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On $\lambda^D - R_0$ and $\lambda^D - R_1$ spaces

Sarhad F. Namiq
Department of Mathematics,
College of Education,
University of Garmian,
Kurdistan-Region, Iraq
email: sarhad1983@gmail.com

Ennis Rosas
Departamento de Ciencias
Naturales y Exactas,
Universidad de la Costa, Barranquilla,
Colombia & Departmento de
Matemáticas, Universidad de Oriente,
Cumaná, Venezuela
email: ennisrafael@gmail.com

Abstract. In this paper we introduce the new types of separation axioms called $\lambda^D - R_0$ and $\lambda^D - R_1$ spaces, by using λ^D-open set. The notion $\lambda^D - R_0$ and $\lambda^D - R_1$ spaces are introduced and some of their properties are investigated.

1 Introduction

In 1943, the notion of R_0 topological space was introduced by Shanin [6]. Later, Davis [3] rediscovered it and studied some properties of this weak separation axiom. In the same paper, Davis also introduced the notion of R_1 topological space which are independent of both T_0 and T_1, but strictly weaker than T_2. The notion of λ−open (λ^*−open) sets was introduced by Alais B. Khalaf and Sarhad F. Namiq [1]. The notion of λ^D-open sets was introduced by Sarhad F. Namiq [5]. In this paper, we continue the study of the above mentioned classes of topological spaces satisfying these axioms by introducing two more notions in terms of λ^D−open sets called $\lambda^D - R_0$ and $\lambda^D - R_1$.

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2 Preliminaries

Throughout, X denote a topological space. Let A be a subset of X, the closure and the interior of A are denoted by $Cl(A)$ and $Int(A)$ respectively. A subset A of a topological space (X, τ) is said to be dense set [7] if $Cl(A) = X$. A subset A of a topological space (X, τ) is said to be semi open [4] if $A \subseteq Cl(Int(A))$. The complement of a semi open set is said to be semi closed [4]. The family of all semi open (resp. semi closed) sets in a topological space (X, τ) is denoted by $SO(X, \tau)$ or $SO(X)$ (resp. $SC(X, \tau)$ or $SC(X)$). We consider λ as a function defined on $SO(X)$ into $\mathcal{P}(X)$ and $\lambda : SO(X) \to \mathcal{P}(X)$ is called an s-operation if $V \subseteq \lambda(V)$ for each non-empty semi open set V. It is assumed that $\lambda(\emptyset) = \emptyset$ and $\lambda(X) = X$ for any s-operation λ.

Definition 1 [1] *Let (X, τ) be a topological space and $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation, then a subset A of X is called a λ^*-open set which is equivalent to $\lambda-$open set, if for each $x \in A$, there exists a semi open set U such that $x \in U$ and $\lambda(U) \subseteq A$. The complement of a λ^*-open set is said to be λ^*-closed set which is equivalent to $\lambda-$closed set. The family of all λ^*-open (resp., λ^*-closed) subsets of a topological space (X, τ) is denoted by or $SO_\lambda(X)$ (resp. $SC_\lambda(X, \tau)$ or $SC_\lambda(X)$).*

Definition 2 [5] *Let (X, τ) be a topological space and $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation, then a λ^*-open subset A of X is called a λ^D-open set, if for each $x \in A$, there exists a dense set D such that $x \in D \subseteq A$. The complement of a λ^D-open set is said to be λ^D-closed. The family of all λ^D-open (resp., λ^D-closed) subsets of a topological space (X, τ) is denoted by or $SO_{\lambda^D})(X)$ or $SO_{\lambda^D}(X, \tau)$ (resp. $SC_{\lambda^D}(X, \tau)$ or $SC_{\lambda^D}(X)$).*

Example 1 *Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. The $SO(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}\}$. Define $\lambda : SO(X) \to \mathcal{P}(X)$ as:*

$$\lambda(A) = \begin{cases} A & \text{if } a \in A \\ X & \text{if } a \notin A \end{cases}$$

The $SO_{\lambda^D}(X) = \{\emptyset, X, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$.

