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On Laplacian spectrum of unitary Cayley graphs

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Abstract. Let R be a commutative ring with unity $1 \neq 0$ and let $R^\times$ be the set of all unit elements of R. The unitary Cayley graph of R, denoted by $G_R = \mathrm{Cay}(R, R^\times)$, is a simple graph whose vertex set is R and there is an edge between two distinct vertices x and y of R if and only if $x - y \in R^\times$. In this paper, we determine the Laplacian and signless Laplacian eigenvalues for the unitary Cayley graph of a commutative ring. Also, we compute the Laplacian and signless Laplacian energy of the graph G_R and its line graph.

1 Introduction

We consider finite commutative rings R with unit element $1 \neq 0$. Let $R^\times$ be the set of all unit elements of R. We know that an Artinian ring R can be written as $R \cong R_1 \times \cdots \times R_t$, where R_i is a finite local ring with maximal ideal $\mathfrak{M}_i$, for all $1 \leqslant i \leqslant t$. This decomposition is unique up to permutation of factors. We

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denote the (finite) residue field $\frac{R_i}{\mathfrak{M}_i}$ by K_i and $f_i = |K_i| = \frac{|R_i|}{|\mathfrak{M}_i|}$. Also, assume that $f_1 \leqslant f_2 \leqslant \cdots \leqslant f_t$.

A simple graph G consists of a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. We call $|V(G) = n|$ and $|E(G)| = m$, respectively, as the order and the size of the graph G. The complement of G, denoted by $\overline{G}$, is the graph whose vertex set is same as that of G and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A complete graph on n vertices is denoted by K_n. A graph G is multipartite if its vertex set can be partitioned into non-empty subsets, called partite sets, such that no two vertices in the same part are adjacent. A multipartite graph is complete if every vertex of a partite set is adjacent to each vertex of the other partite sets. A complete multipartite graph with k parts is denoted by $K_{n_1,n_2,\ldots,n_k}$ where n_i is the number of vertices in the i-th part of the graph.

The join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{xy; x \in V(G_1), y \in V(G_2)\}$. The direct product of G_1 and G_2, denoted by $G_1 \otimes G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ in which (u_1, v_1) and (u_2, v_2) are adjacent if u_1 and u_2 are adjacent in G_1 and v_1 and v_2 are adjacent in G_2. For other undefined notations and terminology from graph theory and spectral graph theory, the readers are referred to [6, 18].

The unitary Cayley graph of R, denoted by $G_R = \mathrm{Cay}(R, R^\times)$, is a (simple) graph whose vertex set is R and two distinct vertices x and y of R are adjacent if and only if $x - y \in R^\times$. Some recent results on unitary Cayley graphs can be seen in [16]. If $G = \mathbb{Z}_n$ is the finite cyclic group of order n and the set S consists of two elements, the standard generator of G and its inverse, then the Cayley graph is the cycle C_n. More generally, the Cayley graphs of finite cyclic groups are exactly the circulant graphs. Some examples of unitary Cayley graphs are given in Figure 1.

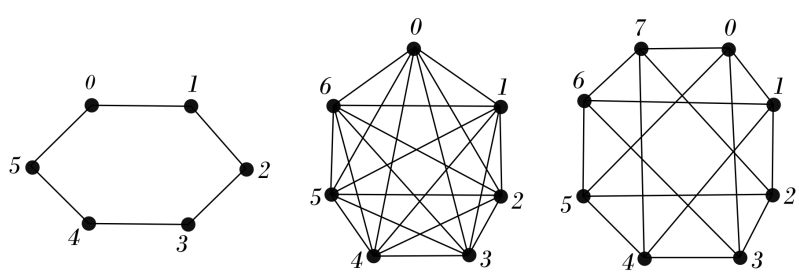

Figure 1: The unitary Cayley graphs for $\mathbb{Z}_6$, $\mathbb{Z}_7$, $\mathbb{Z}_8$

The following proposition is a basic consequence of the definition and it was illustrated in [1].

