

On multigraphic and potentially multigraphic sequences

Dedicated to the memory of Antal Iványi

Shariefuddin PIRZADA

University of Kashmir
Department of Mathematics
Srinagar, Kashmir, India
email:

pirzadasd@kashmiruniversity.ac.in

Bilal Ahmad CHAT

University of Kashmir
Srinagar, Kashmir, India
email: bilalchat99@gmail.com

Uma Tul SAMEE

Islamia College for Science and Commerce,
Srinagar, Kashmir
email: pzsamee@yahoo.co.in

Abstract. An r -graph (or a multigraph) is a loopless graph in which no two vertices are joined by more than r edges. An r -complete graph on n vertices, denoted by $K_n^{(r)}$, is an r -graph on n vertices in which each pair of vertices is joined by exactly r edges. A non-increasing sequence $\pi = (d_1, d_2, \dots, d_n)$ of non-negative integers is said to be r -graphic if it is realizable by an r -graph on n vertices. An r -graphic sequence π is said to be potentially $S_{L,M}^{(r)}$ -graphic if it has a realization containing $S_{L,M}^{(r)}$ as a subgraph. We obtain conditions for an r -graphic sequence to be potentially $S_{L,M}^{(r)}$ -graphic. These are generalizations from split graphs to p -tuple r -split graph.

Computing Classification System 1998: G.2.2

Mathematics Subject Classification 2010: 05C07

Key words and phrases: multigraph, multigraphic sequence, potentially multigraphic sequences, split graph

1 Introduction

For a positive integer r , an r -graph(or multigraph) is a loopless graph in which no two vertices are joined by more than r edges. An r -complete graph on n vertices, denoted by $K_n^{(r)}$, is an r -graph on n vertices in which each pair of vertices is joined by exactly r edges. Clearly, $K_n^{(1)} = K_n$. A non-increasing sequence $\pi = (d_1, d_2, \dots, d_n)$ of non-negative integers is said to be r -graphic if it is the degree sequence of an r -graph G on n vertices, and such an r -graph G is referred to as a realization of π . We take $\sigma(\pi) = \sum_{i=1}^n d_i$. For graph theoretical notations and definitions we refer to [9].

Let $\pi = (d_1, d_2, \dots, d_n)$ be a non-increasing sequence of non-negative integers with $d_1 \leq \sum_{i=2}^n \min\{r, d_i\}$. Define $\pi'_k = (d'_1, d'_2, \dots, d'_{n-1})$ to be the non-increasing rearrangement of the sequence obtained from

$$(d_1, d_2, \dots, d_{k-1}, d_{k+1}, \dots, d_n)$$

by reducing by 1 the remaining largest terms that have not been reduced r times, and repeating the procedure d_k times. π'_k is called the residual sequence obtained from π by laying off d_k .

The following three results due to Chungphaisian [2] are generalizations from 1-graphs to r -graphs of three well-known results, one by Erdős and Gallai [3], one by Kleitman and Wang [6] and one by Fulkerson, Hoffman and Mcandrew [5].

Theorem 1 [2] *Let $\pi = (d_1, d_2, \dots, d_n)$ be a non-increasing sequence of non-negative integers, where $\sigma(\pi)$ is even. Then π is r -graphic if and only if for each positive integer $t \leq n$,*

$$\sum_{i=1}^t d_i \leq rt(t-1) + \sum_{i=t+1}^n \min\{rt, d_i\}.$$

Theorem 2 [2] *π is r -graphic if and only if π'_k is r -graphic.*

Let the subgraph H on the vertices v_i, v_j, v_k, v_l of a multigraph G contain the edges $v_i v_j$ and $v_k v_l$. The operation of deleting these edges and introducing a pair of new edges $v_i v_l$ and $v_j v_k$, or $v_i v_k$ and $v_j v_l$ is called an elementary degree preserving transformation. If this operation is performed r times on the same edge set, it is called r -exchange.

Theorem 3 [2] *Let π be an r -graphic sequence, and let G and G' be realizations of π . Then there is a sequence of r -exchanges, $E_1, \dots, E_k$ such that the application of these r -exchanges to G in order will result in G' .*

An r -graphic sequence π is said to be potentially $K_{m+1}^{(r)}$ if there exists a realization of π containing $K_{m+1}^{(r)}$ as a subgraph. If π has a realization G containing $K_{m+1}^{(r)}$ on the $m+1$ vertices of highest degree in G , then π is said to be potentially $A_{m+1}^{(r)}$ -graphic. As a special case of Lemma 2.1 in [13], Yin showed that an r -graphic sequence is potentially $K_{m+1}^{(r)}$ -graphic if and only if it is potentially $A_{m+1}^{(r)}$ -graphic.