Definition 3 [5] *Let (X, τ) be a topological space and let A be a subset of X. Then:*

1. *The λ-closure of A (denoted by $\lambda^D Cl(A)$) is the intersection of all λ^D- closed sets containing A.*

2. *The λ-interior of A (denoted by $\lambda^D Int(A)$) is the union of all λ^D-open sets of X contained in A.*

Proposition 1 [5] *For each point $x \in X$, $x \in \lambda^D Cl(A)$ if and only if $V \cap A \neq \emptyset$, for every $V \in SO_{\lambda^D}(X)$ such that $x \in V$.*

3 On $\lambda^D - R_0$ and $\lambda^D - R_1$ spaces

We introduce the following definitions.

Definition 4 *For any s-operation $\lambda : SO(X) \to \mathcal{P}(X)$ and any subset A of a space (X, τ) the λ^D-kernel of A, denoted by $\lambda^D \operatorname{Ker}(A)$ is defined as:*

$$\lambda^D \operatorname{Ker}(A) = \cap\{G \in SO_{\lambda^D}(X) : A \subseteq G\}.$$

Lemma 1 *Let (X, τ) be a topological space, $A \subseteq X$ and $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation. Then $\lambda^D \operatorname{Ker}(A) = \{x \in X : \lambda^D Cl(\{x\}) \cap A \neq \emptyset\}$.*

Proof. Let $x \in \lambda^D \operatorname{Ker}(A)$ such that $\lambda^D Cl(\{x\}) \cap A = \emptyset$. Since $x \notin X \setminus \lambda^D Cl(\{x\})$ which is a λ^D-open set containing A. Thus $x \notin \lambda^D \operatorname{Ker}(A)$ a contradiction.

Conversely, let $x \in X$ be such that $\lambda^D Cl(\{x\}) \cap A \neq \emptyset$. If possible, let $x \notin \lambda^D \operatorname{Ker}(A)$. Then there exist a λ^D-open set G such that $x \notin G$ and $A \subseteq G$. Let $y \in \lambda^D Cl(\{x\}) \cap A$. This implies that $y \in \lambda^D Cl(\{x\})$ and $y \in G$, which gives $x \in G$, a contradiction. □

Theorem 1 *Let (X, τ) be a topological space, A and B be subsets of X. Then:*

(1) $x \in \lambda^D \operatorname{Ker}(A)$ *if and only if* $A \cap F \neq \emptyset$; *for any λ^D-closed set F containing x.*

(2) $A \subseteq \lambda^D \operatorname{Ker}(A)$ *and* $A = \lambda^D \operatorname{Ker}(A)$ *if A is λ^D-open.*

(3) *If $A \subseteq B$, then $\lambda^D \operatorname{Ker}(A) \subseteq \lambda^D \operatorname{Ker}(B)$.*

Proof. Obvious. □

Definition 5 *Let $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation, a topological space (X, τ) is called $\lambda^D - R_0$, if $U \in SO_{\lambda^D}(X)$ and $x \in U$ then $\lambda^D Cl(\{x\}) \subseteq U$.*

Example 2 *Let $X = \{a, b, c, d\}$, and $\tau = \mathcal{P}(X)$. We define an s-operation $\lambda : SO(X) \to \mathcal{P}(X)$ as:*

$\lambda(A) = A$, *for every subset A of X.*

$SO(X) = \mathcal{P}(X) = SO_{\lambda^D}(X) = SC_{\lambda^D}(X)$.

Theorem 2 *For any topological space* X *and any s-operation* $\lambda : SO(X) \to \mathcal{P}(X)$*, the following statements are equivalent:*

(1) X *is* $\lambda^D - R_0$.

(2) $F \in SC_{\lambda^D}(X)$ *and* $x \notin F$ *implies that* $F \subseteq U$ *and* $x \notin U$ *for some* $U \in SO_{\lambda^D}(X)$.

(3) $F \in SC_{\lambda}(X)$ *and* $x \notin F$ *implies that* $F \cap \lambda^D Cl(\{x\}) \neq \emptyset$.

(4) *For any two distinct points* x, y *of* X, *either* $\lambda^D Cl(\{x\}) = \lambda^D Cl(\{y\})$ *or* $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\}) = \emptyset$.

Proof.
(1) $\Rightarrow$ (2): Let $F \in SC_{\lambda^D}(X)$ and $x \notin F$. This implies that $x \in X \backslash F \in SO_{\lambda^D}(X)$, then $\lambda^D Cl(\{x\}) \subseteq X \backslash F$ (by (1)). Put $U = X \backslash \lambda^D Cl(\{x\})$. Then $x \notin U \in SO_{\lambda^D}(X)$ and $F \subseteq U$.