Proposition 1 *Let* R *be a commutative ring.*

(a) Then G_R *is a* $|R^{\times}|$*-regular graph.*

(b) If R *is a local ring with maximal ideal* $\mathfrak{M}$*, then* G_R *is a complete multipartite graph whose partite sets are the cosets of* $\mathfrak{M}$ *in* R*. In particular,* G_R *is a complete graph if and only if* R *is a field.*

(c) If R *is an Artinian ring and* $R \cong R_1 \times \ldots \times R_t$ *as a product of local rings, then* $G_R \cong \otimes_{i=1}^{t} G_{R_i}$*. Hence,* G_R *is a direct product of complete multipartite graphs.*

The adjacency matrix A of a graph G is a $(0,1)$-square matrix of order n whose (i,j)-entry is equal to 1, if v_i is adjacent to v_j and equal to 0, otherwise. The eigenvalues of A are the eigenvalues of the graph G. The set of all eigenvalues of G is called the *spectrum* of G. If $\lambda_1 \geq \cdots \geq \lambda_k$ are the eigenvalues of G with multiplicities $r_1, \ldots, r_k$, respectively, the spectrum of G is denoted by $\text{Spec}(G) = \begin{pmatrix} \lambda_1 & \ldots & \lambda_k \\ r_1 & \ldots & r_k \end{pmatrix}$. The energy of a graph was introduced by Gutman [13] and is defined as the sum of the absolute values of all the eigenvalues of a graph G and it is denoted by $E(G)$.

Kiani et al. [15] obtained the following result about the eigenvalues of the unitary Cayley graph. Also, they computed the energy of the unitary Cayley graph of a finite commutative ring R.

Theorem 2 [15] *Let* R *be a finite ring.*

(a) If R *is a finite local ring with the maximal ideal* $\mathfrak{M}$ *of size* m *and* $\frac{|R|}{m} = f$*, then*

$$\textit{Spec}(G_R) = \begin{pmatrix} |R^{\times}| & 0 & -m \\ 1 & |R| - f & f - 1 \end{pmatrix}.$$

In particular, if $\mathbb{F}_q$ *is the field with* q *elements, then*

$$\textit{Spec}(G_{\mathbb{F}_q}) = \begin{pmatrix} q-1 & -1 \\ 1 & q-1 \end{pmatrix}.$$

(b) *Let* R *be a finite commutative ring, where* $R \cong R_1 \times R_2 \times \ldots \times R_t$ *and* R_i *is a local ring with maximal ideal* $\mathfrak{M}_i$ *of size* m_i *for all* $1 \leqslant i \leqslant t$. *Then the eigenvalues of* G_R *are:*

(b-1) $(-1)^{|C|}\dfrac{|R^{\times}|}{\prod_{j\in C}|R_j^{\times}|/m_j}$ *with multiplicity* $\prod_{j\in C}|R_j^{\times}|/m_j$ *for all subsets* C *of the set* $\{1, 2, \ldots, t\}$.

(b-2) 0 *with multiplicity* $|R| - \prod_{i=1}^{t}(1 + |R_i^{\times}|/m_i)$

Theorem 3 [15] *Let* $R \cong R_1 \times R_2 \times \ldots \times R_t$ *be a finite commutative ring where* R_i *is a local ring for all* $1 \leqslant i \leqslant t$. *Then* $E(G_R) = 2^t|R^{\times}|$.