The r -join (complete product) of two r -graphs G_1 and G_2 is a graph $G = G_1 \vee G_2$ with vertex set $V(G_1) \cup V(G_2)$ and the edge set consisting of all edges of G_1 and G_2 together with the edges joining each vertex of G_1 with every vertex of G_2 by exactly r edges. Let $K_l^{(r)}$ and $K_m^{(r)}$ be complete r -graphs with l and m vertices respectively, that is the complete graphs having exactly r edges between every two vertices. The r -split graph of $K_l^{(r)}$ and $\overline{K_m^{(r)}}$ denoted by $\overline{S}_{l,m}^{(r)}$ is the graph $K_l^{(r)} \vee \overline{K_m^{(r)}}$ having $l+m$ vertices, where $\overline{K_m^{(r)}}$ (having no edges) is the complement of $K_m^{(r)}$. [14]. If π has a realization G containing $\overline{S}_{l,m}^{(r)}$ on the $l+m$ vertices of highest degree in G , then π is said to be potentially $\overline{A}_{l,m}$ -graphic.

The following two results due to Yin [13] are generalizations from 1-graphs to r -graphs of two well-known results given by A. R. Rao [12].

Theorem 4 [13] *Let $n \geq l+1$ and $\pi = (d_1, d_2, \dots, d_n)$ be an r -graphic sequence with $d_{l+1} \geq rl$. Then π is potentially $A_{l+1}^{(r)}$ -graphic if and only if π_{l+1} is r -graphic.*

Theorem 5 [13] *Let $n \geq l+1$ and $\pi = (d_1, d_2, \dots, d_n)$ be an r -graphic sequence with $d_{l+1} \geq 2rl - 1$, then π is potentially $K_{l+1}^{(r)}$.*

An extremal problem for 1-graphic sequences to be potentially $K_l^{(1)}$ -graphic was considered by Erdős, Jacobson and Lehel [4] and solved by Li et al. [7, 8]. Yin [13] generalized this extremal problem and the Erdős-Jacobson-Lehel conjecture from 1-graphs to r -graphs.

In 2014, the authors [10] proved the following assertion.

Theorem 6 [10] *If G_1 is a realization of $\pi_1 = (d_1^1, \dots, d_m^1)$ containing K_p as a subgraph and G_2 is a realization of $\pi_2 = (d_1^2, \dots, d_n^2)$ containing K_q as a subgraph, then the degree sequence $\pi = (d_1, \dots, d_{m+n})$ of the join of G_1 and G_2 is potentially K_{p+q} -graphic.*

The following two results for simple graphs are due to Yin [14].

Theorem 7 [14] *π is potentially $\overline{A}_{l,m}$ -graphic if and only if π_l is graphic.*

Theorem 8 [14] *Let $n \geq l + m$ and let $\pi = (d_1, d_2, \dots, d_n)$ be a non-increasing graphic sequence. If $d_{l+m} \geq 2l + m - 2$, then π is potentially $\overline{A}_{l,m}$ -graphic.*

A condition for a graphic sequence π to be potentially $K_4 - e$ graphic can be found in [11], where $K_4 - e$ is the graph obtained from the complete graph K_4 by deleting one edge e .

2 Bounds on the sum of squares of degrees of a multigraph

From the Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^n |a_i b_i| \leq \left(\sum_{i=1}^n |a_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^n |b_i|^2 \right)^{\frac{1}{2}},$$

Taking $a_i = d_i$ and $b_i = 1$, we have $\left(\sum_{i=1}^n d_i \right)^2 \leq n \sum_{i=1}^n d_i^2$ which implies $\frac{1}{n} \left(\sum_{i=1}^n d_i \right)^2 \leq \left(\sum_{i=1}^n d_i^2 \right)$. From this and the hand shaking Lemma $\sum_{i=1}^n d_i = 2|E|$, we have $\frac{4|E|^2}{n} = \frac{1}{n} \left(\sum_{i=1}^n d_i \right)^2 \leq \sum_{i=1}^n d_i^2$.

Now we have the following observation, the proof is by using the same argument as in Theorem 1 of [1].

Lemma 9 *For an r -graph G , $\sum_{i=1}^n d_i^2 \leq |E|(r(n-2) + \frac{2|E|}{n-1})$.*

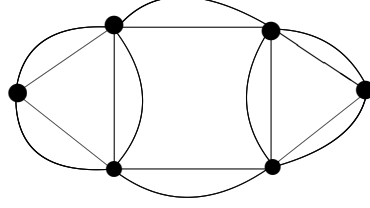


Figure 1: A 2-graph

Remark 10 From Lemma 9, we observe that

$$\frac{4|E|^2}{n} \leq \sum_{i=1}^n d_i^2 \leq |E|(r(n-2) + \frac{2|E|}{n-1}).$$

The following example shows that the equality does not hold in the above inequality.

Example 11 Consider the 2-graph as shown in Figure 1.

Here, $\frac{4|E|^2}{n} = \frac{4 \times 16^2}{6} = \frac{512}{3} < 4^2 + 6^2 + 6^2 + 6^2 + 6^2 + 4^2 = 176 < 16(2(6-2) + \frac{2 \times 16}{6-1}) = \frac{1152}{5}$.