(2) $\Rightarrow$ (3): $F \in SC_{\lambda^D}(X)$ and $x \notin F$ then there exists $U \in SO_{\lambda^D}(X)$ such that $x \notin U$ and $F \subseteq U$ (by(2)), then $U \cap \lambda^D Cl(\{x\}) = \emptyset$ and $F \cap \lambda^D Cl(\{x\}) = \emptyset$.

(3) $\Rightarrow$ (4): Suppose that for any two distinct points x, y of X, if $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$ Then, without loss of generality, we suppose that there exists some $z \in \lambda^D Cl(\{x\})$ such that $z \notin \lambda^D Cl(\{y\})$. Thus, there exists a λ^D-open set V such that $z \in V$ and $y \notin V$ but $x \in V$. Thus $x \notin \lambda^D Cl(\{y\})$. Hence by (3), $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\}) = \emptyset$.

(4) $\Rightarrow$ (1): Let $U \in SO_{\lambda^D}(X)$ and $x \in U$. Then for each $y \notin U$, $x \notin \lambda^D Cl(\{y\})$. Thus $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$. Hence by (4), $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\}) = \emptyset$, for each $y \in X \backslash U$. So $\lambda^D Cl(\{x\}) \cap [\cup \{\lambda^D Cl(\{y\}) : y \in X \backslash U\}] = \emptyset$. Now, $U \in SO_{\lambda^D}(X)$ and $y \in X \backslash U$ then $\{y\} \subseteq \lambda^D Cl(\{y\}) \subseteq \lambda^D Cl(X \backslash U) = X \backslash U$. Thus $X \backslash U = \cup \{\lambda^D Cl(\{y\}) : y \in X \backslash U\}$. Hence, $\lambda^D Cl(\{y\}) \cap X \backslash U = \emptyset$ then $\lambda^D Cl(\{x\}) \subseteq U$. This showing that (X, τ) is $\lambda^D - R_0$. □

Lemma 2 *Let* $\lambda : SO(X) \to \mathcal{P}(X)$ *be an s-operation. Then* $y \in \lambda^D \operatorname{Ker}(\{x\})$ *if and only if* $x \in \lambda^D Cl(\{y\})$.

Proof. Let $y \notin \lambda^D \operatorname{Ker}(\{x\})$. Then there exists $V \in SO_{\lambda^D}(X)$ containing x such that $y \notin V$. Therefore $x \notin \lambda^D Cl(\{y\})$. The converse part can be proved in a similar way. □

Theorem 3 *Let* $\lambda : SO(X) \to \mathcal{P}(X)$ *be an s-operation. Then for any two points* x, y *in* X, $\lambda^D \operatorname{Ker}(\{x\}) \neq \lambda^D \operatorname{Ker}(\{y\})$ *if and only if* $\lambda^D Cl(\{y\}) \neq \lambda^D Cl(\{x\})$.

Proof. Suppose that $\lambda^D\,\mathrm{Ker}(\{x\}) \neq \lambda^D\,\mathrm{Ker}(\{y\})$. Then there exists $z \in \lambda^D\,\mathrm{Ker}(\{x\})$ such that $z \notin \lambda^D\,\mathrm{Ker}(\{y\})$. Now, $z \in \lambda^D\,\mathrm{Ker}(x)$ if and only if $x \in \lambda^D\,\mathrm{Ker}(\{z\})$ by Lemma 2 and $z \notin \lambda^D\,\mathrm{Ker}(\{y\})$ if and only if $y \in \lambda^D Cl(\{x\})$ by Lemma 2. Hence $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$.

Conversely, suppose that $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$. Then there exists $z \in X$ such that $z \in \lambda^D Cl(\{x\})$ and $z \notin \lambda^D Cl(\{y\})$ so there exists $U \in SO_{\lambda^D}(X)$ such that $z \in U$, $y \notin U$ and $x \in U$. Then $y \notin \lambda^D\,\mathrm{Ker}(\{x\})$. Thus $\lambda^D\,\mathrm{Ker}(\{x\}) \neq \lambda^D\,\mathrm{Ker}(\{y\})$. □

Theorem 4 *Let $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation. Then (X, τ) is $\lambda^D - R_0$ if and only if for any two points $x, y \in X$, $\lambda^D\,\mathrm{Ker}(\{x\}) \notin \lambda^D\,\mathrm{Ker}(\{y\})$, implies that $\lambda^D\,\mathrm{Ker}(\{x\}) \cap \lambda^D\,\mathrm{Ker}(\{y\}) = \emptyset$.*

