Acta Univ. Sapientiae, Mathematica, **13**, 2 (2021) 483–493

DOI: 10.2478/ausm-2021-0030

On generalized γ_μ-closed sets and related continuity

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Abstract. In this paper our main interest is to introduce a new type of generalized open sets defined in terms of an operation on a generalized topological space. We have studied some properties of this newly defined sets. As an application, we have introduced some weak separation axioms and discussed some of their properties. Finally, we have studied some preservation theorems in terms of some irresolute functions.

1 Introduction

In 1979, Kasahara [5] introduced the notion of an operation on a topological space and introduced the concept of α-closed graph of a function. After then Janković defined [4] the concept of α-closed sets and investigated some properties of functions with α-closed graphs. On the other hand, in 1991 Ogata [7] introduced the notion of γ-open sets to investigate some new separation axioms on a topological space. The notion of operations on the family of all semi-open sets and pre-open sets are investigated by Krishnan et al. [6] and Van An et al. [11]. Recently, the concept of γ_μ-Lindelöf spaces was studied by Roy and Noiri in [9, 10].

2010 Mathematics Subject Classification: 54A10, 54C08
Key words and phrases: operation, μ-open set, γ_μ-open set, $\gamma_\mu g$-closed set, $(\gamma_\mu, \beta_\lambda)$-irresolute function

In this paper our aim is to study an operation based on generalized γ_μ-closed like sets, where the operation is defined on the collection of generalized open sets. The most common properties of different open like sets or weakly open sets are that they are closed under arbitary unions and contain the null set. Observing these, Császár introduced the concept of generalized open sets. We now recall some notions defined in [1]. Let X be a non-empty set. A subcollection $\mu \subseteq \mathcal{P}(X)$ (where $\mathcal{P}(X)$ denotes the power set of X) is called a generalized topology [1], (briefly, GT) if $\varnothing \in \mu$ and any union of elements of μ belongs to μ. A set X with a GT μ on the set X is called a generalized topological space (briefly, GTS) and is denoted by (X, μ). If for a GTS (X, μ), $X \in \mu$, then (X, μ) is known as a strong GTS. Throughout the paper, we assume that (X, μ) and (Y, λ) are strong GTS's. The elements of μ are called μ-open sets and μ-closed sets are their complements. The μ-closure of a set $A \subseteq X$ is denoted by $c_\mu(A)$ and defined by the smallest μ-closed set containing A which is equivalent to the intersection of all μ-closed sets containing A. We use the symbol $i_\mu(A)$ to mean the μ-interior of A and it is defined as the union of all μ-open sets contained in A i.e., the largest μ-open set contained in A (see [3, 2, 1]).

2 $\gamma_\mu g$-closed sets and their related properties

Definition 1 [9] *Let (X, μ) be a GTS. An operation γ_μ on a generalized topology μ is a mapping from μ to $\mathcal{P}(X)$ (where $\mathcal{P}(X)$ is the power set of X) with $G \subseteq \gamma_\mu(G)$, for each $G \in \mu$. This operation is denoted by $\gamma_\mu : \mu \to \mathcal{P}(X)$. Note that $\gamma_\mu(A)$ and A^{γ_μ} are two different notations for the same set.*

Definition 2 [9] *Let (X, μ) be a GTS and γ_μ be an operation on μ. A subset G of (X, μ) is called γ_μ-open if for each point x of G, there exists a μ-open set U containing x such that $\gamma_\mu(U) \subseteq G$.*

A subset of a GTS (X, μ) is called γ_μ-closed if its complement is γ_μ-open in (X, μ). We shall use the symbol γ_μ to mean the collection of all γ_μ-open sets of the GTS (X, μ).

Definition 3 [9] *Let (X, μ) be a GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation. It is easy to see that the family of all γ_μ-open sets forms a GT on X. The γ_μ-closure of a set A of X is denoted by $c_{\gamma_\mu}(A)$ and is defined as $c_{\gamma_\mu}(A) = \cap\{F : F$ is a γ_μ-closed set and $A \subseteq F\}$.*

It is easy to check that for each $x \in X$, $x \in c_{\gamma_\mu}(A)$ if and only if $V \cap A \neq \varnothing$, for any $V \in \gamma_\mu$ with $x \in V$. Note that if $\gamma_\mu = id_\mu$, then $c_{\gamma_\mu}(A) = c_\mu(A)$.