Now, we have the following result.

Lemma 12 A multigraph G is regular if and only if $\frac{4|E|^2}{n} = \sum_{i=1}^n d_i^2$.

Proof. Suppose an r -graph G is regular of degree b . Then $2|E| = nb$ and $d_i = b$ for all $i = 1, 2, \dots, n$. We know that $\sum_{i=1}^n d_i^2 = nb^2$ and $\frac{4|E|^2}{n} = \frac{1}{n} 4 \frac{1}{4} n^2 b^2 = nb^2$.

These together give $\sum_{i=1}^n d_i^2 = \frac{4|E|^2}{n}$.

Conversely, suppose that $\sum_{i=1}^n d_i^2 = \frac{4|E|^2}{n}$. Then $\frac{4|E|^2}{n} = \sum_{i=1}^n d_i^2$. This implies that $\frac{1}{n}(d_1^2 + d_2^2 + \dots + d_n^2 + 2(d_1 d_2 + d_1 d_3 + \dots + d_1 d_n) + \dots + 2(d_{n-2} d_{n-1} + d_{n-2} d_n) + 2(d_{n-1} d_n)) - (d_1^2 + d_2^2 + \dots + d_n^2) = 0$, which on simplification gives

$\frac{1}{n} \left((d_1 - d_2)^2 + (d_1 - d_3)^2 + \dots + (d_1 - d_n)^2 + (d_2 - d_3)^2 + (d_2 - d_4)^2 + \dots + (d_2 - d_n)^2 + \dots + (d_{n-1} - d_n)^2 \right) = 0$. From this, we see that each term on left side is non-negative for every i, j and right side is equal to zero. Therefore the above equation is possible when $d_i = d_j$ for every $i, j = 1, 2, \dots, n$ and hence G is a regular graph. $\square$

Now, we have the following observation.

Lemma 13 *Let G be an r -graph with $n > 2$ vertices. Then G is a complete graph K_n^r if and only if $\frac{4|E|^2}{n} = \sum_{i=1}^n d_i^2 = |E|(r(n-2) + \frac{2|E|}{n-1})$.*

Proof. First we note that an r -graph G is a complete r -graph if and only if $|E| = \frac{1}{2}rn(n-1)$. Moreover, we know that $|E| = \frac{1}{2}nr(n-1)$, which implies that $2|E|(n-2) + 2|E|n = nr(n-1)(n-2) + 2|E|n$ and on simplification gives $\frac{4|E|^2}{n} = |E|(r(n-2) + \frac{2|E|}{n-1})$. Thus the result follows. $\square$

The following result partially answers the question raised in Remark 10.

Theorem 14 *A bipartite multigraph $G = K_{l,m}^{(r)}$, where $m > 1$, is an r -star graph $K_{1,n-1}^{(r)}$ if and only if $\sum_{i=1}^n d_i^2 = |E|(r(n-2) + \frac{2|E|}{n-1})$.*

Proof. Let $K_{l,m}^{(r)}$ be an r -complete bipartite graph, where $m > 1$, $n = l + m$ and $|E| = rlm$. There are l -vertices each of whose degree is $r \times m$ and m vertices each of whose degree is $r \times l$, so $\sum_{i=1}^n d_i^2 = l(rm)^2 + m(rl)^2 = lr^2m^2 + mr^2l^2 = r^2(lm^2 + ml^2) = r^2lm(l+m)$. Therefore, we have $|E|(r(n-2) + \frac{2|E|}{n-1}) = rlm(r(l+m-2) + \frac{2rlm}{n-1}) = r^2lm(\frac{l^2+m^2+4lm-3l-3m+2}{l+m-1})$. Therefore, $r^2lm(l+m) = r^2lm(\frac{l^2+m^2+4lm-3l-3m+2}{l+m-1})$, which gives $l = 1$. Hence the result follows. $\square$

3 Potentially r -graphic sequences

Definition 15 *Let $\bar{S}_{r_1, s_1}^{(r)}, \bar{S}_{r_2, s_2}^{(r)}, \dots, \bar{S}_{r_p, s_p}^{(r)}$ be r -split graphs, respectively with $r_1 + s_1, r_2 + s_2, \dots, r_p + s_p$ vertices. Let $L = \sum_{i=1}^p r_i$ and $M = \sum_{i=1}^p s_i$. Then*

the p -tuple r -split graph, denoted by $S_{L,M}^{(r)}$, is the graph

$$S_{L,M}^{(r)} = S_{\sum_{i=1}^p r_i, \sum_{i=1}^p s_i}^{(r)} = \bar{S}_{r_1, s_1}^{(r)} \vee \bar{S}_{r_2, s_2}^{(r)} \vee \dots \vee \bar{S}_{r_p, s_p}^{(r)}.$$

Clearly $S_{L,M}^{(r)}$ has vertex set $\bigcup_{i=1}^p V(\bar{S}_{r_i, s_i}^{(r)})$ and the edge set consists of all edges of $\bar{S}_{r_1, s_1}^{(r)}, \bar{S}_{r_2, s_2}^{(r)}, \dots, \bar{S}_{r_p, s_p}^{(r)}$ together with the edges joining each vertex of $\bar{S}_{r_i, s_i}^{(r)}$ with every vertex of $\bar{S}_{r_j, s_j}^{(r)}$ by exactly r -edges for every i, j with $i \neq j$.

