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On graphs with minimal distance signless Laplacian energy

S. Pirzada
Department of Mathematics,
University of Kashmir, Srinagar, India
email:
pirzadasd@kashmiruniversity.ac.in

Bilal A. Rather
Department of Mathematics,
University of Kashmir, Srinagar, India
email: bilalahmadrr@gmail.com

Rezwan Ul Shaban
Department of Mathematics,
University of Kashmir, Srinagar, India
email: rezwanbhat21@gmail.com

Merajuddin
Department of Applied Mathematics,
Aligarh Muslim University,
Aligarh, India
email: meraj1957@rediffmail.com

Abstract. For a simple connected graph G of order n having distance signless Laplacian eigenvalues $\rho_1^Q \geq \rho_2^Q \geq \cdots \geq \rho_n^Q$, the distance signless Laplacian energy $DSLE(G)$ is defined as $DSLE(G) = \sum_{i=1}^{n} \left| \rho_i^Q - \frac{2W(G)}{n} \right|$, where $W(G)$ is the Weiner index of G. We show that the complete split graph has the minimum distance signless Laplacian energy among all connected graphs with given independence number. Further, we prove that the graph $K_k \vee (K_t \cup K_{n-k-t})$, $1 \leq t \leq \lfloor \frac{n-k}{2} \rfloor$ has the minimum distance signless Laplacian energy among all connected graphs with vertex connectivity k.

1 Introduction

A simple and finite graph is denoted by $G(V(G), E(G))$ (or simply by G when there is no confusion), where $V(G) = \{v_1, v_2, \ldots, v_n\}$ is its vertex set and $E(G)$

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is its edge set. The cardinality of $V(G)$ and $E(G)$ are respectively the *order* and *size* of G, and are denoted by n and m. The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to $v \in V(G)$, and its cardinality is the *degree* of v, denoted by $d_G(v)$ (we simply write d_v if it is clear from the context). Throughout this paper, G will be connected. The adjacency matrix $A = [a_{ij}]$ of G is a $(0,1)$-square matrix of order n whose (i,j)-entry is equal to 1, if v_i is adjacent to v_j and equal to 0, otherwise. The diagonal matrix of vertex degrees $d_i = d_G(v_i)$, $i = 1, 2, \ldots, n$ associated to G is $Deg(G) = diag[d_1, d_2, \ldots, d_n]$. The real symmetric and positive semi-definite matrices $L(G) = Deg(G) - A(G)$ and $Q(G) = Deg(G) + A(G)$ are respectively the Laplacian and the signless Laplacian matrices and their spectrum are respectively the Laplacian spectrum and signless Laplacian spectrum of the graph G. Recent work on signless Laplacian spectrum can be seen in [11, 20, 21, 22]. We use standard terminology, K_n denotes a complete graph, $K_{a,n-a}$ is a complete bipartite graph with partite sets of cardinality a and $n-a$. For other undefined notations and terminology, the readers are referred to [5, 13, 15, 16, 23].

In a connected graph G, the *distance* between two vertices $v_1, v_2 \in V(G)$, denoted by $d(v_1, v_2)$, is the length of a shortest path between v_1 and v_2. The *diameter* of G is the maximum distance between any two pair of vertices of G. The *distance matrix* of G, denoted by $D(G)$, is defined as $D(G) = [d(v_i, v_j)]$ where $v_i, v_j \in V(G)$. The *transmission* $Tr_G(v)$ (or simply by $Tr(v)$, when graph under consideration is clear) of a vertex v is defined to be the sum of the distances from v to all other vertices in G, that is, $Tr(v) = \sum\limits_{u \in V(G)} d_{uv}$. The *transmission* number or Wiener index of a graph G, denoted by $W(G)$, is the sum of distances between all unordered pairs of vertices in G. Clearly, $W(G) = \frac{1}{2} \sum\limits_{v \in V(G)} Tr(v)$. For any vertex $v_i \in V(G)$, the transmission $Tr(v_i)$ is called the *transmission degree*, shortly denoted by Tr_i and the sequence $\{Tr_1, Tr_2, \ldots, Tr_n\}$ is called the *transmission degree sequence* of the graph G.

If $Tr(G) = diag[Tr_1, Tr_2, \ldots, Tr_n]$ is the diagonal matrix of vertex transmissions of G, the matrices $D^L(G) = Tr(G) - D(G)$ and $D^Q(G) = Tr(G) + D(G)$ are respectively called as the *distance Laplacian matrix* and the *distance signless Laplacian matrix* of G [3].

If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the adjacency eigenvalues of a graph G, the *energy* of G [12], denoted by $E(G)$, is defined as $E(G) = \sum\limits_{i=1}^{n} |\lambda_i|$. The reader is referred to the book [15] and for some recent work to [4, 9, 10].

Let $\rho_1^D \geq \rho_2^D \geq \ldots \geq \rho_n^D$ and $\rho_1^Q \geq \rho_2^Q \geq \ldots \geq \rho_n^Q$ be respectively, the distance, and distance signless Laplacian eigenvalues of the graph G. The distance energy [14] of a graph G is the sum of the absolute values of the distance eigenvalues of G, that is, $DE(G) = \sum_{i=1}^{n} |\rho_i^D|$. For some recent works on distance energy, we refer to [2, 6, 8, 18] and the references therein. The distance signless Laplacian energy $DSLE(G)$ [6] of a connected graph G is defined as

$$DSLE(G) = \sum_{i=1}^{n} |\rho_i^Q - \frac{2W(G)}{n}|.$$

Let σ' be the largest positive integer such that $\rho_{\sigma'}^Q \geq \frac{2W(G)}{n}$ and let $B_b^Q(G) = \sum_{i=1}^{b} \rho_i^Q$ be the sum of b largest distance signless Laplacian eigenvalues of G. Then, using $\sum_{i=1}^{n} \rho_i^Q = 2W(G)$, in [6], it is shown that

$$\begin{aligned} DSLE(G) &= 2\left(B_{\sigma'}^Q(G) - \frac{2\sigma' W(G)}{n}\right) = 2 \max_{1\leq j\leq n}\left(\sum_{i=1}^{j} \rho_i^Q(G) - \frac{2jW(G)}{n}\right) \\ &= 2 \max_{1\leq j\leq n}\left(B_j^Q(G) - \frac{2jW(G)}{n}\right). \end{aligned}$$

For some recent works on $DSLE(G)$, see [6, 8, 19].