Definition 4 *Let (X, μ) be a GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation. A subset A of X is said to be $\gamma_\mu g$-closed if $c_{\gamma_\mu}(A) \subseteqq U$, whenever $A \subseteqq U$ and U is a γ_μ-open set in (X, μ).*

Every γ_μ-closed set is $\gamma_\mu g$-closed but the converse is not true as shown in the next example. Also note that if $\gamma_\mu = id_\mu$, then $\gamma_\mu g$-closed set reduces to a μg-closed set [8].

Example 1 *Let $X = \{1, 2, 3\}$ and $\mu = \{\varnothing, \{1\}, \{1, 2\}, \{2, 3\}, X\}$. Then (X, μ) is a GTS. Now $\gamma_\mu : \mu \to \mathcal{P}(X)$ defined by*

$$\gamma_\mu(A) = \begin{cases} A, & \text{if } 1 \in A \\ \{2, 3\}, & \text{otherwise} \end{cases}$$

is an operation. It can be easily checked that $\{1, 3\}$ is a $\gamma_\mu g$-closed set but not a γ_μ-closed set.

Theorem 1 *Let (X, μ) be a GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation. A subset A of X is $\gamma_\mu g$-closed if and only if $c_{\gamma_\mu}(\{x\}) \cap A \neq \varnothing$, holds for every $x \in c_{\gamma_\mu}(A)$.*

Proof. First let the given condition hold and let U be a γ_μ-open set with $A \subseteqq U$ and $x \in c_{\gamma_\mu}(A)$. As $c_{\gamma_\mu}(\{x\}) \cap A \neq \varnothing$, there exists a $z \in c_{\gamma_\mu}(\{x\})$ such that $z \in A \subseteqq U$. Thus $U \cap \{x\} \neq \varnothing$. Hence $x \in U$. Thus $c_{\gamma_\mu}(A) \subseteqq U$, proving A to be a $\gamma_\mu g$-closed set.

Conversely, let A be a $\gamma_\mu g$-closed subset of X and $x \in c_{\gamma_\mu}(A)$ with $c_{\gamma_\mu}(\{x\}) \cap A = \varnothing$. Then $A \subseteqq X \setminus c_{\gamma_\mu}(\{x\})$ which implies that $c_{\gamma_\mu}(A) \subseteqq X \setminus c_{\gamma_\mu}(\{x\})$ (as A is $\gamma_\mu g$-closed), which is a contradiction to the fact that $x \in c_{\gamma_\mu}(\{x\})$. Thus $c_{\gamma_\mu}(\{x\}) \cap A \neq \varnothing$. □

Theorem 2 *Let (X, μ) be a GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation. If $c_{\gamma_\mu}(\{x\}) \cap A \neq \varnothing$ for every $x \in c_{\gamma_\mu}(A)$, then $c_{\gamma_\mu}(A) \setminus A$ does not contain any non-empty γ_μ-closed set.*

Proof. If possible, let there exist a non-empty γ_μ-closed set F such that $F \subseteqq c_{\gamma_\mu}(A) \setminus A$. Let $x \in F$. Then $x \in c_{\gamma_\mu}(A)$. Since $\varnothing \neq c_{\gamma_\mu}(\{x\}) \cap A \subseteqq F \cap A$, we have $F \cap A \neq \varnothing$, which is a contradiction. □

Corollary 1 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. A subset* A *of a GTS* (X, μ) *is* $\gamma_\mu g$*-closed if and only if* $A = F \setminus N$*, where* F *is* γ_μ*-closed and* N *contains no non-empty* γ_μ*-closed subset of* X*.*

Proof. One part of the theorem follows from Theorems 1 and 2 by taking $F = c_{\gamma_\mu}(A)$ and $N = c_{\gamma_\mu}(A) \setminus A$.