An r -graphic sequence π is said to be potentially $S_{L,M}^{(r)}$ -graphic if there exists a realization of π containing $S_{L,M}^{(r)}$ as a subgraph. If π has a realization G containing $S_{L,M}^{(r)}$ on the $L + M$ vertices of highest degree in G , then π is said to be potentially $A_{L,M}^{(r)}$ -graphic.

Let $n \geq L + M$ and let $\pi = (d_1, \dots, d_n)$ be a non-increasing sequence of non-negative integers with $d_L \geq r(L + M) - 1$ and $d_{L+M} \geq rL$. We define sequences $\pi_1, \dots, \pi_L$ as follows. Construct the sequence

$$\pi_1 = (d_2 - r, \dots, d_L - r, d_{L+1} - r, \dots, d_{L+M} - r, d_{L+M+1}^1, \dots, d_n^1)$$

from π by reducing 1 from the largest term that have not been already reduced r times, and then reordering the last $n - L - M$ terms to be non-increasing. For $2 \leq i \leq r$, construct

$$\pi_i = (d_{i+1} - ir, \dots, d_L - ir, d_{L+1} - ir, \dots, d_{L+M} - ir, d_{L+M+1}^i, \dots, d_n^i)$$

from

$$\begin{aligned} \pi_{i-1} = & (d_i - (i-1)r, \dots, d_L - (i-1)r, d_{L+1} - (i-1)r, \dots, \\ & d_{L+M} - (i-1)r, d_{L+M+1}^{i-1}, \dots, d_n^{i-1}) \end{aligned}$$

by deleting $d_i - (i-1)r$, reducing the first $d_i - (i-1)r$ remaining terms of π_{i-1} by one that have not been already reduced r times, and then reordering the last $n - L - M$ terms to be non-increasing.

We start with the following lemma.

Lemma 16 If $\pi = (d_1, d_2, \dots, d_n)$ is the graphic sequence of $S_{L,M}^{(r)}$, then

$$\pi = \left(\left(\sum_{i=1}^m r(r_i + s_i - 1) \right)^{r_j}, \left(\sum_{i=1}^m r r_i + \sum_{i=1, i \neq j}^m r s_i \right)^{s_j} \right), \quad \text{for } j = 1, 2, \dots, m.$$

Proof. To prove the result we use induction on m .

For $m = 1$, the result is obviously true. For $m = 2$, we have $S_{\sum_{i=1}^2 r_i, \sum_{i=1}^2 s_i}^{(r)}$. Therefore for every $i = 1, 2, \dots, r_1$ and $i = 1, 2, 3, \dots, r_2$ and $j = 1, 2, 3, \dots, s_1$ and $j = 1, 2, 3, \dots, s_2$

$$\overline{d}_i = d_i + r(r_2 + s_2) \quad (1)$$

and

$$\overline{d}_j = r(r_1 + r_2 + s_2), \quad (2)$$

where $\overline{d}_i$ and $\overline{d}_j$ are respectively the degree of $\overline{v}_i^{\text{th}}$ and $\overline{v}_j^{\text{th}}$ vertex in $S_{r_1+r_2, s_1+s_2}$ and d_i is the degree of i^{th} vertex in K_{r_1} . Equations (1) and (2) hold for every i, j . Thus the graphic sequence π^2 of $S_{r_1+r_2, s_1+s_2}$ is

$$\begin{aligned} \pi^2 &= \left(\left(r(r_1 + s_1 - 1) + r(r_2 + s_2) \right)^{r_1}, \left(r(r_1 + s_1 - 1) + r(r_2 + s_2) \right)^{r_2}, \right. \\ &\quad \left. \left(r(r_1 + r_2 + s_2) \right)^{s_1}, \left(r(r_1 + r_2 + s_2) \right)^{s_2} \right) \\ &= \left(\left(\sum_{i=1}^2 r(r_i + s_i - 1) \right)^{r_j}, \left(\sum_{i=1}^m r r_i + \sum_{i=1, i \neq j}^m r s_i \right)^{r_j} \right), \quad \text{for } j = 1, 2. \end{aligned}$$

This shows that the result is true for $m = 2$. Assume that the result holds for $m = k - 1$, therefore for all $j = 1, 2, \dots, k - 1$,

$$\pi^{k-1} = \left(\left(\sum_{i=1}^{k-1} r(r_i + s_i - 1) \right)^{r_j}, \left(\sum_{i=1}^{k-1} r r_i + \sum_{i=1, i \neq j}^{k-1} r s_i \right)^{r_j} \right), \quad \text{for } j = 1, 2.$$

