*,* $2^{19} - 1 = 524287$ *(Mersenne primes).*

*– Let* $f(n) = \sigma^*(n), g(n) = \sigma(n)$ *in Theorem 2. Then (15) has the form* $\sigma(p^\alpha) = q^\beta$*. Since* $\alpha = 1$*, one has* $p + 1 = q^\beta$*, i.e.* $p = q^\beta - 1$*. For* $q \geq 3$ *this is even, so we must have* $q = 2$*, when* $p = 2^\beta - 1$ *is Mersenne prime. When* $q = 3$ *for* $\beta = 1$ *we get the prime 2, the first* $\sigma^* \circ \sigma$*-plus aperfect number.*

**Corollary 8** *The only* $\sigma \circ \sigma$*-plus aperfect number* $n$ *is* $n = 2$.

– *Let* $f(n) = g(n) = \sigma(n)$*. Then we get* $\alpha = \beta = 1$ *so* $\sigma(p) = q$*, i.e.* $p + 1 = q$ *with* $p, q$ *This is possible only for* $p = 2, q = 3$.

**Corollary 9** *The only* $\sigma^{**} \circ \sigma^{**}$*-plus aperfect numbers* $n$ *are* $n = 2, 3, 4$.

– *Since the equality* $\sigma^{**}(n) = n + 1$ *is satisfied only if* $n = p$ *or* $n = p^2$ *(*$p$ *prime), we must study the equality:*

$$\sigma^{**}(p^\alpha) = q^\beta \tag{16}$$

*for* $\alpha, \beta \in \{1, 2\}$.
*If* $\alpha = 1$*, then* $\beta = 1$ *implies* $p + 1 = q$*, which is possible only for* $p = 2, q = 3$*. Now for* $\alpha = 1, \beta = 2$ *we get* $p + 1 = q^2$*, so* $p = q^2 - 1 = (q - 1)(q + 1)$*, which is possible only for* $q = 2$ *and* $p = 3$*. Thus* $p = 3$ *is acceptable too.*
*If* $\alpha = 1, \beta = 2$*, we get* $q^2 + 1 = p$*, i.e.* $q^2 = p - 1$*. Here* $p = 2$ *is not possible, while for* $p \geq 3, p - 1$ *is even, thus* $2 | q^2$*. This means* $q = 2$*. So* $p = 5$ *is another solution. For* $q^2 + 1 = p^2$ *we get* $q^2 = (p - 1)(p + 1)$*, which for* $p = 2$ *gives* $q^2 = 3$*, which is impossible. For* $p \geq 3$ *we get* $q = 3$*, so* $5 = p^2$*, which is again impossible. Then result follows.*

Similarly to Theorem 2, we may prove the following:

**Theorem 3** *Let* $f, g : N \to N$ *be two arithmetic functions, such that* $g(n) \leq n - 1$ *for* $n > 1$*, with equality only for* $n = p^\alpha$ *(*$p$ *prime,* $\alpha \geq 1$ *integer) and* $f(m) \leq m - 1$*, for* $m > 1$*, with equality only for* $m = q^\beta$ *(*$q$ *prime,* $\beta \geq 1$ *integer). Then one has the inequality:*

$$f\big(g(n)\big) \leq n - 2 \tag{17}$$

*for all* $n > 2$*, and* $n$ *is* $f \circ g$*-minus aperfect only if the prime powers* $p^\alpha$ *and* $q^\beta$ *satisfy the equation*

$$g(p^\alpha) = q^\beta. \tag{18}$$

**Proof.** From the stated properties one can write $f\big(g(n)\big) \leq g(n) - 1 \leq (n - 1) - 1 = n - 2$, with equality only if $n = p^\alpha$ and $g(n) = q^\beta$, so (18) follows. ■

**Corollary 10** *All $\varphi \circ \varphi^*$-minus aperfect numbers $n$ are $n = 3$, or have the form $n = 2^a$, where $2^a - 1$ is a Mersenne prime.*

– *The first $\varphi \circ \varphi^*$-minus aperfect numbers are:* $3$, $4 = 2^2$, $8 = 2^3$, $32 = 2^5$, $128 = 2^7$, $8192 = 2^{13}$, *where* $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$, $2^7 - 1 = 127$, $2^{13} - 1 = 8191$ *are Mersenne primes.*

– *Let $f(n) = \varphi(n), g(n) = \varphi^*(n)$ in Theorem 3. As $\varphi(m) = m - 1$ only for $m =$ prime, we have $\beta = 1$, so (18) becomes $\varphi^*(p^\alpha) = q$, i.e. $p^\alpha - 1 = q$. Then $p = 2$, so $q = 2^\alpha - 1$ is a Mersenne prime. Here $n = 2^\alpha$, so the result follows. When $p = 3$ and $\alpha = 1$, then $q = 1$, and we obtain the first $\varphi \circ \varphi^*$-minus aperfect number: $3$.*

**Corollary 11** *All $\varphi^* \circ \varphi$-minus aperfect numbers $n$ have the form $n = 2^a + 1 =$ Fermate prime.*

– *The first $\varphi^* \circ \varphi$-minus aperfect numbers are:* $3 = 2^1 + 1$, $5 = 2^2 + 1$, $17 = 2^4 + 1$, $257 = 2^8 + 1$ *Fermat primes.*

– *Put $f(n) = \varphi^*(n), g(n) = \varphi(n)$ in Theorem 3. Now $\alpha = 1$, so $\varphi(p) = q^\beta$, i.e. $p - 1 = q^\beta$, implying $p = q^\beta + 1$. Since $p, q$ are primes, one must have $q = 2$. Thus $p = 2^\beta + 1$ and $n = p$, which implies the assertion.*

**Remark 3** *At the present state of the science, there are only 5 Fermat primes known, namely $n = 3, 5, 17, 257, 65537$ (see [5], [3]).*

**Corollary 12** *All $\varphi^* \circ \varphi^*$-minus aperfect numbers are $n = 9$ or $n = 2^\alpha$ with $2^\alpha - 1$ is Mersenne prime, or $n = 2^\beta + 1$ is Fermat prime.*

– *The first $\varphi^* \circ \varphi^*$-minus aperfect numbers are: 3, 4, 5, 8, 9, 17, 32, 128, 257, 8192.*

– *We have $\varphi^*(n) = n - 1$ only if $n = p^\alpha$, so we must solve the equation $\varphi^*(p^\alpha) = p^\alpha - 1 = q^\beta$.*

***Case 1)*** *If $q \geq 3$, then as $p^\alpha = q^\beta + 1 =$ even, we get $2 | p^\alpha$, so $p = 2$. We get the equation:*

$$q^\beta = 2^\alpha - 1. \tag{19}$$

*Equation (19) has been studied in [9] (Lemma 6'), so we get $\beta = 1, q = 2^\alpha - 1$ is Mersenne prime.*

***Case 2)*** *If $q = 2$, then we get the equation:*

$$p^\alpha = 2^\beta + 1, \tag{20}$$

*studied in [9] (Lemma 4). Thus we have: a)* $p = 3, \alpha = 2, \beta = 3$*, in which case* $n = p^{\alpha} = 3^2 = 9$*; b)* $\alpha = 1, p = 2^{\beta} + 1$ *is Fermat prime.*
*This finishes the proof of Corollary 12.*

**Remark 4** *It is easy to see that the only* $\varphi \circ \varphi$*-minus aperfect number is* $n = 3$*.*

**Remark 5** *Since the result of Corollary 7 is a characterisation of odd solutions, it could be used as a Mersenne prime test, too; and Corollary 11 could be used as a Fermat prime test, too.*

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*Received: March 25, 2009*