Conversely, suppose that $A = F \setminus N$ and $A \subseteq U$, where U is γ_μ-open. Then $F \cap (X \setminus U)$ is a γ_μ-closed subset of N and hence it must be empty. Thus $c_{\gamma_\mu}(A) \subseteq F \subseteq U$. □

Theorem 3 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Let* A *be a* $\gamma_\mu g$*-closed subset of* X *and* $A \subseteq B \subseteq c_{\gamma_\mu}(A)$*. Then* B *is also* $\gamma_\mu g$*-closed.*

Proof. Let A be a $\gamma_\mu g$-closed set such that $A \subseteq B \subseteq c_{\gamma_\mu}(A)$ and U be a γ_μ-open set with $B \subseteq U$. Then $A \subseteq U$ and hence $c_{\gamma_\mu}(A) \subseteq U$. Thus $c_{\gamma_\mu}(B) \subseteq U$. Thus B is $\gamma_\mu g$-closed. □

Theorem 4 *Let* (X, μ) *be a strong GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. For each* $x \in X$*, either* $\{x\}$ *is* γ_μ*-closed or* $X \setminus \{x\}$ *is a* $\gamma_\mu g$*-closed set in* (X, μ)*.*

Proof. If $\{x\}$ is γ_μ-closed, then we have nothing to prove. Suppose that $\{x\}$ is not γ_μ-closed. Then $X \setminus \{x\}$ is not γ_μ-open. Let U be any γ_μ-open set such that $X \setminus \{x\} \subseteq U$. Hence $U = X$. Thus $c_{\gamma_\mu}(X \setminus \{x\}) \subseteq U$. Thus $X \setminus \{x\}$ is a $\gamma_\mu g$-closed set. □

Definition 5 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is said to be* γ_μ-$T_{\frac{1}{2}}$ *if every* $\gamma_\mu g$*-closed set is* γ_μ*-closed.*

Theorem 5 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is* γ_μ-$T_{\frac{1}{2}}$ *if and only if* $\{x\}$ *is either* γ_μ*-closed or* γ_μ*-open.*

Proof. Suppose that (X, μ) be a γ_μ-$T_{\frac{1}{2}}$ space and $\{x\}$ is not γ_μ-closed. Then by Theorem 4, $X \setminus \{x\}$ is a $\gamma_\mu g$-closed set and hence a γ_μ-closed set. So $\{x\}$ is a γ_μ-open set.

Conversely, suppose that F be a $\gamma_\mu g$-closed set in (X, μ). Let $x \in c_{\gamma_\mu}(F)$. Then $\{x\}$ is either γ_μ-open or γ_μ-closed. If $\{x\}$ is γ_μ-open, then $\{x\} \cap F \neq \varnothing$. Hence $x \in F$. Thus $c_{\gamma_\mu}(F) \subseteq F$, which implies that F is γ_μ-closed. If $\{x\}$ is γ_μ-closed, suppose that $x \notin F$. Then $x \in c_{\gamma_\mu}(F) \setminus F$, which is impossible by Theorem 2. Thus $x \in F$. Hence $c_{\gamma_\mu}(F) \subseteq F$, so that F is γ_μ-closed. □

Definition 6 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is said to be*

(i) γ_μ-T_0 *if for any two distinct points* x *and* y *of* X*, there exists a* γ_μ*-open set* U *containing* x *but not containing* y *or a* γ_μ*-open set* V *containing* y *but not containing* x.

(ii) γ_μ-T_1 *if for any two distinct points* x *and* y *of* X*, there exist two* γ_μ*-open sets* U *and* V *such that* $x \in U$, $y \notin U$ *and* $y \in V$, $x \notin U$.

(iii) γ_μ-T_2 *if for any two distinct points* x *and* y *of* X*, there exist two disjoint* γ_μ*-open sets* U *and* V *such that* $x \in U$ *and* $y \in V$.