Proof. Let x, y be any two points in a $\lambda^D - R_0$ space X such that $\lambda^D\,\mathrm{Ker}(\{x\}) \neq \lambda^D\,\mathrm{Ker}(\{y\})$. Hence by Theorem 3, $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$. We show that $\lambda^D\,\mathrm{Ker}(\{x\}) \cap \lambda^D\,\mathrm{Ker}(\{y\}) = \emptyset$. In fact, if $z \in \lambda^D\,\mathrm{Ker}(\{x\}) \cap \lambda^D\,\mathrm{Ker}(\{y\})$, then by Lemma 2, we have $x, y \in \lambda^D Cl(z)$ and by Theorem 2, we obtain that $\lambda^D Cl(\{x\}) = \lambda^D Cl(\{z\}) = \lambda^D Cl(\{y\})$ which is impossible.

Conversely, suppose that for any points $x, y \in X$, $\lambda^D\,\mathrm{Ker}(\{x\}) \neq \lambda^D\,\mathrm{Ker}(\{y\})$. Thus $\lambda^D\,\mathrm{Ker}(\{x\}) \cap \lambda^D\,\mathrm{Ker}(\{y\}) = \emptyset$. Hence we get $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\}) = \emptyset$. In fact $z \in \lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\})$, this implies that $x, y \in \lambda^D\,\mathrm{Ker}(\{z\})$. Thus $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{z\}) \neq \emptyset$. Hence by hypothesis, we get $\lambda^D\,\mathrm{Ker}(\{x\}) = \lambda^D\,\mathrm{Ker}(\{z\})$. By similar way it follows that $\lambda^D\,\mathrm{Ker}(\{x\}) = \lambda^D\,\mathrm{Ker}(\{z\})$. Thus $\lambda^D\,\mathrm{Ker}(\{x\}) \neq \lambda^D\,\mathrm{Ker}(\{y\})$ which is a contradiction. Hence $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\}) \neq \emptyset$ and then by Theorem 2, the space X is $\lambda^D - R_0$. □

Theorem 5 *Let (X, τ) be a topological space and for any s-operation $\lambda : SO(X) \to \mathcal{P}(X)$ the following statements are equivalent:*

(1) *X is a $\lambda^D - R_0$ space.*

(2) *For any non-empty set A in X and any $G \in SO_{\lambda^D}(X)$ such that $A \cap G \neq \emptyset$ there exists $F \in SC_{\lambda^D}(X)$ such that $A \cap F \neq \emptyset$ and $F \subseteq G$.*

(3) *For any $G \in SO_{\lambda^D}(X)$, $G = \cup\{F \in SC_{\lambda^D}(X) : F \subseteq G\}$.*

(4) *For any $F \in SC_{\lambda^D}(X)$, $F = \cap\{G \in SO_{\lambda^D}(X) : F \subseteq G\}$.*

(5) *For any $x \in X$, $\lambda^D Cl(\{x\}) \subseteq \lambda^D\,\mathrm{Ker}(\{x\})$.*

Proof.

(1) $\Rightarrow$ (2): Let A be a non-empty subset of X and $G \in SO_{\lambda^D}(X)$ such that $A \cap G \neq \emptyset$. Let $x \in A \cap G$. Then as $x \in G \in SO_{\lambda^D}(X)$, by (1), we get $\lambda^D Cl(\{x\}) \subseteq G$. Put $F = \lambda^D Cl(\{x\})$. Then $F \in SC_{\lambda^D}(X)$, $F \subseteq G$ and $A \cap F \neq \emptyset$.

(2) $\Rightarrow$ (3): Let $G \in SO_{\lambda^D}(X)$. Then $\bigcup\{F \in SC_{\lambda^D}(X) : F \subseteq G\} \subseteq G$. Let $x \in G$. Then there exists $F \in SC_{\lambda^D}(X)$ such that $x \in F$ and $F \subseteq G$. Thus $x \in F \cup \{K \in SC_{\lambda^D}(X) : K \subseteq G\}$. Hence (3) follows.

(3) $\Rightarrow$ (4): Straight forward.