In the next section, we show that the complete split graph has the minimum distance signless Laplacian energy among all connected graphs with given independence number. Further, we show that among all connected graphs with given vertex connectivity k, the graph $K_k \vee (K_t \cup K_{n-k-t})$, $1 \leq t \leq \lfloor \frac{n-k}{2} \rfloor$ has the minimum distance signless Laplacian energy.

2 Distance signless Laplacian energy of graphs with given independence number and connectivity

Let $e = v_i v_j$ be an edge of a graph G such that $G - e$ is connected. Then removing the edge e increases the distance by at least one unit. Similarly adding an edge decreases the distance by at least one unit. By Perron-Frobenius theorem, if each entry of the first non negative matrix majorizes the second non negative matrix, then their spectrum is also majorized. This is summarized in next useful result, which can be found in [3].

Lemma 1 *Let* G *be a connected graph of order* n *and size* m*, where* $m \geq n$ *and let* $G' = G - e$ *be a connected graph obtained from* G *by deleting an edge. Let* $\rho_1^Q(G) \geq \rho_2^Q(G) \geq \cdots \geq \rho_n^Q(G)$ *and* $\rho_1^Q(G') \geq \rho_2^Q(G') \geq \cdots \geq \rho_n^Q(G')$ *be respectively, the distance signless Laplacian eigenvalues of* G *and* G'*. Then* $\rho_i^Q(G') \geq \rho_i^Q(G)$ *holds for all* $1 \leq i \leq n$.

Motivated by Lemma 1, we have the following observation, which says that the complete graph has minimum distance signless Laplacian energy among all graphs of order n.

Theorem 1 *Let* G *be a connected graph of order* n*. Then*

$$DSLE(G) \geq 2\left(n + b(n-2) - \frac{2W(G)}{n}\right),$$

equality occurs if and only if $G \cong K_n$.

Proof. By Lemma 1, $\rho_i^Q(G) \geq \rho_i^Q(K_n)$ for each $i = 1, 2, \ldots, n$. So using the definition of $B_b^Q(G)$, we have

$$B_b^Q(G) \geq B_b^Q(K_n) = 2n - 2 + (b-1)(n-2), \tag{1}$$

with equality if and only if $G \cong K_n$. Let σ' be the positive integer such that $\rho_{\sigma'}^Q \geq \frac{2W(G)}{n}$. Then using (1) and the definition of distance signless Laplacian energy, we have

$$\begin{aligned} DSLE(G) &= 2\left(\sum_{i=1}^{\sigma'} \rho_i^Q(G) - \frac{2\sigma' W(G)}{n}\right) = 2 \max_{1 \leq j \leq n}\left(\sum_{i=1}^{j} \rho_i^Q(G) - \frac{2jW(G)}{n}\right) \\ &\geq 2 \max_{1 \leq j \leq n}\left(\sum_{i=1}^{j} \rho_i^Q(K_n) - \frac{2jW(G)}{n}\right) = 2\left(n + b(n-2) - \frac{2(b-1)W(G)}{n}\right). \end{aligned}$$

By Lemma 1 and Inequality (1), equality holds if and only if $G \cong K_n$. □

A graph is complete split, denoted by $CS_{n,\alpha}$, if it can be partitioned into an independent set (a subset of vertices of a graph is said to be an independent set if the subgraph induced by them is an empty graph) on α vertices and a clique on $n - \alpha$ vertices, such that each vertex of the independent set is adjacent to every vertex of the clique.

The following result [17] gives the distance signless Laplacian spectrum of $CS_{n,\alpha}$.

Lemma 2 *Let $CS_{n,\alpha}$ be the complete split graph with independence number α. Then the distance signless Laplacian spectrum of $CS_{n,\alpha}$ is given by* $\left\{\frac{3n+2\alpha-6\pm\sqrt{n^2+12\alpha^2-\alpha(4n+16)+4n+4}}{2}, (n+\alpha-4)^{[\alpha-1]}, (n-2)^{[n-\alpha-1]}\right\}$.

Since independence number of the complete graph K_n is 1 and its distance signless Laplacian energy is discussed in Theorem 1, so we assume $2 \le \alpha \le n-2$, and discuss $\alpha = n-1$ separately. The following theorem shows that among all connected graphs with given independence number α, the complete split graph $CS_{n,\alpha}$ has the minimum distance signless Laplacian energy.

Theorem 2 *Let G be a connected graph of order $n \ge 3$ having independence number α, where $\frac{n+1-\sqrt{n^2+1-10n}}{2} < \alpha < \frac{n+1+\sqrt{n^2+1-10n}}{2}$. Then*

$$DSLE(G) \ge \begin{cases} 2\Big(2n + \alpha(n-3) + \alpha^2 - 2 - \frac{2(\alpha+1)W(G)}{n}\Big), & \text{if } \alpha \le \frac{n}{2}, \\ n + \sqrt{\theta} + \alpha(2n-8) + 2\alpha^2 + 2 - \frac{4\alpha W(G)}{n}, & \text{if } \alpha > \frac{n}{2}, \end{cases}$$

where $\theta = n^2 + 12\alpha^2 + 4n - \alpha(4n+16) + 4$. Equality occurs in each of the inequalities if and only if $G \cong CS_{n,\alpha}$.