Definition 7 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. A subset* A *of* (X, μ) *is said to be a* γ_μ-D_μ *set if there exist two* γ_μ*-open sets* U *and* V *such that* $U \neq X$ *and* $A = U \setminus V$.

It follows from the definition that every γ_μ-open set (other than X) is a γ_μ-D_μ set.

Definition 8 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is said to be a*

(i) γ_μ-D_0 *space if for any pair of distinct points, there exists a* γ_μ-D_μ *set containing* x *but not containing* y *or a* γ_μ-D_μ *set containing* y *but not containing* x.

(ii) γ_μ-D_1 *space if for any pair of distinct points* x *and* y *of* X*, there exist a* γ_μ-D_μ *set containing* x *but not* y *and a* γ_μ-D_μ *set containing* y *but not containing* x.

(iii) γ_μ-D_2 *space if for any two two distinct points* x *and* y *of* X*, there exist disjoint* γ_μ-D_μ *sets* U *and* V *containing* x *and* y *respectively.*

Remark 1 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then the following hold:*

(i) *If* (X, μ) *is* γ_μ-T_i*, then it is* γ_μ-T_{i-1} *for* $i = 1, 2$.

(ii) *If* (X, μ) *is* γ_μ-T_i*, then it is* γ_μ-D_i *for* $i = 0, 1, 2$.

(iii) *If* (X, μ) *is* γ_μ-D_i*, then it is* γ_μ-D_{i-1} *for* $i = 1, 2$.

Proposition 1 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is* γ_μ-T_1 *if and only if every singleton is* γ_μ*-closed.*

Proof. Obvious. □

Definition 9 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is said to be* γ_μ*-symmetric if for* $x, y \in X$, $x \in c_{\gamma_\mu}(\{y\}) \Rightarrow y \in c_{\gamma_\mu}(\{x\})$.

Proposition 2 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is a* γ_μ*-symmetric space if and only if* $\{x\}$ *is a* $\gamma_\mu g$*-closed set, for each* $x \in X$.

Proof. Assume that $\{x\} \subseteqq U$, where U is a γ_μ-open set. If possible, let $c_{\gamma_\mu}(\{x\}) \nsubseteq U$. Then $c_{\gamma_\mu}(\{x\}) \cap (X \setminus U) \neq \varnothing$. Let $y \in c_{\gamma_\mu}(\{x\}) \cap (X \setminus U) \neq \varnothing$. Then by hypothesis, $x \in c_{\gamma_\mu}(\{y\}) \subseteqq X \setminus U$ and hence $x \notin U$, which is a contradiction. Therefore $\{x\}$ is a $\gamma_\mu g$-closed set.

Conversely, assume that $x \in c_{\gamma_\mu}(\{y\})$ but $y \notin c_{\gamma_\mu}(\{x\})$. Then $y \in X \setminus c_{\gamma_\mu}(\{x\})$. Thus $c_{\gamma_\mu}(\{y\}) \subseteqq X \setminus c_{\gamma_\mu}(\{x\})$. Thus $x \notin c_{\gamma_\mu}(\{x\})$, which is a contradiction. Thus (X, μ) is γ_μ-symmetric. □

Corollary 2 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. If* (X, μ) *is* γ_μ*-*T_1*, then it is* γ_μ*-symmetric.*

Proof. It follows from Proposition 1 that in a γ_μ-T_1 space every singleton is γ_μ-closed and hence $\gamma_\mu g$-closed. □

Corollary 3 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is* γ_μ*-*T_1 *if and only if it is* γ_μ*-*T_0 *and* γ_μ*-symmetric.*

Proof. One part of the theorem follows from Remark 1 and Corollary 2. Let (X, μ) be a γ_μ-T_0 and γ_μ-symmetric space and x, y be any two distinct points of X. We may assume that $x \in U$ but $y \notin U$, for some γ_μ-open set U of X. Thus $x \notin c_{\gamma_\mu}(\{y\})$ and hence $y \notin c_{\gamma_\mu}(\{x\})$. Thus there exists a γ_μ-open set V containing y such that $x \notin V$. Thus (X, μ) is γ_μ-T_1. □

Proposition 3 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then*

(a) *Every* γ_μ*-*T_1 *space is* γ_μ*-*$T_{\frac{1}{2}}$ *and every* γ_μ*-*$T_{\frac{1}{2}}$ *space is* γ_μ*-*T_0.