(4) $\Rightarrow$ (5): Let $x \in X$. Now, $y \notin \lambda^D \operatorname{Ker}(\{x\})$ implies there exists $V \in SO_{\lambda^D}(X)$ such that $x \in V$ and $y \notin V$ then $\lambda^D Cl(\{y\}) \cap V = \emptyset$. This implies by (4) $[\cap\{G \in SO_{\lambda^D}(X) : \lambda^D Cl(\{y\}) \subseteq G\}] \cap V = \emptyset$. Then there exists $G \in SO_{\lambda^D}(X)$ such that $x \in G$ and $\lambda^D Cl(\{y\}) \subseteq G$, so $y \notin \lambda^D Cl(\{x\})$.

(5) $\Rightarrow$ (1): Let $G \in SO_{\lambda^D}(X)$ and $x \in G$. Let $y \in \lambda^D \operatorname{Ker}(\{x\})$. Then $x \in \lambda^D Cl(\{y\})$ and hence $y \in G$. This implies that $\lambda^D \operatorname{Ker}(\{x\}) \subseteq G$. Thus $x \in \lambda^D Cl(\{x\}) \subseteq \lambda^D \operatorname{Ker}(\{x\}) \subseteq G$. Hence X is $\lambda^D - R_0$. □

Corollary 1 *Let $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation. Then X is $\lambda^D - R_0$ if and only if $\lambda^D Cl(\{x\}) = \lambda^D \operatorname{Ker}(\{x\})$, for all $x \in X$.*

Proof. Suppose that X is $\lambda^D - R_0$. By Theorem 5, $\lambda^D Cl(\{x\}) \subseteq \lambda^D \operatorname{Ker}(\{x\})$. For each $x \in X$. Let $y \in \lambda^D \operatorname{Ker}(\{x\})$. Then $x \in \lambda^D Cl(\{y\})$ (by Lemma 2), and hence by Theorem 2, $\lambda^D Cl(\{x\}) = \lambda^D Cl(\{y\})$. Thus $y \in \lambda^D Cl(\{x\})$ and hence $\lambda^D \operatorname{Ker}(\{x\}) \subseteq \lambda^D Cl(\{x\})$. Thus $\lambda^D Cl(\{x\}) = \lambda^D \operatorname{Ker}(\{x\})$. □

The converse is obvious in view of Theorem 5.

Theorem 6 *Let (X, τ) be a topological space and $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation. A space X is $\lambda^D - R_0$ if and only if for any $x, y \in X$, whenever $x \in \lambda^D Cl(\{y\})$ implies $y \in \lambda^D Cl(\{x\})$ and conversely.*

Proof. Suppose that a topological space (X, τ) is $\lambda^D - R_0$. Let $x \in \lambda^D Cl(\{y\})$. Then by Theorem 5, we have $\lambda^D Cl(\{y\}) \subseteq \lambda^D \operatorname{Ker}(\{x\})$. Thus $x \in \lambda^D \operatorname{Ker}(\{y\})$. Hence by Lemma 1, we have $y \in \lambda^D Cl(\{x\})$.

Conversely, let $U \in SO_{\lambda^D}(X)$ and $x \in U$. Let $y \in \lambda^D Cl(\{x\})$ hence by hypothesis, $x \in \lambda^D Cl(\{y\})$. Since $x \in U$, so $y \in U$. Hence $\lambda^D Cl(\{x\}) \subseteq U$. Thus X is $\lambda^D - R_0$. □

Theorem 7 *Let X be a topological space and $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation. Then the following statements are equivalent:*

(1) X *is* $\lambda^D - R_0$.

(2) *If* $F \in SC_{\lambda^D}(X)$ *then* $F = \lambda^D \operatorname{Ker}(F)$.

(3) *If* $F \in SC_{\lambda^D}(X)$ *and* $x \in F$, *then* $\lambda^D \operatorname{Ker}(\{x\}) \subseteq F$.

(4) *If* $x \in X$, *then* $\lambda^D \operatorname{Ker}(\{x\}) \subseteq \lambda^D Cl(\{x\})$.

Proof.

(1) $\Rightarrow$ (2): Follows from Theorem 5.

(2) $\Rightarrow$ (3): Follows from the fact that $x \in F$ then $\lambda^D \operatorname{Ker}(\{x\}) \subseteq \lambda^D \operatorname{Ker}(F) = F$ by part 3 of Theorem 1.