Let $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix associated to the graph G, where $d_i = deg(v_i)$ is the degree of the vertex v_i, for all $1 \leqslant i \leqslant n$. The matrices $L(G) = D(G) - A(G)$ and $|L|(G) = D(G) + A(G)$ are respectively, called the Laplacian and the signless Laplacian matrices of G. Their spectrum are respectively, the Laplacian spectrum and the signless Laplacian spectrum of the graph G. We denote the Laplacian spectrum and the signless Laplacian spectrum of the graph G by $\mathrm{Spec}_L(G)$ and $\mathrm{Spec}_{|L|}(G)$, respectively. Both the matrices $L(G)$ and $|L|(G)$ are real symmetric, positive semi-definite and therefore their eigenvalues are non-negative real numbers. Let $0 = \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$ and $\mu_n^+ \leqslant \mu_{n-1}^+ \leqslant \cdots \leqslant \mu_1^+$ be respectively, the Laplacian spectrum and the signless Laplacian spectrum of G. It is known that the smallest eigenvalue of $L(G)$ is 0 with multiplicity equal to the number of connected components of G. So, $\mu_2 > 0$ if and only if G is connected. Also, the least eigenvalue of the signless Laplacian matrix of a connected graph is 0 if and only if the graph is bipartite. In this case, 0 is a simple eigenvalue. Furthermore, it is easy to see that $\mathrm{tr}(L(G)) = \sum_{i=1}^{n}\mu_i = 2m$ and tr(—L—(G)) $= \sum_{i=1}^{n}\mu_i^+ = 2m$. Recent work on Laplacian eigenvalues can be seen in [2, 5, 9, 10, 11, 12]. The Laplacian energy of a graph G defined by Gutman and Zhou [14] is $LE(G) = \sum_{i=1}^{n}|\mu_i - \frac{2m}{n}|$. The Laplacian energy, which is an extension of graph energy concept, has found remarkable chemical applications (see [24]). For recent development on $LE(G)$ see [7, 8] and the references therein. The signless Laplacian energy $|L|E(G)$ of G, in analogy to $LE(G)$, is defined as $|L|E(G) = \sum_{i=1}^{n}|\mu_i^+ - \frac{2m}{n}|$. Recent work on Laplacian eigenvalues can be seen in [19].

The rest of the paper is organized as follows. In Section 2, we determine the Laplacian spectrum and the Laplacian energy of the unitary Cayley graph G_R. Also, we completely obtain the signless Laplacian spectrum of the graph G_R

and compute the signless Laplacian energy of G_R. Further, we compute the Laplacian and signless Laplacian energy of the line graph of G_R.

2 Laplacian spectrum of unitary Cayley graphs

We begin with the following theorem, which gives the Laplacian spectrum of the join of two graphs G_1 and G_2.

Theorem 4 [17] *Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, respectively. Suppose that $0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{n_1}$ and $0 = \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_{n_2}$ are the Laplacian eigenvalues of G_1 and G_2, respectively. Then the Laplacian eigenvalues of the graph $G_1 \vee G_2$ are*

(i) 0 *with multiplicity* 1,

(ii) $\lambda_i + n_2$ *with multiplicity* 1 *for all* $2 \leqslant i \leqslant n_1$,

(iii) $\mu_j + n_1$ *with multiplicity* 1 *for all* $2 \leqslant j \leqslant n_2$,

(iv) $n_1 + n_2$ *with multiplicity* 1.

Now, we have the following observation.

Lemma 5 *If $G = K_{n_1,n_2,\ldots,n_k}$, where $n_i \in \mathbb{N}$ for all $1 \leqslant i \leqslant k$, then the Laplacian eigenvalues of G are*

(i) 0 *with multiplicity* 1,

(ii) $\alpha_i = \sum\limits_{\substack{j=1 \\ j \neq i}}^{k} n_j$ *with multiplicity* $n_i - 1$ *for all* $1 \leqslant i \leqslant k$,

(iii) $n_1 + n_2 + \cdots + n_k$ *with multiplicity* $k - 1$.

Proof. We induct on k. For $k = 2$, we have $G = \overline{K}_{n_1} \vee \overline{K}_{n_2}$. So, by Theorem 4, we have that

$$\mathrm{Spec}_L(K_{n_1,n_2}) = \begin{pmatrix} 0 & n_1 & n_2 & n_1 + n_2 \\ 1 & n_2 - 1 & n_1 - 1 & 1 \end{pmatrix}.$$

Assume that the hypothesis is true for $K_{n_1,n_2,\ldots,n_k}$.

We prove it for the graph $K_{n_1,n_2,\ldots,n_k,n_{k+1}}$.

Clearly, $K_{n_1,n_2,\ldots,n_k,n_{k+1}} \cong K_{n_1,n_2,\ldots,n_k} \vee \overline{K}_{n_k}$.