(b) *For a* γ_μ*-symmetric space the following are equivalent:*

(i) (X, μ) *is a* γ_μ*-*T_0 *space.*

(ii) (X, μ) *is a* γ_μ-T_1 *space.*
(iii) (X, μ) *is a* γ_μ-$T_{\frac{1}{2}}$ *space.*
(iv) (X, μ) *is a* γ_μ-D_1 *space.*

Proof. (a) Follows from Theorem 5 and Proposition 1.
(b) If (X, μ) is γ_μ-symmetric and a γ_μ-T_0 space, then by Corollary 3, (X, μ) is a γ_μ-T_1 space and hence (by (a) above) (X, μ) is γ_μ-$T_{\frac{1}{2}}$ and again by (a) above, (X, μ) is γ_μ-T_0. Thus (i), (ii) and (iii) are equivalent.
Again by Remark 1, (ii) $\Rightarrow$ (iv) is obvious.
(iv)$\Rightarrow$ (i) : Let (X, μ) be a γ_μ-D_1 space. Hence (X, μ) is a γ_μ-D_0 space. Thus for each pair of distinct points $x, y \in X$, at least one of x, y, say x, belongs to a γ_μ-D_μ set S but $y \notin S$. Let $S = U_1 \setminus U_2$, where U_1 and U_2 are γ_μ-sets and $U_1 \neq X$. Then $x \in U_1$. If $y \notin U_1$, then the proof is complete. If $y \in U_1 \cap U_2$, then $y \in U_2$ but $x \notin U_2$. Thus (X, μ) is γ_μ-T_0. □

Theorem 6 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is* γ_μ-D_2 *if and only if it is* γ_μ-D_1.

Proof. One part follows from Remark 1. Conversely, let x and y be two distinct points of X. Then there exist γ_μ-D_μ sets G_1 and G_2 in X such that $x \in G_1$, $y \notin G_1$ and $y \in G_2$, $x \notin G_2$. Let $G_1 = U_1 \setminus U_2$ and $G_2 = U_3 \setminus U_4$, where U_1, U_2, U_3 and U_4 are γ_μ-open sets in X and $U_1 \neq X$, $U_3 \neq X$. From $x \notin G_2$, it follows that either $x \in U_3 \cap U_4$ or $x \notin U_3$. We will discuss the two cases separately.
Case - 1: $x \in U_3 \cap U_4$: Then $y \in G_2$ and $x \in U_4$, with $G_2 \cap U_4 = \varnothing$.
Case - 2: $x \notin U_3$: By $y \notin G_1$ the following two cases may arise. If $y \notin U_1$, as $x \in U_1 \setminus U_2$, it follows that $x \in U_1 \setminus (U_2 \cup U_3)$ with $y \in U_3 \setminus (U_1 \cup U_4)$ and $(U_1 \setminus (U_2 \cup U_3)) \cap (U_3 \setminus (U_1 \cup U_4)) = \varnothing$. In the case if $y \in U_1 \cap U_2$, we have $x \in U_1 \setminus U_2$ and $y \in U_2$ such that $(U_1 \setminus U_2) \cap U_2 = \varnothing$. □

Definition 10 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then the* γ_μ*-kernel of a subset* A *of* X *is denoted by* $\ker_{\gamma_\mu}(A) = \cap\{U : A \subseteq U$ *and* U *is* γ_μ*-open*$\}$.

Proposition 4 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* $y \in \ker_{\gamma_\mu}(\{x\})$ *if and only if* $x \in c_{\gamma_\mu}(\{y\})$.