Now for $m = k$,

$$\begin{aligned} G &= S_{r_1, s_1}^{(r)} \vee S_{r_2, s_2}^{(r)} \vee \dots \vee S_{r_{k-1}, s_{k-1}}^{(r)} \vee S_{r_k, s_k}^{(r)} \\ &= A \vee S_{r_k, s_k}^{(r)}, \quad \text{where } A = S_{r_1, s_1}^{(r)} \vee S_{r_2, s_2}^{(r)} \vee \dots \vee S_{r_{k-1}, s_{k-1}}^{(r)}. \end{aligned}$$

Since the result is proved for all $m = k - 1$ and using the fact that the result is proved for each pair and since the result is already proved for $k = 2$, it follows by induction hypothesis that result holds for $m = k$ also. That is,

$$\pi = \left(\left(\sum_{i=1}^k r(r_i + s_i - 1) \right)^{r_j}, \left(\sum_{i=1}^k r r_i + \sum_{i=1, i \neq j}^k r s_i \right)^{s_j} \right), \quad \text{for } j = 1, 2, \dots, k$$

This proves the lemma. $\square$

Lemma 17 *A non-increasing integer sequence $\pi = (d_1, \dots, d_n)$ is potentially $A_{L,M}^{(r)}$ -graphic if and only if it is potentially $S_{L,M}^{(r)}$ -graphic.*

Proof. We only need to prove that if $\pi = (d_1, \dots, d_n)$ is potentially $S_{L,M}^{(r)}$ -graphic, then it is potentially $A_{L,M}^{(r)}$ -graphic. We choose a realization G of π with vertex set $V(G) = \{v_1, \dots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \leq i \leq n$, the induced r -subgraph $G[\{v_1, \dots, v_{L+M}\}]$ of $\{v_1, \dots, v_{L+M}\}$ in G contains $S_{L,M}^{(r)}$ as its r -subgraph and $|V(K_L^{(r)}) \cap \{v_1, \dots, v_L\}|$ is maximum. Denote $H = G[\{v_1, \dots, v_{L+M}\}]$. If $|V(K_L^{(r)}) \cap \{v_1, \dots, v_L\}| = L$, that is, $V(K_L^{(r)}) = \{v_1, \dots, v_L\}$, then π is potentially $A_{L,M}^{(r)}$ -graphic. Assume that $|V(K_L^{(r)}) \cap \{v_1, \dots, v_L\}| < L$. Then there exists $v_i \in \{v_1, \dots, v_L\} \setminus V(K_L^{(r)})$ and a $v_j \in V(K_L^{(r)}) \setminus \{v_1, \dots, v_L\}$. Let $A = N_H(v_j) \setminus (\{v_i\} \cup N_H(v_i))$ and $B = N_G(v_i) \setminus (\{v_j\} \cup N_G(v_j))$. Since $d_G(v_i) \geq d_G(v_j)$, we have $|B| \geq |A|$. Let C be any subset of B such that $|C| = |A|$. Now form a new realization G' of π by a sequence of r -exchanges to the r -edges of the star centralized at v_j with end vertices in A with the non r -edges of the star centralized at v_j with end vertices in C , and by a sequence of r -exchange the r -edges of the star centralized at v_i with end vertices in C with the non r -edges of the star centralized at v_i with end vertices in A . It is easy to see that G' contains $S_{L,M}^{(r)}$ on $\{v_1, \dots, v_{L+M}\}$ so that $|V(K_L^{(r)}) \cap \{v_1, \dots, v_L\}|$ is larger than that of G , which contradicts to the choice of G . $\square$

We use the Havel-Hakimi procedure to test whether or not an r -graphic sequence π is potentially $A_{L,M}^{(r)}$ -graphic.

Theorem 18 *For $r \geq 1$ and $n \geq 1$, an r -graphic sequence $\pi = (d_1, \dots, d_n)$ is potentially $A_{L,M}^{(r)}$ -graphic if and only if π_L is r -graphic.*

Proof. Assume that π is potentially $A_{L,M}^{(r)}$ -graphic. Then π has a realization G with the vertex set $V(G) = \{v_1, \dots, v_n\}$ such that $d_G(v_i) = d_i$ for $(1 \leq i \leq n)$