Proof. Let G be a connected graph of order $n \ge 3$ having independence number α. Let $CS_{n,\alpha}$ be the complete split graph having independence number α. Clearly, G is a spanning subgraph of $CS_{n,\alpha}$. Therefore, by Lemma 1, we have $\rho_i^Q(G) \ge \rho_i^Q(CS_{n,\alpha})$. Let σ' be the largest positive integer such that $\rho_{\sigma'}^Q(G) \ge \frac{2W(G)}{n}$. With this information, and using the equivalent definition of distance signless Laplacian energy, we have

$$\begin{aligned} DSLE(G) &= 2\left(\sum_{i=1}^{\sigma'} \rho_i^Q(G) - \frac{2\sigma' W(G)}{n}\right) = 2\max_{1\le j\le n}\left(\sum_{i=1}^{j} \rho_i^Q(G) - \frac{2jW(G)}{n}\right) \\ &\ge 2\max_{1\le j\le n}\left(\sum_{i=1}^{j} \rho_i^Q(CS_{n,\alpha}) - \frac{2jW(G)}{n}\right). \end{aligned} \tag{2}$$

By using Lemma 2, the trace is $n^2+\alpha^2-n-\alpha$ and the average Wiener index is $\frac{2W(CS_{n,\alpha})}{n} = \frac{n^2+\alpha^2-n-\alpha}{n}$. Therefore, it follows that $\frac{3n+2\alpha-6+\sqrt{n^2+12\alpha^2-\alpha(4n+16)+4n+4}}{2}$ is the distance signless Laplacian spectral radius of $CS_{n,\alpha}$. Next, for the eigenvalue $n+\alpha-4$, we have

$$n + \alpha - 4 < \frac{2W(CS_{n,\alpha})}{n} = \frac{n^2+\alpha^2-n-\alpha}{n},$$

provided

$$\alpha^2 - (n+1)\alpha + 3n > 0. \tag{3}$$

Consider the polynomial $f(t) = t^2 - (n+1)t + 3n$, for $1 \leq t \leq n-1$. The zeros of this polynomial are

$$x_1 = \frac{n+1-\sqrt{n^2+1-10n}}{2} \text{ and } x_2 = \frac{n+1+\sqrt{n^2+1-10n}}{2}.$$

This implies that $f(t) > 0$ for all $t < x_1$ and $f(t) > 0$ for all $t > x_2$. From this, for

$$\frac{n+1-\sqrt{n^2+1-10n}}{2} < \alpha < \frac{n+1+\sqrt{n^2+1-10n}}{2},$$

we have $n + \alpha - 4 \geq \frac{2W(CS_{n,\alpha})}{n}$. Similarly, for the second smallest distance signless Laplacian eigenvalue, we have

$$\frac{3n+2\alpha-6-\sqrt{n^2+12\alpha^2-\alpha(4n+16)+4n+4}}{2} < \frac{2W(CS_{n,\alpha})}{n},$$

which after simplification implies that

$$(12-8\alpha)n^3+(-12-4\alpha+12\alpha^2)n^2+(16\alpha-24\alpha^2+8\alpha^3)n+8\alpha^3-4\alpha^2-4\alpha^4 > 0. \tag{4}$$

Inequality (4) is a function of two variables, and putting conditions on the independence number α we have verified that (4) holds true for $\alpha \leq \frac{n}{2}$. Also, the smallest distance signless Laplacian eigenvalue $n-2$ is always less than $\frac{2W(CS_{n,\alpha})}{n}$. Therefore, we have the following cases to consider.

Case (i). If $\alpha \leq \frac{n}{2}$, then $\sigma' = \alpha$. Thus, from (2), it follows that

$$\begin{aligned} DSLE(G) &\geq 2 \max_{1\leq j\leq n} \left(\sum_{i=1}^{j} \rho_i^Q(CS_{n,\alpha}) - \frac{2jW(G)}{n} \right) \\ &\geq 2 \left(\sum_{i=1}^{\alpha} \rho_i^Q(CS_{n,\alpha}) - \frac{2\alpha W(G)}{n} \right) \\ &= 2 \left(\frac{3n+2\alpha-6+\sqrt{\theta}}{2} + (\alpha-1)(n+\alpha-4) - \frac{2\alpha W(G)}{n} \right) \\ &= n + \alpha(2n-8) + 2\alpha^2 + 2 + \sqrt{\theta} - \frac{4\alpha W(G)}{n}, \end{aligned}$$

where $\theta = n^2 + 12\alpha^2 + 4n - \alpha(4n+16) + 4$.
Case (ii). If $\alpha > \frac{n}{2}$, then $\sigma' = \alpha + 1$. So, from (2), it follows that

$$\begin{aligned}
DSLE(G) &\geq 2 \max_{1\leq j\leq n} \left(\sum_{i=1}^{j} \rho_i^Q(CS_{n,\alpha}) - \frac{2jW(G)}{n} \right) \\
&\geq 2 \left(\sum_{i=1}^{\alpha+1} \rho_i^Q(CS_{n,\alpha}) - \frac{2(\alpha+1)W(G)}{n} \right) \\
&= 2 \left(3n + 2\alpha - 6 + (\alpha-1)(n+\alpha-4) - \frac{2(\alpha+1)W(G)}{n} \right) \\
&= 2 \left(2n + \alpha(n-3) + \alpha^2 - 2 - \frac{2(\alpha+1)W(G)}{n} \right).
\end{aligned}$$

Equality occurs in all the inequalities above if and only if equality occurs in Inequality (2). It is clear that equality occurs in (2) if and only if $G \cong CS_{n,\alpha}$. This shows that equality occurs in all the inequalities above if and only if $G \cong CS_{n,\alpha}$. This completes the proof. □

When order n of graph G increases, we observe that $\frac{n+1-\sqrt{n^2+1-10n}}{2} \approx 3$ and $\frac{n+1+\sqrt{n^2+1-10n}}{2} \approx n-2$. These remaining cases of independence are discussed as follows.