Proof. Suppose that $y \notin \ker_{\gamma_\mu}(\{x\})$. Then there exists a γ_μ-open set V containing x such that $y \notin V$. Therefore we have, $x \notin c_{\gamma_\mu}(\{y\})$. The other part can be done in the similar manner. □

Proposition 5 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then for any subset* A *of* X, $\ker_{\gamma_\mu}(A) = \{x : c_{\gamma_\mu}(\{x\}) \cap A \neq \varnothing\}$.

Proof. Let $x \in \ker_{\gamma_\mu}(A)$ and $c_{\gamma_\mu}(\{x\}) \cap A = \varnothing$. Then $A \subseteqq X \setminus c_{\gamma_\mu}(\{x\})$, where $X \setminus c_{\gamma_\mu}(\{x\})$ is a γ_μ-open set not containing x.

Conversely, let $x \in X$ and $c_{\gamma_\mu}(\{x\}) \cap A \neq \varnothing$, with $x \notin \ker_{\gamma_\mu}(A)$. Then there exists a γ_μ-open set V containing A such that $x \notin V$. Let $y \in c_{\gamma_\mu}(\{x\}) \cap A$. Then V is a γ_μ-open set containing y (as $A \subseteqq V$), but not containing x. □

Proposition 6 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. If a singleton set* $\{x\}$ *is a* γ_μ-D_μ *set of* X, *then* $\ker_{\gamma_\mu}(\{x\}) \neq X$.

Proof. Let $\{x\}$ be a γ_μ-D_μ set. Then there exist two γ_μ-open sets V_1 and V_2 such that $\{x\} = V_1 \setminus V_2$ and $V_1 \neq X$. Thus $\ker_{\gamma_\mu}(\{x\}) \subseteqq V_1 \neq X$. □

Proposition 7 *Let* (X, μ) *be a* γ_μ-$T_{\frac{1}{2}}$ *GTS having at least two points, where* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is* γ_μ-D_1 *if and only if* $\ker_{\gamma_\mu}(\{x\}) \neq X$, *for every point* $x \in X$.

Proof. Let $x \in X$ and X be γ_μ-D_1. For any point y other than x, there exists a γ_μ-D_μ set V such that $x \in V$ and $y \notin V$. Then $V = V_1 \setminus V_2$, where V_1 and V_2 are γ_μ-open sets such that $V_1 \neq X$. Thus $\{x\} \subseteqq V_1$ and $V_1 \neq X$, where V_1 is a γ_μ-open set. Hence $\ker_{\gamma_\mu}(\{x\}) \neq X$.

Conversely, let x and y be two distinct points of X. Using Theorem 5, we have the following cases :
Case -1 : $\{x\}$ and $\{y\}$ both are γ_μ-open : Then the case is obvious, as every γ_μ-open set is a γ_μ-D_μ set.
Case -2: $\{x\}$ and $\{y\}$ both are γ_μ-closed : In this case $x \in X \setminus \{y\}$, $y \in X \setminus \{x\}$, $y \notin X \setminus \{y\}$, $x \notin X \setminus \{x\}$ and $X \setminus \{x\}$, $X \setminus \{y\}$ both are γ_μ-D_μ sets.
Case-3: $\{x\}$ is γ_μ-open and $\{y\}$ is γ_μ-closed : Since $\ker_{\gamma_\mu}(\{y\}) \neq X$, there exists a γ_μ-open set V containing y such that $V \neq X$. Clearly $\{y\} = V \setminus (X \setminus \{y\})$, showing $\{y\}$ to be a γ_μ-D_μ set. Thus $\{x\}$ and $\{y\}$ are the two γ_μ-D_μ sets such that $y \notin \{x\}$ and $x \notin \{y\}$.
Case-4 : $\{x\}$ is γ_μ-closed and $\{y\}$ is γ_μ-open : Can be proved as in case 3. Thus (X, μ) is a γ_μ-D_1 space. □

3 $(\gamma_\mu, \beta_\lambda)$-irresolute functions

Throughout the rest of the paper, (X, μ) and (Y, λ) will denote GTS's and $\gamma_\mu : \mu \to \mathcal{P}(X)$ and $\beta_\lambda : \lambda \to \mathcal{P}(Y)$ will denote operations on μ and λ respectively.