and G contains $S_{L,M}^{(r)}$ on the vertices $v_1, \dots, v_{L+M}$, where $L + M \leq n$, so that $V^{(r)}(K_L) = \{v_1, \dots, v_L\}$ and $V(\bar{K}_M^{(r)}) = \{v_{L+1}, \dots, v_{L+M}\}$. By applying a sequence of r -exchanges to G in order we will show that there is one such realization G' such that $G' \setminus v_1$ has degree sequence π_1 . If not, we may choose such a realization H of r -graphic sequence π such that the number of vertices adjacent to v_1 in $\{v_{L+M+1}, \dots, v_{d_1+1}\}$ is maximum. Let $v_i \in \{v_{L+M+1}, \dots, v_{d_1+1}\}$ and assume that there is no edge between v_1 and v_i and let $v_j \in \{v_{d_1+2}, \dots, v_n\}$ and there are r edges between v_1 and v_j . We may assume that $d_i > d_j$. Hence there is a vertex $v_t, t \neq i, j$ such that there are r edges between v_i and v_t and no edge between v_j and v_t . Clearly $G = (H \setminus \{v_1^{(r)} v_j, v_i^{(r)} v_t\}) \cup \{v_1^{(r)} v_i, v_j^{(r)} v_t\}$ (where $v_i^{(r)} v_j$ means that there are r edges between v_i and v_j) is a realization of π such that $d_G(v_i) = d_i$ for $1 \leq i \leq n$, G contains $S_{L,M}^{(r)}$ on $v_1, \dots, v_{L+M}$ with $V^{(r)}(K_L) = \{v_1, \dots, v_L\}$ and $V(\bar{K}_M^{(r)}) = \{v_{L+1}, \dots, v_{L+M}\}$ and G has the number of vertices adjacent to v_1 in $\{v_{L+M+1}, \dots, v_{d_1+1}\}$ larger than that of H . This contradicts the choice of H . Repeating this procedure, we can see that π_i is potentially $A_{L-i}^{(r)}$ -graphic successively for $i = 2, \dots, L$. In particular, π_L is r -graphic.

Conversely, suppose that π_L is r -graphic and is realized by a graph G_L with a vertex set $V(G_L) = \{v_{L+1}, \dots, v_n\}$ such that $d_{G_L}(v_i) = d_i$ for $L + 1 \leq i \leq n$. For $i = L, L - 1, \dots, 1$ form G_{i-1} from G_i by adding a new vertex v_i that is adjacent to each of $v_{i+1}, \dots, v_{L+M}$ with r -edges and also to the vertices of G_i with degrees $s_{L+M+1}^{i-1} - r, \dots, d_{d_1+1}^{i-1} - r$. Then for each i , G_i has degrees given by π_i and G_i contains $S_{L-i,M}^{(r)}$ on $L + M - i$ vertices $v_{i+1}, \dots, v_{L+M}$ whose degrees are $d_{i+1} - ir, \dots, d_{L+M} - ir$ so that $V(K_{L-i}^{(r)}) = \{v_{i+1}, \dots, v_L\}$ and $V(\bar{K}_M^{(r)}) = \{v_{L+1}, \dots, v_{L+M}\}$. In particular, G_0 has degrees given by π and contains $S_{L,M}^{(r)}$ on $L + M$ vertices $v_1, \dots, v_{L+M}$ whose degrees are $d_1, \dots, d_{L+M}$ so that $V(K_L^{(r)}) = \{v_1, \dots, v_L\}$ and $V(\bar{K}_M^{(r)}) = \{v_{L+1}, \dots, v_{L+M}\}$. Hence the result follows. $\square$

The following is a sufficient condition for an r -graphic sequence to be potentially $A_{L,M}^{(r)}$ -graphic.

Theorem 19 *Let $n \geq L + M$ and let $\pi = (d_1, \dots, d_n)$ be an r -graphic sequence. If $d_{L+M} \geq 2rL + rM - 2$, then π is potentially $A_{L,M}^{(r)}$ -graphic.*

Proof. Let $n \geq L + M$ and let $\pi = (d_1, \dots, d_n)$ be a non-increasing r -graphic sequence with $d_{L+M} \geq 2rL + rM - 2$. By using the argument similar

to Theorem 8, π is potentially $K_L^{(r)}$ -graphic and hence by Lemma 17, $A_L^{(r)}$ -graphic. Therefore, we assume that G is a realization of π with a vertex set $V(G) = (v_1, \dots, v_n)$ such that $d_G(v_i) = d_i$, ($1 \leq i \leq n$) and G contains $K_L^{(r)}$ on $(v_1, \dots, v_L)$, that is, $V(K_L^{(r)}) = \{v_1, \dots, v_L\}$ and

$$t = e_G(\{v_1, \dots, v_{r_1}, \dots, v_L\}, \{v_{L+1}, \dots, v_{L+s_1}, \dots, v_{L+M}\})$$

(that is, the number of edges between $\{v_1, \dots, v_L\}$ and $\{v_{L+1}, \dots, v_{L+M}\}$) is maximum. If $t = rLM + rs_1s_2 + s_j \sum_{i=1}^{j-1} rs_i$, for $j = 3, 4, \dots, p$, then the cardinality of the edge set of $S_{L,M}^{(r)}$ is same as t and therefore G contains $S_{L,M}^{(r)}$ on the vertices $v_1, v_2, \dots, v_{L+M}$ with $V^{(r)}(K_M) = \{v_1, v_2, \dots, v_L\}$ and

$$V(\overline{K}^{(r)}_M) = \{v_{L+1}, v_{L+2}, \dots, v_{L+s_1}, \dots, v_{L+M}\}.$$