Proposition 1 *Let G be a graph of order $n \geq 3$ with independence number $\alpha \in \{2,3\}$. Then*

$$DSLE(G) \geq \begin{cases} 2\Big(3n - 2 - \frac{4W(G)}{n}\Big), & \text{if } \alpha = 2, \\ 2\left(3n - \frac{4W(G)}{n}\right), & \text{if } \alpha = 3, \end{cases}$$

equality occurs in first and second inequality if and only $G \cong CS_{n,2}$ and $CS_{n,3}$ respectively.

Proof. By substituting $\alpha = 2$ in Lemma 2, the distance signless Laplacian spectrum of $CS_{n,2}$ is given by $\left\{\frac{1}{2}(3n-2 \pm \sqrt{n^2-4n+20}), (n-2)^{[n-2]}\right\}$ and the Wiener index can be calculated to be $\frac{2W(G)}{n} = \frac{n^2-n+2}{n}$. Clearly, $\frac{1}{2}(3n - 2 + \sqrt{n^2-4n+20})$ is the spectral radius and it is always greater or equal to Wiener index. Next $\frac{1}{2}(3n-2-\sqrt{n^2-4n+20}) < \frac{n^2-n+2}{n}$ implies that $n^2(n^2-4n+20) - 16 > 0$ which is true for each $n \geq 1$. Also, the smallest distance signless Laplacian eigenvalue is always strictly less than $\frac{2W(G)}{2}$. Thus, we have $\sigma' = 2$ and the distance signless Laplacian energy is given by

$$\mathrm{DSLE}(G) \geq 2\left(\sum_{i=1}^{2} \rho_i^Q(G) - \frac{4W(G)}{n}\right) = 2\left(3n - 2 - \frac{4W(G)}{2}\right). \quad (5)$$

By using similar arguments, we can easily prove the second inequality. Equality holds as in Theorem 2. □

Now, we obtain a lower bound for the distance signless Laplacian energy when independence number is $\alpha = n-2,$ or $n-1$.

Proposition 2 *Let G be a graph of order* $n \geq 6$ *with independence number* $\alpha \in \{n-2, n-1\}$. *Then*

$$\mathrm{DSLE}(G) \geq \begin{cases} n(4n-19) + \sqrt{9n^2 - 52n + 84} + 26 - \frac{4W(G)}{n}, & \text{if } \alpha = n-2, \\ 5n + \sqrt{9n^2 - 32n + 32} - 8 - \frac{4W(G)}{n}, & \text{if } \alpha = n-1, \end{cases}$$

equality occurs in first and second inequality if and only $G \cong CS_{n,n-2}$ *and* $CS_{n,n-1}$ *respectively.*

Proof. From Lemma 2, the distance signless Laplacian spectrum of $CS_{n,n-2}$ with independence number $n-2$ is given by

$$\left\{\frac{1}{2}(5n - 10 \pm \sqrt{9n^2 - 52n + 84}), (2n-6)^{[n-3]}, n-2\right\}$$

and Wiener index is $\frac{2W(G)}{n} = \frac{2n^2-6n+6}{n}$. Now, it is clear that $\frac{1}{2}(5n - 10 + \sqrt{9n^2 - 52n + 84})$ is the spectral radius and is always greater or equal to $\frac{2W(G)}{n}$. Also, $2n-6 < \frac{2W(G)}{n}$ implies that $6 > 0$, which is always true. For the second smallest distance signless Laplacian eigenvalue $\frac{1}{2}(5n-10-\sqrt{9n^2-52n+84})$, we have $\frac{1}{2}(5n - 10 - \sqrt{9n^2 - 52n + 84}) < \frac{n^2-n+2}{n}$, which implies that $n^4 - 7n^3 + 13n^2 + 6n - 18 > 0$, and is true for each $n \geq 2$. Also, the smallest distance signless Laplacian eigenvalue is always strictly less than $\frac{2W(G)}{2}$. Thus, we have $\sigma' = n-2$ and the distance Laplacian energy is given by

$$\mathrm{DSLE}(G) \geq 2\left(\sum_{i=1}^{n-2} \rho_i^Q(G) - \frac{2(n-2)W(G)}{n}\right)$$
$$= 2\left(\frac{5n - 10 + \sqrt{9n^2 - 52n + 84}}{2} + (n-3)(2n-6) - \frac{2(n-2)W(G)}{2}\right)$$

$$=n(4n-19)+26+\sqrt{9n^2-52n+84}-\frac{4(n-2)W(G)}{2}$$

By using similar arguments, we can easily prove the second inequality. By Lemma 1, equality holds as in Theorem 2. □

The vertex connectivity of a graph G, denoted by $\kappa(G)$, is the minimum number of vertices of G whose deletion disconnects G. Let $\mathcal{F}_n$ be the family of simple connected graphs on n vertices and let

$$\mathcal{V}_n^k=\{G\in\mathcal{F}_n:\kappa(G)\le k\},$$

that is, $\mathcal{V}_n^k$ is the family of graphs with vertex connectivity at most k.