(3) $\Rightarrow$ (4): Since $x \in \lambda^D Cl(\{x\}) \in SC_{\lambda^D}(X)$ we have by (3), $\lambda^D \operatorname{Ker}(\{x\}) \subseteq \lambda^D Cl(\{x\})$ and (4) follows.

(4) $\Rightarrow$ (1): Let $U \in SO_{\lambda^D}(X)$ and $x \in U$. To show $\lambda^D Cl(\{x\}) \subseteq U$. If possible, suppose that, there exists $y \in \lambda^D Cl(\{x\})$ such that $y \notin U$. Then $y \in X \backslash U$. This by (4) implies that $\lambda^D \operatorname{Ker}(\{y\}) \subseteq X \backslash U$. Therefore $U \subseteq X \backslash \lambda^D \operatorname{Ker}(\{x\})$. So $x \notin \lambda^D \operatorname{Ker}(\{y\})$. Then, there exists a λ^D-open set G such that $y \in G$ but $x \notin G$. This implies that $y \notin \lambda^D Cl(\{x\})$ which is impossible. Hence $\lambda^D Cl(\{x\}) \sqsubseteq U$. Thus X is a $\lambda^D - R_0$ space. □

Definition 6 *Let* (X, τ) *be a topological space and* $\lambda : SO(X) \to \mathcal{P}(X)$ *be an s-operation. The space* X *is said to be* $\lambda^D - R_1$ *if for* $x, y \in X$ *with* $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(y)$ *there exist disjoint* λ^D*-open sets* U *and* V *such that* $\lambda^D Cl(\{x\}) \subseteq U$ *and* $\lambda^D Cl(\{y\}) \subseteq V$.

Remark 1 *A space* X *in Example 2 is* $\lambda^D - R_1$.

Theorem 8 *Every* $\lambda^D - R_1$ *space is a* $\lambda^D - R_0$ *space.*

Proof. Let $U \in SO_{\lambda^D}(X)$ and $x \in U$. If $y \notin U$ then $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$ (as $x \notin \lambda^D Cl(\{y\})$). Hence there exists $V \in SO_{\lambda^D}(X)$ such that $\lambda^D Cl(\{y\}) \subseteq V$ and $x \notin V$. This gives $y \notin \lambda^D Cl(\{y\})$, proving that $\lambda^D Cl(\{x\}) \subseteq U$. So X is a $\lambda^D - R_0$ space. □

The converse of Theorem 8 is not true, we can show it by the following example:

Example 3 *Let* $X = \{a, b, c, d\}$, *and* $\tau = \mathcal{P}(X)$. *We define an s-operation* $\lambda : SO(X) \to \mathcal{P}(X)$ *as:*

$$\lambda(A) = \begin{cases} X & \text{Otherwise} \\ A & \text{if } A = \emptyset \text{ or} \{b, c\} \text{ or } \{a, c\} \text{ or } \{a, b\}. \end{cases}$$

Now:

$SO(X) = \mathcal{P}(X)$.

$SO_{\lambda^D}(X) = \{\emptyset, \{b, c\}, \{a, c\}, \{a, b\}, X\}$.

$SC_{\lambda^D}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, X\}$.

Clearly X *is* $\lambda^D - R_0$ *but it is not* $\lambda^D - R_1$.

Theorem 9 *Let* (X, τ) *be a topological space and* $\lambda : SO(X) \to \mathcal{P}(X)$ *be an s-operation. Then the following statements are equivalent:*

(1) X *is* $\lambda^D - R_1$.

(2) *For any* $x, y \in X$*, one of the following holds:*

a) For $U \in SO_\lambda(X)$*,* $x \in U$ *if and only if* $y \in U$*;*

b) There exist disjoint λ^D*-open sets* U *and* V *such that* $x \in U$*,* $y \in V$*.*

(3) *If* $x, y \in X$*, such that* $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$ *then there exist* λ^D*-closed sets* F *and* H *such that* $x \in F$*,* $y \notin F$*,* $y \in H$*,* $x \notin H$ *and* $X = F \cup H$*.*

Proof.