In other-words, π is potentially $\overline{A}_{L,M}^{(r)}$ -graphic. Assume that $t < \{rLM + rs_1s_2 + s_j \sum_{i=1}^{j-1} rs_i\}$, for $j = 3, 4, \dots, p$. Then there exists a $v_k \in \{v_1, v_2, \dots, v_{s_i}\}$ and $v_m \in \{v_{s_i+1}, v_{s_i+2}, \dots, v_{s_i+s_j}\}$, ($i \neq j$) such that $v_k^r v_m \notin E(G)$. Let

$$A = N_{G \setminus \{v_{s_i+1}, v_{s_i+2}, \dots, v_{s_i+s_j}\}}(v_k) \setminus N_{G \setminus \{v_1, v_2, \dots, v_{s_i}\}}(v_m)$$

and

$$B = N_{G \setminus \{v_{s_i+1}, v_{s_i+2}, \dots, v_{s_i+s_j}\}}(v_k) \cap N_{G \setminus \{v_1, v_2, \dots, v_{s_i}\}}(v_m).$$

Then $e_G(x, y) = r$ for $x \in N_{G \setminus \{v_1, \dots, v_L\}}(v_m)$ and $y \in N_{G \setminus \{v_1, \dots, v_{L+M}\}}(v_k)$. Otherwise, if $e_G(x, y) < r$, then $G' = (G \setminus \{v^{(r)}y, v_m^{(r)}x\}) \cup \{v_k^{(r)}v_m, x^{(r)}y\}$ is a realization of π and contains $\overline{S}_{L,M}^{(r)}$ on $v_1, \dots, v_{L+M}$ with $V(K_L^{(r)}) = \{v_1, \dots, v_L\}$ and $(\overline{K}_M^{(r)}) = \{v_{L+1}, \dots, v_{L+M}\}$ such that

$$e_{G'}(\{v_1, \dots, v_L\}, \{v_{L+1}, \dots, v_{L+M}\}) > t,$$

which contradicts the choice of G . Thus B is r -complete. We consider the following cases.

Let $A = \emptyset$. Then $2rL + rM - 2 \leq d_k = d_G(v_k) < rL + rM - 1 + r|B|$, and so $|B| \geq rL$. Since each vertex in $N_{G \setminus \{v_1, \dots, v_L\}}(v_m)$ is adjacent to each vertex in B by r edges and $|N_{G \setminus \{v_1, \dots, v_L\}}(v_m)| \geq 2rL + rM - 2 = rL + rM - 1$. It can be easily seen that the r induced subgraph of $N_{G \setminus \{v_1, \dots, v_L\}}(v_m) \cup \{v_m\}$ in G contains $\overline{S}_{L,M}^{(r)}$

as a subgraph. Thus π is potentially $\overline{A}_{L,M}^{(r)}$ -graphic.

Let $A \neq \emptyset$. Let $a \in A$. If there are $x, y \in N_{G \setminus \{v_1, \dots, v_L\}}(v_m)$ such that $e_G(x, y) < r$ then $G' = (G \setminus \{v_m^{(r)}x, v_m^{(r)}y, v_k^{(r)}a\}) \cup \{v_k^{(r)}v_m, a^{(r)}v_m, x^{(r)}y\}$ is a realization of π and contains $\overline{S}_{L,M}^{(r)}$ on $v_1, \dots, v_{L+M}$ with $V(K_L^{(r)}) = \{v_1, \dots, v_L\}$ and $V(\overline{K}_M^{(r)}) = \{v_{L+1}, \dots, v_{L+M}\}$ such that $e_{G'}(\{v_1, \dots, v_L\}, \{v_{L+1}, \dots, v_{L+M}\}) > t$ which contradicts the choice of G . Thus $N_{G \setminus \{v_1, \dots, v_L\}}(v_m)$ is r -complete. Since

$$|N_{G \setminus \{v_1, \dots, v_L\}}(v_m)| \geq rL + rM - 1 \quad \text{and} \quad e_G(v_m, z) = r,$$

for any $z \in N_{G \setminus \{v_1, \dots, v_L\}}(v_m)$, it is easy to see that the induced r -subgraph of $N_{G \setminus \{v_1, \dots, v_L\}}(v_m) \cup \{v_m\}$ in G is r -complete, and so contains $\overline{S}_{L,M}^{(r)}$ as a r -subgraph. Thus π is potentially $\overline{A}_{L,M}^{(r)}$ -graphic. $\square$

Theorem 20 *If $\pi = (d_1, d_2, \dots, d_n)$ is an r -graphic sequence such that $\sigma(\pi)$ is at least $(n^2 - 3n + 8)r$, then π is potentially $K_4^{(r)}$ -graphic.*