Let $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ be two graphs on disjoint vertex sets V_1 and V_2 with orders n_1 and n_2, respectively. Then their *union* is the graph $G_1\cup G_2=(V_1\cup V_2,E_1\cup E_2)$. The *join* of graphs G_1 and G_2, denoted by $G_1\vee G_2$, is the graph consisting of $G_1\cup G_2$ and all edges joining the vertices in V_1 and the vertices in V_2. In other words, the join of two graphs G_1 and G_2, denoted by $G_1\vee G_2$, is the graph obtained from G_1 and G_2 by joining each vertex of G_1 to every vertex of G_2.

The following result [17] gives the distance signless Laplacian spectrum of the join of a connected graph G_1 with the union of two connected graphs G_2 and G_3, in terms of the adjacency spectrum of the graphs G_1,G_2 and G_3.

Theorem 3 *Let G_i be r_i regular graphs of orders n_i, having adjacency eigenvalues $\lambda_{i,1}=r_i\ge\lambda_{i,2}\ge\cdots\ge\lambda_{i,n_i}$, for $i=1,2,3$. Then the distance signless Laplacian eigenvalues of $G_1\vee(G_2\cup G_3)$ are $(n+n_1-r_1-\lambda_{1,k}-4)^{[n_1-1]}$, $(2n-n_1-r_2-\lambda_{2,k}-4)^{[n_2-1]}$, $(2n-n_1-r_3-\lambda_{3,k}-4)^{[n_3-1]}$, where $k=2,3,\dots,n_i$, for $i=1,2,3$ and $n=n_1+n_2+n_3$. The remaining three eigenvalues are given by the equitable quotient matrix*

$$\begin{bmatrix} n+3n_1-2r_1-4 & n_2 & n_3\\ n_1 & 2n+2n_2-n_1-2r_2-4 & 2n_3\\ n_1 & 2n_2 & 2n+2n_3-n_1-2r_3-4\end{bmatrix}.$$

Corollary 1 *Let $G=K_k\vee(K_t\cup K_{n-t-k})$, where $\vee$ is the join and $\cup$ is the union, be the connected graph with connectivity k. Then the distance signless Laplacian spectrum of G consists of the eigenvalue $\frac{4n-k-4\pm\sqrt{k^2+16nt-16kt-16t^2}}{2}$, the eigenvalue $(2n-k-t-2)$ with multiplicity $t-1$, the eigenvalue $(n+t-2)$ with multiplicity $n-k-t-1$ and the eigenvalue $(n-2)$ with multiplicity k.*

Proof. Let $G_1 = K_k, G_2 = K_t$ and $G_3 = K_{n-t-k}$. Then substituting $r_1 = k-1, r_2 = t-1$, and $r_3 = n-k-t-1$ and noting that the adjacency spectrum of K_ω is $\{n-1, (-1)^{[\omega]}\}$, the result follows by Theorem 3. $\square$

The following lemma says that for each $G \in \mathcal{V}_n^k$, the graph $K_k \vee (K_t \cup K_{n-t-k})$ has the minimum value of $B_i^Q, 1 \le i \le n-1$, that is, the sum of i^{th} largest distance signless Laplacian eigenvalues.

Lemma 3 *Let* G *be a connected graph of order* n *with vertex connectivity* k*,* $1 \le k \le n-1$*. Then*

$$B_i^Q(G) \ge B_i^Q(K_k \vee (K_t \cup K_{n-t-k})),$$

with equality if and only if $G \cong K_k \vee (K_t \cup K_{n-t-k})$.

Proof. Let G be a connected graph of order n with vertex connectivity k, $1 \le k \le n-1$. We first show that $B_i^Q(G) \ge B_i^Q(K_k \vee (K_t \cup K_{n-t-k}))$, for all $i = 1, 2, \dots, n$. Suppose that $1 \le k \le n-2$. Then G is the connected graph of order n with vertex connectivity k for which the spectral parameter $B_i^Q(G)$ has the minimum possible value. It is clear that $G \in \mathcal{V}_n^k$ and $B_i^Q(G)$ attains the minimum value for G. Let $U \subseteq V(G)$ be such that $G - U$ is disconnected and has r connected components, say $G_1, G_2, \dots, G_r$. We are required to show that $r = 2$. For if, $r > 2$, then we can construct a new graph $G' = G + e$ by adding an edge between any two components, say G_1 and G_2 of $G - S$, which is such that $G' \in \mathcal{V}_n^k$. By Lemma 1, we have $B_i^Q(G) > B_i^Q(G')$. This is a contradiction to the fact $B_i^Q(G)$ attains the minimum possible value for G. Therefore, we must have $r = 2$. Further, we claim that each of the components G_1, G_2 and the vertex induced subgraph $\langle U \rangle$ are cliques. For if one among them say G_1 is not a clique, then adding an edge between the two non adjacent vertices of G_1 gives a graph $H \in \mathcal{V}_n^k$ and by Lemma 1, we have $B_i^Q(G) > B_i^Q(H)$. This is again a contradiction, as $B_i^Q(G)$ attains minimum possible value for G. Again $|U| \le k$, and we prove that $|U| = k$. Assume that $|U| < k$. In a similar way, we can form a new graph $G + e = L \in \mathcal{V}_n^k$, where e is adjacent to a vertex of G_1 with a vertex of G_2. Thus, by Lemma 1, $B_i^Q(G) > B_i^Q(H)$, which is not true. Hence G must be of the form $G = K_k \vee (K_t \cup K_{n-k-t})$, $1 \le t \le \lfloor \frac{n-k}{2} \rfloor$. This shows that for all $G \in \mathcal{V}_n^k$, the spectral parameter $B_i^Q(G)$ has the minimum possible value for the graph $K_k \vee (K_t \cup K_{n-k-t})$. $\square$

As $1 \le k =\le n-1$ and $t \le n-k-t$, we have $t \le \lfloor \frac{n-k}{2} \rfloor$. Also, the distance signless Laplacian energy for $k = n-1$ is given by Theorem 1, so we avoid the case $k = n-1$, and thus $1 \le t \le \lfloor \frac{n-k}{2} \rfloor$ makes sense.