Now, by Theorem 4, it is easy to see that the Laplacian eigenvalues of $K_{n_1,n_2,\ldots,n_k,n_{k+1}}$ are

(i) 0 with multiplicity 1,

(ii) $\sum_{\substack{j=1 \\ j\neq i}}^{k+1} n_j$ with multiplicity $n_i - 1$ for all $1 \leqslant i \leqslant k+1$,

(iii) $n_1 + n_2 + \cdots + n_k + n_{k+1}$ with multiplicity k. □

At first, we assume that R is a local ring.

Proposition 6 *Let $(R, \mathfrak{M})$ be a local ring with $|\mathfrak{M}| = m$ and $|\frac{R}{\mathfrak{M}}| = f$. Then*

$$Spec_L(G_R) = \begin{pmatrix} 0 & |R^\times| & |R| \\ 1 & |R|-f & f-1 \end{pmatrix}.$$

In particular, if $R = \mathbb{F}_q$ is the field with q elements, then

$$Spec_L(G_{\mathbb{F}_q}) = \begin{pmatrix} 0 & q \\ 1 & q-1 \end{pmatrix}.$$

Proof. It is easy to see that G_R is a complete multipartite graph in which every partite set is a coset of $\mathfrak{M}$. So, G_R is the join of f copies of the empty graph $\overline{K}_m$. Now, by Lemma 5, we have

$$\text{Spec}_L(G_R) = \begin{pmatrix} 0 & |R|-m & |R| \\ 1 & |R|-f & f-1 \end{pmatrix}.$$

Since $|R| - m = |R^\times|$, therefore

$$\text{Spec}_L(G_R) = \begin{pmatrix} 0 & |R^\times| & |R| \\ 1 & |R|-f & f-1 \end{pmatrix}.$$

□

The Laplacian spectrum of the direct product of graphs has been described completely only when the factor graphs are regular. The Laplacian eigenvalues of the direct product of two regular graphs are listed in the following theorem.

Theorem 7 [3] *Let G_1 be an r_1-regular graph with n_1 vertices and G_2 be an r_2-regular graph with n_2 vertices. Let $Spec_L(G_1) = (\lambda_1, \lambda_2, \ldots, \lambda_{n_1})$ and $Spec_L(G_2) = (\mu_1, \mu_2, \ldots, \mu_{n_2})$. Then the eigenvalues of the graph $G_1 \otimes G_2$ are $r_1\mu_j + r_2\lambda_i - \mu_j\lambda_i$ for all $1 \leqslant i \leqslant n_1$ and $1 \leqslant j \leqslant n_2$.*

In the following theorem, we obtain the Laplacian eigenvalues of G_R with their multiplicities. Here, $|R_S^\times|$ stands for $|R_{s_1}^\times \times R_{s_2}^\times \ldots \times R_{s_k}^\times|$, where $S = \{s_1, \ldots, s_k\} \subseteq \{1, 2, \ldots, t\}$ (if $S = \emptyset$, then we define $|R_S^\times| = 1$).

Theorem 8 *Let R be a finite commutative ring such that $R \cong R_1 \times R_2 \times \cdots \times R_t$, where $(R_i, \mathfrak{M}_i)$ is a local ring with $|\mathfrak{M}_i| = m_i$ and $|\frac{R_i}{\mathfrak{M}_i}| = f_i$. Then the Laplacian eigenvalues of G_R are*

(i) 0 *with multiplicity* 1,

(ii) $|R^\times|$ *with multiplicity* $|R| - \prod_{i=1}^t f_i$,

(iii) λ_A *with multiplicity* $\prod_{i \in A'}(f_i - 1)$ *for all* $A \subsetneq \{1, 2, \ldots, t\}$, *where*

$$\lambda_A = |R_A^\times| \sum_{\substack{C=\{i_1, i_2, \ldots, i_k\} \subseteq A' \\ k=1}}^{|A'|} (-1)^{|C|-1} |R_{i_1}||R_{i_2}| \ldots |R_{i_k}| \frac{|R_{A'}^\times|}{|R_C^\times|}$$

and A' is the complement of A.