Proof. Let $\pi = (d_1, d_2, \dots, d_n)$ be an r -graphic sequence such that $d_1 \geq d_2 \geq \dots \geq d_n \geq 1$ and $\sigma(\pi) = (n^2 - 3n + 8)r$. Suppose G is a graphical realization of π and $e(G)$ is the size of G . Then $2e(G) = \sigma(\pi)$ and $2e(G^c) = nb(n-1) - \sigma(\pi) = nr(n-1) - (n^2 - 3n + 6)r = r(2n-6)$, so that $e(G^c) = r(n-3)$, where G^c is the complement of the r -graph G . An extremal problem is r -graph G is obtained by deleting $r(n-3)$ independent edges from the complete r -graph $K_n^{(r)}$ of order n . Hence the largest vertex number of independent sets in G^c is 3. This implies that the largest possible complete r -subgraph of G is of order 3. As $1 \leq n-3 \leq 3$. Hence there is no complete r -subgraph of order 4 in G . Therefore, we have

$$\sigma(K_4^{(r)}, n) \geq (n^2 - 3n + 6)r + 2r = (n^2 - 3n + 8)r$$

Now Suppose that $\pi = (d_1, d_2, \dots, d_n)$ is r -graphic sequence with $d_1 \geq d_2 \geq \dots \geq d_n \geq r$ and $\sigma(\pi) \geq (n^2 - 3n + 8)r$. Then every graphical realization G of π is obtained by removing at most $r(n-4)$ edges from the r -complete graph $K_n^{(r)}$. Hence the maximal complete subgraph of G has order at least $n - (n-4) = 4$. Thus G is potentially $K_4^{(r)}$. In other words,

$$\sigma(K_4^{(r)}, n) \leq (n^2 - 3n + 8)r \tag{3}$$

Combining (3) and (4), the result follows. $\square$

Acknowledgements. The authors thank the anonymous referee for his useful comments and suggestions.

References

- [1] [D. de Caen](#), An upper bound on the sum of squares of degrees in a graph, *Discrete Math.* **185** (1998) 245–248. [⇒38](#)
- [2] V. Chungphaisan, Conditions for a sequences to be r -graphic, *Discrete Math.* **7** (1974) 31–39. [⇒36, 37](#)
- [3] [P. Erdős](#), [T. Gallai](#), Graphs with prescribed degrees (in Hungarian) *Matematikai Lapok* **11**(1960) 264–274. [⇒36](#)
- [4] [P. Erdős](#), [M. S. Jacobson](#), [J. Lehel](#), Graphs realizing the same degree sequences and their respective clique numbers, in: *Graph Theory, Combinatorics and Applications*, vol. 1, John Wiley and Sons, New York, 1991, 439–449. [⇒37](#)
- [5] [D. R. Fulkerson](#), [A. J. Hoffman](#), [M. H. McAndrew](#), Some properties of graphs with multiple edges, *Canad. J. Math.* **17** (1965) 166–177. [⇒36](#)
- [6] [D. J. Kleitman](#), D. L. Wang, Algorithm for constructing graphs and digraphs with given valencies and factors, *Discrete Math.* **6** (1973) 79–88. [⇒36](#)
- [7] [J. S. Li](#), [Z. X. Song](#), R. Luo, The Erdős-Jacobson-Lehel conjecture on potentially p_k -graphic sequences is true, *Sci. China Ser. A* **41** (1998) 510–520. [⇒37](#)
- [8] [J. S. Li](#), [Z. X. Song](#), An extremal problem on the potentially p_k -graphic sequence, *Discrete Math.* **212** (2000) 223–231. [⇒37](#)
- [9] [S. Pirzada](#), *An Introduction to Graph Theory*, Universities Press, Orient Blackswan, Hyderabad, India 2012. [⇒36](#)
- [10] [S. Pirzada](#), [B. A. Chat](#), Potentially graphic sequences of split graphs, *Kragujevac J. Math.* **38**, **1** (2014) 73–81. [⇒37, 38](#)
- [11] [S. Pirzada](#), [B. A. Chat](#), [F. A. Dar](#), Graphical sequences of some family of induced subgraphs, *J. Algebra Comb. Discrete Appl.* **2(2)** (2015) 95–109. [⇒38](#)
- [12] A. R. Rao, The clique number of a graph with a given degree sequence, *Proc. Symposium on Graph Theory* (ed. A. R. Rao, Macmillan and Co. India Ltd, I.S.I. Lecture Notes Series, **4** (1979) 251–267. [⇒37](#)
- [13] [J. H. Yin](#), Conditions for r -graphic sequences to be potentially K_{m+1}^r -graphic, *Discrete Math.* **309** (2009) 6271–6276. [⇒37](#)
- [14] [J. H. Yin](#), A Havel-Hakimi type procedure and a sufficient condition for a sequence to be potentially $S_{r,s}$ -graphic, *Czechoslovak Math. J.* **62,3** (2012) 863–867. [⇒37, 38](#)

Received: November 30, 2016 • Revised: January 19, 2017