Now, we prove that among all connected graphs with given vertex connectivity k, the graph $K_k \vee (K_t \cup K_{n-k-t})$ has the minimum distance signless Laplacian energy.

Theorem 4 *Let* $G \in \mathcal{V}_n^k$ *be a connected graph of order* $n \geq 4$ *with vertex connectivity number* k *satisfying* $a_2 \leq k \leq a_1$*. Then*

$$DSLE(G) \geq \begin{cases} \sqrt{D} + 2t(2n-k-t-1) + k - \frac{4tW(G)}{n}, \\ \sqrt{D} + 2n^2 + n(4t-2k-6) - 4kt - 4t^2 + 5k + 4 - \frac{4(n-k-1)W(G)}{n}, \end{cases}$$

according as $k < \frac{n(t+1)}{2t} - t$ *or* $k \geq \frac{n(t+1)}{2t} - t$, *where* $a_i = \frac{n^2(10t+1)-n^3-n(10t^2+4t)+8t^3 \pm \sqrt{n^4-n^3(12t+2)+n^2(40t^2+12t+1)+n(8t^3-36t^2)+4t^4}}{4(nt-2t^2)}$ *and* $D = k^2 + 16nt - 16kt - 16t^2$. *Equality occurs in each of these inequalities if and only if* $G \cong K_k \vee (K_t \cup K_{n-k-t})$ *with* $1 \leq t \leq \lfloor \frac{n-k}{2} \rfloor$.

Proof. Let G be a connected graph of order n with vertex connectivity k, $2 \leq k \leq n-2$. Then, by Lemma 3, for each $G \in \mathcal{V}_n^k$, the spectral parameter $B_i^Q(G)$ has the minimum possible value for the graph $K_k \vee (K_t \cup K_{n-k-t})$. That is, for all $G \in \mathcal{V}_n^k$, we have $B_i^Q(G) \geq B_i^Q(K_k \vee (K_t \cup K_{n-t-k}))$. With this, from the definition of distance signless Laplacian energy, it follows that

$$\begin{aligned} DSLE(G) &= 2\left(B_{\sigma'}(G) - \frac{2\sigma' W(G)}{n}\right) = 2 \max_{1 \leq j \leq n} \left(\sum_{i=1}^{j} \rho_i^Q(G) - \frac{2jW(G)}{n}\right) \\ &\geq 2 \max_{1 \leq j \leq n} \left(\sum_{i=1}^{j} \rho_i^Q(K_k \vee (K_t \cup K_{n-t-k})) - \frac{2jW(G)}{n}\right). \end{aligned} \tag{6}$$

By Corollary 1, the distance signless Laplacian spectrum of the graph $K_k \vee (K_t \cup K_{n-t-k})$ is
$\left\{\frac{4n-k-4 \pm \sqrt{k^2+16nt-16kt-16t^2}}{2}, (2n-k-t-2)^{[t-1]}, (n+t-2)^{[n-k-t-1]}, (n-2)^k\right\}$.
Let σ' be the number of distance signless Laplacian eigenvalues of $K_k \vee (K_t \cup K_{n-t-k})$ which are greater than or equal to that $\frac{2W(K_k \vee (K_t \cup K_{n-t-k}))}{n} = \frac{n^2-n+2nt-2t^2-2kt}{n}$. Clearly, $\frac{4n-k-4+\sqrt{k^2+16nt-16kt-16t^2}}{2}$ is the distance signless Laplacian spectral radius of the graph $K_k \vee (K_t \cup K_{n-t-k})$ and is always greater than $\frac{2W(K_k \vee (K_t \cup K_{n-t-k}))}{n}$. Now, for the eigenvalue $2n-k-t-2$, we have

$$2n-k-t-2 \geq \frac{2W(K_k \vee (K_t \cup K_{n-t-k}))}{n} = \frac{n^2-n+2nt-2t^2-2kt}{n}$$

which implies that

$$2t^2-(3n-2k)t+(n^2-n-kn)\geq 0. \tag{7}$$

The roots of the polynomial $g_1(t)=2t^2-(3n-2k)t+(n^2-n-kn)=0$ are

$$r_1=\frac{3n-2k+\sqrt{(n-2k)^2+8n}}{2} \text{ and } r_2=\frac{3n-2k-\sqrt{(n-2k)^2+8n}}{2}.$$