Proof. We use induction on t. For $t = 1$ and the local ring $R \cong R_1$, by Proposition 6, we have

$$\mathrm{Spec}_L(G_R) = \begin{pmatrix} 0 & |R^\times| & |R| \\ 1 & |R| - f & f - 1 \end{pmatrix}.$$

Note that $\emptyset$ is the only proper subset of $\{1\}$ and $\lambda_\emptyset = |R|$. So, we are done in this case. Now, assume that the Laplacian eigenvalues of $R_1 \times R_2 \times \cdots \times R_{t-1}$ are

(i) 0 with multiplicity 1,

(ii) $|R_1^\times \times R_2^\times \times \cdots \times R_{t-1}^\times|$ with multiplicity $|R_1 \times R_2 \times \cdots \times R_{t-1}| - \prod_{i=1}^{t-1} f_i$,

(iii) λ_A with multiplicity $\prod_{i \in A'}(f_i - 1)$ for all $A \subsetneq \{1, 2, \ldots, t-1\}$, where

$$\lambda_A = |R_A^\times| \sum_{\substack{C=\{i_1, i_2, \ldots, i_k\} \subseteq A' \\ k=1}}^{|A'|} (-1)^{|C|-1} |R_{i_1}||R_{i_2}| \ldots |R_{i_k}| \frac{|R_{A'}^\times|}{|R_C^\times|}.$$

Now, we determine the Laplacian eigenvalues of G_R when $R \cong R_1 \times R_2 \times \cdots \times R_{t-1} \times R_t$. We know that $G_R \cong G_{R_1 \times R_2 \times \cdots \times R_{t-1}} \otimes G_{R_t}$. Note that the graphs $G_{R_1 \times R_2 \times \cdots \times R_{t-1}}$ and G_{R_t} are regular, so we can use Theorem 7. Since

$$\mathrm{Spec}_L(G_{R_t}) = \begin{pmatrix} \mu_1 = 0 & \mu_2 = |R_t^\times| & \mu_3 = |R_t| \\ 1 & |R_t| - f_t & f_t - 1 \end{pmatrix},$$

we have the following cases to consider.

Case 1. For $\mu_1 = 0$, we have the following eigenvalues.

1.1. 0 with multiplicity 1,

1.2. $|R_1^\times \times R_2^\times \times \cdots \times R_{t-1}^\times| \times |R_t^\times|$ with multiplicity $|R_1 \times R_2 \times \cdots \times R_{t-1}| - \prod_{i=1}^{t-1} f_i$,

1.3. $\lambda_A \times |R_t^\times|$ with multiplicity $\prod_{i \in A'}(f_i - 1)$ for all $A \subsetneq \{1, 2, \dots, t-1\}$.

Case 2. For $\mu_2 = |R_t^\times|$, we obtain the following eigenvalues.

2.1. $|R_1^\times \times R_2^\times \times \cdots \times R_{t-1}^\times||R_t^\times|$ with multiplicity $|R_t| - f_t$,

2.2. $|R_1^\times \times R_2^\times \times \cdots \times R_{t-1}^\times||R_t^\times|$ with multiplicity $(|R_1 \times R_2 \times \cdots \times R_{t-1}| - \prod_{i=1}^{t-1} f_i)(|R_t| - f_t)$,

2.3. $|R_1^\times \times R_2^\times \times \cdots \times R_{t-1}^\times||R_t^\times|$ with multiplicity $\sum_{A \subsetneq \{1,2,\dots,t-1\}} \prod_{i \in A'}(f_i - 1)(|R_t| - f_t)$.

Therefore, in this case, we see that the eigenvalue is equal to $|R_1^\times \times R_2^\times \times \cdots \times R_{t-1}^\times||R_t^\times|$ and this implies that $|R_1^\times \times R_2^\times \times \cdots \times R_{t-1}^\times||R_t^\times|$ is an eigenvalue with multiplicity $|R_1 \times R_2 \times \cdots \times R_{t-1}|(|R_t| - f_t)$.