Definition 11 *A function* $f : (X, \mu) \to (Y, \lambda)$ *is said to be* $(\gamma_\mu, \beta_\lambda)$*-irresolute if for each* $x \in X$ *and each* β_λ*-open set* V *containing* $f(x)$*, there is a* γ_μ*-open set* U *containing* x *such that* $f(U) \subseteqq V$.

Theorem 7 *Let* $f : (X, \mu) \to (Y, \lambda)$ *be a function. Then the following are equivalent:*

(i) f *is* $(\gamma_\mu, \beta_\lambda)$*-irresolute.*

(ii) $f(c_{\gamma_\mu}(A)) \subseteqq c_{\beta_\lambda}(f(A))$, *holds for every subset* A *of* X.

(iii) $f^{-1}(B)$ *is* γ_μ*-closed, for every* β_λ*-closed set* B *of* Y.

Proof. Obvious. □

Definition 12 *A mapping* $f : (X, \mu) \to (Y, \lambda)$ *is said to be* $(\gamma_\mu, \beta_\lambda)$*-closed if for any* γ_μ*-closed set* A *of* (X, μ), $f(A)$ *is* β_λ*-closed in* Y.

Theorem 8 *Suppose that* $f : (X, \mu) \to (Y, \lambda)$ *is* $(\gamma_\mu, \beta_\lambda)$*-irresolute and* $(\gamma_\mu, \beta\lambda)$*-closed.*

(i) *If* A *is* $\gamma_\mu g$*-closed, then* $f(A)$ *is* $\beta_\lambda g$*-closed.*

(ii) *If* B *be any* $\beta_\lambda g$*-closed set of* Y*, then* $f^{-1}(B)$ *is a* $\gamma_\mu g$*-closed set in* X.

Proof. (i) Let V be any β_λ-open set such that $f(A) \subseteqq V$. Then by Theorem 7, $f^{-1}(V)$ is γ_μ-open in X. Since $A \subseteqq f^{-1}(V)$ and A is $\gamma_\mu g$-closed, $c_{\gamma_\mu}(A) \subseteqq f^{-1}(V)$ and hence $f(c_{\gamma_\mu}(A)) \subseteqq V$. Now by the assumption, $f(c_{\gamma_\mu}(A))$ is a β_λ-closed set in Y. Thus $c_{\beta_\lambda}(f(A)) \subseteqq c_{\beta_\lambda}(f(c_{\gamma_\mu}(A))) = f(c_{\gamma_\mu}(A)) \subseteqq V$. Hence $f(A)$ is $\beta_\lambda g$-closed.

(ii) Let U be any γ_μ-open set of X with $f^{-1}(B) \subseteqq U$. Let $F = c_{\gamma_\mu}(f^{-1}(B)) \cap (X \setminus U)$. Then F is a γ_μ-closed set in X. Since f is $(\gamma_\mu, \beta_\lambda)$-closed, $f(F)$ is β_λ-closed. Since $f(F) \subseteqq f(c_{\gamma_\mu}(f^{-1}(B))) \cap f(X \setminus U) \subseteqq c_{\beta_\lambda}(f(f^{-1}(B))) \cap f(X \setminus U) \subseteqq c_{\beta_\lambda}(B) \cap (Y \setminus B) = c_{\beta_\lambda}(B) \setminus B$ by Corollary 1, it then follows that $f(F) = \varnothing$ and hence $F = \varnothing$. □

Theorem 9 *Suppose that* $f : (X, \mu) \to (Y, \lambda)$ *is* $(\gamma_\mu, \beta_\lambda)$*-irresolute and* $(\gamma_\mu, \beta_\lambda)$*-closed.*

(i) *If* f *is injective and* (Y, λ) *is* $\beta\lambda$-$T_{\frac{1}{2}}$*, then* (X, μ) *is* γ_μ-$T_{\frac{1}{2}}$.

(ii) *If* f *is surjective and* (X, μ) *is* γ_μ-$T_{\frac{1}{2}}$*, then* (Y, λ) *is* β_λ-$T_{\frac{1}{2}}$.