This shows that $g_1(t)\geq 0$, for all $t\leq r_2$ and $t\geq r_1$. Since,

$$t=\frac{n-k}{2}<\frac{3n-2k-\sqrt{(n-2k)^2+8n}}{2}=r_2$$

gives $k\leq n-2$, which is the maximum value for connectivity. Thus, $g_1(t)\geq 0$, for all $t\leq\frac{n-k}{2}$. For the eigenvalue $n+t-2$, we have

$$n+t-2\geq\frac{2W(K_k\vee(K_t\cup K_{n-t-k}))}{n}=\frac{n^2-n+2nt-2t^2-2kt}{n}$$

which implies that $k\geq\frac{n(t+1)}{2t}-t$. This shows that

$$n+t-2\geq\frac{2W(K_k\vee(K_t\cup K_{n-t-k}))}{n},$$

for all $k\geq\frac{n(t+1)}{2t}-t$ and

$$n+t-2<\frac{2W(K_k\vee(K_t\cup K_{n-t-k}))}{n},$$

for all $k<\frac{n(t+1)}{2t}-t$. For the second smallest distance signless Laplacian eigenvalue

$$\frac{4n-k-4-\sqrt{k^2+16nt-16kt-16t^2}}{2},$$

we have

$$\begin{aligned}\frac{4n-k-4+\sqrt{k^2+16nt-16kt-16t^2}}{2}&\geq\frac{2W(K_k\vee(K_t\cup K_{n-t-k}))}{n}\\&=\frac{n^2-n+2nt-2t^2-2kt}{n}\end{aligned}$$

implying that

$$\begin{aligned}f(k)=\,&k^2(16t^2-8nt)+k(4n^2-4n^3-16nt+40n^2t-40nt^2+32t^3)\\&-8n^3+4n^4+4n^2+16n^2t-32n^3t-16nt^2+48n^2t^2-32nt^3+16t^4\\&\geq 0.\end{aligned}$$

which in turn implies that

$$\frac{4n-k-4-\sqrt{k^2+16nt-16kt-16t^2}}{2} < \frac{2W(K_k \vee (K_t \cup K_{n-t-k}))}{n}$$

for $a_2 < k < a_1$, where
$a_i = \frac{n^2(10t+1)-n^3-n(10t^2+4t)+8t^3 \pm \sqrt{n^4-n^3(12t+2)+n^2(40t^2+12t+1)+n(8t^3-36t^2)+4t^4}}{4(nt-2t^2)}$,
$i = 1, 2$, are the zeros of $f(k)$. From these calculations it follows that, if $k < \frac{n(t+1)}{2t} - t$, then $\sigma' = t$, and if $k \geq \frac{n(t+1)}{2t} - t$, then $\sigma' = n-k-1$. Therefore, for $k < \frac{n(t+1)}{2t} - t$, it follows from Inequality (6) that

$$\begin{aligned}
DSLE(G) &\geq 2 \max_{1\leq j\leq n} \left(\sum_{i=1}^{j} \rho_i^Q(K_k \vee (K_t \cup K_{n-t-k})) - \frac{2jW(G)}{n} \right) \\
&\geq 2\left(\sum_{i=1}^{t} \rho_i^Q(K_k \vee (K_t \cup K_{n-t-k})) - \frac{2tW(G)}{n} \right) \\
&= 2\Bigg(\frac{4n-k-4+\sqrt{k^2+16nt-16kt-16t^2}}{2} \\
&\quad + (t-1)(2n-k-t-2) - \frac{2tW(G)}{n} \Bigg) \\
&= \sqrt{k^2+16nt-16kt-16t^2} + 2t(2n-k-t-1) + k - \frac{4tW(G)}{n}.
\end{aligned}$$

If $k \geq \frac{n(t+1)}{2t} - t$, from (6), we have

$$\begin{aligned}
&DSLE(G) \geq 2 \max_{1\leq j\leq n} \left(\sum_{i=1}^{j} \rho_i^Q(K_k \vee (K_t \cup K_{n-t-k})) - \frac{2jW(G)}{n} \right) \\
&\geq 2 \left(\sum_{i=1}^{n-k-1} \rho_i^Q(K_k \vee (K_t \cup K_{n-t-k})) - \frac{2(n-k-1)W(G)}{n} \right) \\
&= 2 \left(\frac{4n-k-4+\sqrt{D}}{2} + (t-1)(2n-k-t-2) + (n-k-t-1)(n+t-2) \right) \\
&- \frac{4(n-k-1)W(G)}{n} = \sqrt{D} + 2n^2 + n(4t-2k-6) - 4kt - 4t^2 + 5k + 4 \\
&- \frac{4(n-k-1)W(G)}{n},
\end{aligned}$$

where $D = k^2 + 16nt - 16kt - 16t^2$. By Lemmas 1 and 3, equality holds if and only if $G \cong K_k \vee (K_t \cup K_{n-t-k})$. This completes the proof. □

The next result is the special case of $G \in \mathcal{V}_n^k$, for $t = 1$.

Proposition 3 *Let* $G \in \mathcal{V}_n^k$ *and* $t = 1$. *Then*

$$\mathrm{DSLE}(G) \geq 2\left(4n - k - 4 - \frac{8W(G)}{n}\right)$$

with equality if and only if $G \cong K_k \vee (K_1 \cup K_{n-1-k})$.

Proof. By letting $t = 1$ in Corollary 1, the distance signless Laplacian spectrum of $K_k \vee (K_1 \cup K_{n-1-k})$ is given by

$$\left\{\frac{4n - k - 4 \pm \sqrt{k^2 - 16k + 16n - 16}}{2}, (n-1)^{[n-k-2]}, (n-2)^{[k]}\right\}.$$