Case 3. For $\mu_3 = |R_t|$, the following eigenvalues can be obtained.

3.1. $|R_1^\times \times R_2^\times \times \cdots \times R_{t-1}^\times||R_t|$ with multiplicity $f_t - 1$,

3.2. $|R_1^\times \times R_2^\times \times \cdots \times R_{t-1}^\times||R_t^\times|$ with multiplicity $(|R_1 \times R_2 \times \cdots \times R_{t-1}| - \prod_{i=1}^{t-1} f_i)(f_t - 1)$,

3.3. $|R_1^\times \times R_2^\times \times \cdots \times R_{t-1}^\times||R_t| + \lambda_A |R_t^\times| - \lambda_A |R_t|$ with multiplicity $(\prod_{i \in A'}(f_i - 1))(f_t - 1)$ for all $A \subsetneq \{1, 2, \dots, t-1\}$.

Thus, we conclude the following.

(i) By case (1.1), 0 with multiplicity 1 is a Laplacian eigenvalue of G_R.

(ii) By cases (1.2),(2.1), (2.2),(2.3) and (3.2), $|R_1^\times \times R_2^\times \times \cdots \times R_{t-1}^\times||R_t^\times|$ is a Laplacian eigenvalue of G_R. Its multiplicity is equal to

$$\left(|R_1 \times R_2 \times \cdots \times R_{t-1}| - \prod_{i=1}^{t-1} f_i\right) + \left(|R_1 \times R_2 \times \cdots \times R_{t-1}|(|R_t| - f_t)\right) +$$

$$\left(|R_1 \times R_2 \times \cdots \times R_{t-1}| - \prod_{i=1}^{t-1} f_i\right)(f_t - 1) = |R_1 \times R_2 \times \cdots \times R_t| - \prod_{i=1}^{t} f_i$$

(iii) For $\mathfrak{A} \subsetneq \{1, 2, \ldots, t\}$, three cases (1.3), (3.1) and (3.3) cover all eigenvalues with the type $\lambda_{\mathfrak{A}}$.

(a) From case (1.3), $\lambda_A \times |R_t^\times|$ with multiplicity $\prod_{i\in A'}(f_i - 1)$ is a Laplacian eigenvalue of G_R for all $A \subsetneq \{1, 2, \ldots, t-1\}$. Note that if we set $\mathfrak{A} = A \cup \{t\}$, then $\lambda_{\mathfrak{A}} = \lambda_A \times |R_t^\times|$.

(b) From case (3.1), $|R_1^\times \times R_2^\times \times \cdots \times R_{t-1}^\times||R_t|$ with multiplicity $f_t - 1$ is a Laplacian eigenvalue of G_R. By setting $\mathfrak{A} = \{1, 2, , \ldots, t-1\}$, we have $\lambda_{\mathfrak{A}} = |R_1^\times \times R_2^\times \times \cdots \times R_{t-1}^\times||R_t|$.

(c) From case (3.3), $|R_1^\times \times R_2^\times \times \cdots \times R_{t-1}^\times||R_t| + \lambda_A|R_t^\times| - \lambda_A|R_t|$ with multiplicity $(\prod_{i\in A'}(f_i - 1))(f_t - 1)$ is a Laplacian eigenvalue of G_R, for all $A \subsetneq \{1, 2, \ldots, t-1\}$. This case covers all eigenvalues like $\lambda_{\mathfrak{A}}$, when $\mathfrak{A}$ is a proper subset of the set $\{1, 2, \ldots, t\}$ and $t \notin \mathfrak{A}$. □

Now, we compute the Laplacian energy of the unitary Cayley graph, when R is a finite commutative ring. We start with the local case.

Lemma 9 *Let* R *be a finite local commutative ring. Then* $LE(G_R) = 2|R^\times|$.