Proof. (i) Let A be a $\gamma_\mu g$-closed set of (X, μ). Then by the last theorem, $f(A)$ is $\beta_\lambda g$-closed in Y and hence $f(A)$ is β_λ-closed. Since f is $(\gamma_\mu, \beta_\lambda)$-irresolute, by Theorem 7, $A = f^{-1}(f(A))$ is γ_μ-closed and hence (X, μ) is γ_μ-$T_{\frac{1}{2}}$.

(ii) Let B be any $\beta_\lambda g$-closed set in (Y, λ). It is sufficient to show that B is a β_λ-closed set. By Theorem 8, $f^{-1}(B)$ is a $\gamma_\mu g$-closed set in X. Since (X, μ) is a γ_μ-$T_{\frac{1}{2}}$, $f^{-1}(B)$ is γ_μ-closed. Since f is surjective and $(\gamma_\mu, \beta_\lambda)$-closed, $f(f^{-1}(B)) = B$ is a β_λ-closed set. □

Theorem 10 *Let* $f : (X, \mu) \to (Y, \lambda)$ *be a surjective* $(\gamma_\mu, \beta_\lambda)$*-irresolute function. If* E *be a* β_λ-D_λ *set in* Y*, then* $f^{-1}(E)$ *is a* γ_μ-D_μ *set in* (X, μ).

Proof. Let E be a β_λ-D_λ set in Y. Then there exist two β_λ-open sets U_1 and U_2 in Y such that $E = U_1 \setminus U_2$ and $U_1 \neq Y$. Now by Theorem 7, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are γ_μ-open and $f^{-1}(U_1) \neq X$ (as f is surjective and $U_1 \neq Y$). Thus $f^{-1}(E) = f^{-1}(U_1) \setminus f^{-1}(U_2)$ is a γ_μ-D_μ set. □

Theorem 11 *If* (Y, λ) *is* β_λ-D_1 *and* $f : (X, \mu) \to (Y, \lambda)$ *is a* $(\gamma_\mu, \beta_\lambda)$*-irresolute bijective function, then* (X, μ) *is* γ_μ-D_1.

Proof. Suppose that (Y, λ) is a β_λ-D_1 space. Let x and y be any two distinct points of X. Since f is injective and Y is β_λ-D_1, there exist β_λ-D_λ sets G_x and G_y of Y containing $f(x)$ and $f(y)$ respectively such that $f(x) \notin G_y$ and $f(y) \notin G_x$. Thus by Theorem 10, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are γ_μ-D_μ sets containing x and y respectively such that $x \notin f^{-1}(G_y)$ and $y \notin f^{-1}(G_x)$. Thus X a γ_μ-D_1 space. □

Theorem 12 *A GTS* (X, μ) *is* γ_μ-D_1 *if for each distinct points* x *and* y *in* X*, there exists a* $(\gamma_\mu, \beta_\lambda)$*-irresolute surjective function* $f : (X, \mu) \to (Y, \lambda)$*, where* (Y, λ) *is a* β_λ-D_1 *space such that* $f(x)$ *and* $f(y)$ *are distinct.*

Proof. Let x and y be two distinct points of X. By hypothesis, there exists a $(\gamma_\mu, \beta_\lambda)$-irresolute function f on X onto a β_λ-D_1 space Y such that $f(x) \neq f(y)$. Then there exist β_λ-D_λ sets G_x and G_y containing $f(x)$ and $f(y)$ respectively such that $f(x) \notin G_y$ and $f(y) \notin G_x$. As f is surjective and $(\gamma_\mu, \beta_\lambda)$-irresolute, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are γ_μ-D_μ sets in X (by Theorem 10) containing x and y respectively such that $x \notin f^{-1}(G_y)$ and $y \notin f^{-1}(G_x)$. Hence X is a γ_μ-D_1 space. □

Conclusion: If we replace μ by different GT's or γ_μ by different operators, we can obtain various forms of generalized closed sets and related continuous functions.

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