(1) $\Rightarrow$ (2): Let $x, y \in X$. Then $\lambda^D Cl(\{x\}) = \lambda^D Cl(\{y\})$ or $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$. If $\lambda^D Cl(\{x\}) = \lambda^D Cl(\{y\})$ and $U \in SO_{\lambda^D}(X)$, then for any $U \in SO_{\lambda^D}(X)$, $x \in U$ then $y \in \lambda^D Cl(\{x\}) = \lambda^D Cl(\{y\}) \subseteq U$ then (as X is $\lambda^D - R_0$). If $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$, then there exist $U, V \in SO_{\lambda^D}(X)$ such that $x \in \lambda^D Cl(\{x\}) \subseteq U$, $y \in \lambda^D Cl(\{y\}) \subseteq V$ and $U \cap V = \emptyset$.

(2) $\Rightarrow$ (3): Let $x, y \in X$ such that $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$. Then $x \notin \lambda^D Cl(\{y\})$, so that there exists $G \in SO_{\lambda^D}(X)$, such that $x \in G$ and $y \notin G$. Thus by (2), there exist disjoint λ^D-open sets U and V such that $x \in U$, $y \in V$. Put $F = X \backslash V$ and $H = X \backslash U$. Then $F, H \in SO_{\lambda^D}(X)$, $x \in F$, $y \notin F$, $y \in H$, $x \notin H$ and $X = F \cup H$.

(3) $\Rightarrow$ (1): Let $U \in SO_\lambda(X)$ and $x \in U$. Then $\lambda^D Cl(\{x\}) \subseteq U$. In fact, otherwise there exists $y \in \lambda^D Cl(\{x\}) \cap X \backslash U$. Implies that $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$ (as $x \notin \lambda^D Cl(\{y\})$) and so by (3), there exist $F, H \in SO_{\lambda^D}(X)$ such that $x \in F$, $y \notin F$, $y \in H$, $x \notin H$ and $X = F \cup H$. Then $y \in H \backslash F = X \backslash F$ and $x \notin X \backslash F$, where $X \backslash F \in SO_{\lambda^D}(X)$, which is a contradiction to the fact that $y \in \lambda^D Cl(\{x\})$. Hence $\lambda^D Cl(\{x\}) \subseteq U$. Thus X is $\lambda^D - R_0$. To show X to be $\lambda^D - R_1$. Assume that $a, b \in X$ with $\lambda^D Cl(\{a\}) \neq \lambda^D Cl(\{b\})$. Then as above, there exist $K, L \in SC_{\lambda^D}(X)$ such that $a \in K$, $b \notin K$, $b \in L$, $a \notin L$ and $X = K \cup L$. Thus $a \in K \backslash L \in SO_{\lambda^D}(X)$, $b \in L \backslash K \in SO_{\lambda^D}(X)$. So $\lambda^D Cl(\{a\}) \subseteq K \backslash L$, $\lambda^D Cl(\{b\}) \subseteq L \backslash K$. Thus X is $\lambda^D - R_1$. □

Proposition 2 *Let* (X, τ) *be a topological space and* $\lambda : SO(X) \to \mathcal{P}(X)$ *be an s-operation. Then* X *is* $\lambda^D - R_1$*, if and only if for* $x, y \in X$*, with* $\lambda^D \operatorname{Ker}(\{x\}) \neq \lambda^D \operatorname{Ker}(\{y\})$ *there exist disjoint* λ^D*-open sets* U *and* V *such that* $\lambda^D Cl(\{x\}) \subseteq U$ *and* $\lambda^D Cl(\{y\}) \subseteq V$.

Proof. Follows from Theorem 3 and Definition 6. □

4 Conclusion

Introduced by Alais B. Khalaf and Sarhad F. Namiq [1]. The main results are the following:

(1) Let (X, τ) be a topological space, and $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation and $A \subseteq X$. Then $\lambda^D \operatorname{Ker}(\{A\}) = \{x \in X : \lambda^D Cl(\{x\}) \cap A \neq \emptyset\}$.

(2) For any topological space X and any s-operation $\lambda : SO(X) \to \mathcal{P}(X)$, the following statements are equivalent:

a) X is $\lambda^D - R_0$.

b) $F \in SC_{\lambda^D}(X)$ and $x \in F$ implies that $F \subseteq U$ and $x \in U$ for some $U \in SO_{\lambda^D}(X)$.

c) $F \in SC_{\lambda^D}(X)$ and $x \notin F$ implies that $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\}) = \emptyset$.

d) For any two distinct points x, y of X, either $\lambda^D Cl(\{x\}) = \lambda^D Cl(\{y\})$ or $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\}) = \emptyset$.

(3) Every $\lambda^D - R_1$ space is a $\lambda^D - R_0$ space.

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