Proof. First, note that in the graph G_R, we have $\frac{2m}{n} = |R^\times|$. Since the Laplacian spectrum of G_R is

$$\mathrm{Spec}_L(G_R) = \begin{pmatrix} 0 & |R^\times| & |R| \\ 1 & |R| - f & f - 1 \end{pmatrix},$$

we have $LE(G_R) = 2|R^\times|$. □

Lemma 10 *Let* $R \cong R_1 \times R_2$*, where* $(R_1, \mathfrak{M}_1)$ *and* $(R_2, \mathfrak{M}_2)$ *are local rings. Then*

$$LE(G_R) = 2^2|R^\times|.$$

Proof. We know that $G_R \cong G_{R_1} \otimes G_{R_2}$. Now, let $\text{Spec}_L(G_{R_1}) = (\lambda_1, \lambda_2, \ldots, \lambda_{|R_1|})$ and $\text{Spec}_L(G_{R_2}) = (\mu_1, \mu_2, \ldots, \mu_{|R_2|})$. Then, by Theorem 7, we have

$$\begin{aligned} LE(G_R) &= \sum_{i=1}^{|R_1|} \sum_{j=1}^{|R_2|} \left| |R_1^\times| \mu_j + |R_2^\times| \lambda_i - \mu_j \lambda_i - |R_1^\times||R_2^\times| \right| \\ &= \sum_{i=1}^{|R_1|} \sum_{j=1}^{|R_2|} \left| (\mu_j - |R_2^\times|) \right| \left| (\lambda_i - |R_1^\times|) \right| \\ &= LE(G_{R_1}) LE(G_{R_2}) = (2|R_1^\times|)(2|R_2^\times|) = 2^2 |R^\times|. \end{aligned}$$

□

Theorem 11 *Let R be a finite commutative ring such that $R \cong R_1 \times R_2 \times \cdots \times R_t$, where R_i is a local ring for all $1 \leqslant i \leqslant t$. Then $LE(G_R) = 2^t |R^\times|$.*

Proof. This follows by using induction on t and in view of Lemmas 9 and 10. □

The following results concern about the signless Laplacia spectrum of G_R. The proofs are omitted since they are similar to the proofs on the Laplacian spectrum.

Proposition 12 *Let $(R, \mathfrak{M})$ be a local ring with $|\mathfrak{M}| = m$ and $|\frac{R}{\mathfrak{M}}| = f$. Then*

$$Spec_{|L|}(G_R) = \begin{pmatrix} |R^\times| - m & |R^\times| & 2|R^\times| \\ f-1 & |R| - f & 1 \end{pmatrix}.$$

In particular, if $R = \mathbb{F}_q$ is the field with q elements, then

$$Spec_{|L|}(G_{\mathbb{F}_q}) = \begin{pmatrix} q-2 & 2(q-1) \\ q-1 & 1 \end{pmatrix}.$$

Theorem 13 *Let R be a finite commutative ring such that $R \cong R_1 \times R_2 \times \cdots \times R_t$, where $(R_i, \mathfrak{M}_i)$ is a local ring with $|\mathfrak{M}_i| = m_i$ and $|\frac{R_i}{\mathfrak{M}_i}| = f_i$. Then the signless Laplacian eigenvalues of G_R are*

(i) $2|R^\times|$ *with multiplicity* 1,

(ii) $|R^\times|$ *with multiplicity* $|R| - \prod_{i=1}^{t} f_i$,

(iii) λ_A *with multiplicity* $\prod_{i \in A'} (f_i - 1)$ *for all* $A \subsetneq \{1, 2, \ldots, t\}$ *where*

$$\lambda_A = |R^\times| + (-1)^{|A'|} \prod_{i \in A} |R_i^\times| \prod_{j \in A'} |m_j|.$$

If R be a local finite commutative ring, it is easy to see that the signless Laplacian energy of G_R is given by $|L|E(G_R) = 2|R^\times|$. Further, if $R \cong R_1 \times R_2$, where R_1 and R_2 are local rings, then $|L|E(G_R) = 2^2|R^\times|$.

Thus, we have the following observation.