Clearly, the distance signless Laplacian eigenvalue $\frac{4n-k-4+\sqrt{k^2-16k+16n-16}}{2}$ is the distance signless spectral radius and is always greater than

$$\frac{2W(K_k \vee (K_1 \cup K_{n-1-k}))}{n} = \frac{n^2 + n - 2k - 2}{n}.$$

For the eigenvalue $n - 1$, we have

$$n - 1 < \frac{2W(K_k \vee (K_1 \cup K_{n-1-k}))}{n}$$

if $n + k > 1$, which is always true as $n \geq 4$ and $k \geq 2$.
Lastly, for the eigenvalue $\frac{4n-k-4-\sqrt{k^2-16k+16n-16}}{2}$, we see if

$$\frac{4n - k - 4 - \sqrt{k^2 - 16k + 16n - 16}}{2} < \frac{2W(K_k \vee (K_1 \cup K_{n-1-k}))}{n},$$

then after simplification, we have

$$h(k) = k^2(8n-16) - k(44n^3 - 4n^3 - 56n + 32) - 4n^4 + 40n^3 - 68n^2 + 48n - 16 < 0.$$

The zeros of $h(k)$ are $n - 1$ and $\frac{9n^2-n^3-8n+4}{2(n-2)}$. This implies that

$$\frac{4n - k - 4 - \sqrt{k^2 - 16k + 16n - 16}}{2} \geq \frac{2W(K_k \vee (K_1 \cup K_{n-1-k}))}{n}$$

for

$$\frac{9n^2 - n^3 - 8n + 4}{2(n-2)} \leq k \leq n - 1.$$

Thus, from (6), we have

$$\begin{aligned} DSLE(G) &\geq 2 \max_{1\leq j\leq n} \left(\sum_{i=1}^{j} \rho_i^Q(K_k \vee (K_1 \cup K_{n-1-k})) - \frac{2jW(G)}{n} \right) \\ &\geq 2 \left(\sum_{i=1}^{2} \rho_i^Q(K_k \vee (K_1 \cup K_{n-1-k})) - \frac{2jW(G)}{n} \right) \\ &= 2 \left(4n - k - 4 - \frac{4W(G)}{n} \right). \end{aligned}$$

Clearly, equality occurs by Lemma 1. □

For $G \in \mathcal{V}_n^k$, with $k = t = 1$, we have the following observation.

Corollary 2 *Let* $G \in \mathcal{V}_n^1$. *Then, for* $t = 1$, *we have*

$$DSLE(G) \geq 2 \left(4n - k - 4 - \frac{8W(G)}{n} \right)$$

with equality if and only if $G \cong K_1 \vee (K_1 \cup K_{n-2})$.

Proof. From Corollary 1, the distance signless Laplacian spectrum of $K_1 \vee (K_1 \cup K_{n-2})$ is given by

$$\left\{ \frac{4n - 5 \pm \sqrt{16n - 31}}{2}, (n-1)^{[n-3]}, n - 2 \right\}.$$

It can be easily seen that $\frac{4n-5+\sqrt{16n-31}}{2}$ is the distance signless spectral radius and is always greater than $\frac{2W(K_k \vee (K_1 \cup K_{n-t-k}))}{n} = \frac{n^2+n-4}{n}$. For the eigenvalue $n-1$, we have $n - 1 < \frac{2W(K_k \vee (K_1 \cup K_{n-t-k}))}{n}$ if $n > 2$, which is always true. Next for the eigenvalue $\frac{4n-5-\sqrt{16n-31}}{2}$, we see that $\frac{4n-5-\sqrt{16n-31}}{2} \geq \frac{n^2+n-4}{n}$, which after simplification gives $n^4 - 11n^3 + 28n^2 - 28n + 16 \geq 0$, which is true for $n \geq 8$. Thus, from (6), we have

$$\begin{aligned} DSLE(G) &\geq 2 \max_{1\leq j\leq n} \left(\sum_{i=1}^{j} \rho_i^Q(K_1 \vee (K_1 \cup K_{n-2})) - \frac{2jW(G)}{n} \right) \\ &\geq 2 \left(\sum_{i=1}^{2} \rho_i^Q(K_1 \vee (K_1 \cup K_{n-2})) - \frac{2jW(G)}{n} \right) \\ &= 2 \left(4n - 5 - \frac{4W(G)}{n} \right). \end{aligned}$$

□

Conclusions

We observe that the investigation of the graph invariant $B_b^Q(G) = \sum_{i=1}^{b} \rho_i^Q, 1 \leq b \leq n-1$, that is, the sum of the $b \geq 1$ largest signless Laplacian eigenvalues is an interesting problem. By Lemma 1, Theorem 2, Lemma 3 and Theorem 4, we see that $CS_{n,\alpha}$ and $K_k \vee (K_t \cup K_{n-t-k})$ have minimum value of B_b^Q among the graphs with independence α and connectivity k. In a similar manner, it can be shown that K_n and $K_{a,n-a}$ have minimum value of B_b^Q among all graphs and among all the bipartite graphs. In [1], upper bounds for B_b^Q were discussed for graphs with diameter 1 and 2, split graphs, threshold graphs and a conjecture was also put forward. It will be interesting to find the lower bounds for B_b^Q for an arbitrary graph G and characterization of the extremal graphs. By using Lemma 1 and proceeding as in Theorems 2 and 4, we can show that $K_{a,n-a}$ has the minimum distance signless Laplacian energy among all graphs bipartite graphs. A difficult problem is to investigate the graphs with maximum distance signless Laplacian energy. In particular, it will be interesting to study the graphs with maximum signless Laplacian energy among bipartite graphs, split graphs, graphs with fixed connectivity, perfect matching and other families. The graph invariant σ', that is, the number of distance signless Laplacian eigenvalues which are greater or equal to $\frac{2W(G)}{n}$ is an interesting graph invariant. Several papers exist in the literature in this direction and various open problems were asked in case of Laplacian [7] and signless Laplaian matrices. The same is true for distance signless Laplacian matrix and attractive problems of σ' can be investigated, like characterization of graphs having $\sigma' = 1, 2, \frac{n}{2}$ and $\sigma' = n-1$.

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