Theorem 14 *Let R be a finite commutative ring such that $R \cong R_1 \times R_2 \times \cdots \times R_t$, ($t \geqslant 2$), where R_i is a local ring for all $1 \leqslant i \leqslant t$. Then $|L|E(G_R) = 2^t|R^\times|$.*

Let G be a graph with n vertices and m edges. The line graph $L(G)$ of G is a simple graph whose vertex set is the set of edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges in G have a vertex in common. So, $n_{L(G)}$ (the number of vertices of $L(G)$) equals m. Also, it is easy to see that if G is an r-regular graph, then $L(G)$ is a $(2r-2)$-regular graph.

Theorem 15 [4] *Let G be an r-regular graph ($r \geqslant 2$) with n vertices and m edges. Then*

(a) The Laplacian eigenvalues of the graph $L(G)$ are

- *(i) $2 - \lambda_i$, where λ_i is a Laplacian eigenvalue of G for all $1 \leqslant i \leqslant n$,*
- *(ii) $r - 2$ with multiplicity $m - n$.*

(b) The signless Laplacian eigenvalues of the graph $L(G)$ are

- *(i) $\lambda_i^+ + 2r - 4$, where λ_i^+ is a signless Laplacian eigenvalue of G for all $1 \leqslant i \leqslant n$,*
- *(ii) $2r - 4$ with multiplicity $m - n$.*

Now, we compute the Laplacian energy of the line graph of the unitary Cayley graphs. If $|R^\times| = 1$, then $L(G_R)$ is an empty graph. So in this case, $LE(L(G_R)) = 0$. Thus, we suppose that $|R^\times| \geqslant 2$. Now, by Theorem 15, the spectrum of $L(G)$ consists of the following eigenvalues.

(i) $2 - \lambda_i$, where λ_i is a Laplacian eigenvalue of G_R for all $1 \leqslant i \leqslant |R|$,

(ii) $|R^\times| - 2$ with multiplicity $|R||R^\times|/2 - |R|$.

Proposition 16 *Let R be a finite commutative ring with $|R^\times| \geqslant 2$. Then*

$$LE(L(G_R)) = \frac{|R|\left(|R^\times|^2 + 4|R^\times| - 8\right)}{2}.$$

Proof. Since G_R is $|R^\times|$-reqular, $L(G_R)$ is a $(2|R^\times|-2)$-regular graph. So,

$$2m_{L(G_R)}/n_{L(G_R)} = 2|R^\times|-2,$$

where $n_{L(G_R)}$ and $m_{L(G_R)}$ are the number of vertices and edges of $L(G_R)$, respectively. We have

$$\begin{aligned}
LE(L(G_R)) &= \sum_{i=1}^{|R|} \left|2-\lambda_i-(2|R^\times|-2)\right| + \sum_{i=1}^{|R||R^\times|/2-|R|} \left||R^\times|-2-(2|R^\times|-2))\right| \\
&= \sum_{i=1}^{|R|} \left|-\lambda_i-2|R^\times|+4\right| + \sum_{i=1}^{|R||R^\times|/2-|R|} |R^\times| \\
&= \sum_{i=1}^{|R|} (\lambda_i+2|R^\times|-4) + \sum_{i=1}^{|R||R^\times|/2-|R|} |R^\times| \qquad \text{(Since } |R^\times| \geqslant 2) \\
&= \sum_{i=1}^{|R|} \lambda_i + 2|R||R^\times| - 4|R| + (|R||R^\times|/2-|R|)|R^\times| \\
&= |R||R^\times| + 2|R||R^\times| - 4|R| + (|R||R^\times|/2-|R|)|R^\times| \\
&\text{(Since } \sum_{i=1}^{|R|} \lambda_i = |R||R^\times|) \\
&= \frac{|R|\left(|R^\times|^2+4|R^\times|-8\right)}{2}.
\end{aligned}$$

□

The following result gives the signless Laplacian energy of the line graph of unitary Cayley graphs. The proof is similar to the Laplacian case.

Proposition 17 *Let* R *be a finite commutative ring with* $|R^\times| \geqslant 2$. *Then*

(i) If $f_1 = 2$, *then* $|L|E(L(G_R)) = 2\left(|R|(|R^\times|-2)+1\right)$.

(ii) $|L|E(L(G_R)) = 2|R|(|R^\times|-2)$, *otherwise.*

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