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# On $\gamma-$countably paracompact sets

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**Abstract.** In this paper we introduce and study a new class of sets, namely $\gamma-$countably paracompact sets. We characterize $\gamma-$countably paracompact sets and we study some of its basic properties. We obtain that this class of sets is weaker than $\alpha-$countably paracompact sets and stronger than $\beta-$countably paracompact sets.

## 1 Introduction

In [3] C. E. Aull, presented and studied the concept of $\alpha-$countably paracompact and $\beta-$countably paracompact sets. In connection with the definition of $\alpha-$countably paracompact sets and $\beta-$countably paracompact sets we obtain the definition of $\gamma-$countably paracompact sets. In section 2 of this work, we

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present $\gamma-$countably paracompact sets and then we investigate several characterizations to this types of sets and study some of its basic properties. In section 3 of this work, some of relationships between $\gamma-$countably paracompact sets and other well-known sets are investigated. In particular, we show that this class of sets lies between the classes of $\alpha-$countably paracompact sets and $\beta-$countably paracompact sets. Finally, in section 4, we introduce a class of spaces namely locally $\gamma-$countably paracompact spaces characterized by $\gamma-$countably paracompact sets and study some of their fundamental properties.

Throughout this work a space will always mean a topological space on which no separation axiom is assumed unless explicitly stated. Let $(X, \tau)$ be a space and $A$ be a subset of $X$. The closure of $A$, interior of $A$ and the relative topology on $A$ in $(X, \tau)$ will be denoted by $cl(A)$, $int(A)$ and $\tau_A$, respectively. A space $(X, \tau)$ is called countably paracompact [4] if every countable open cover of $X$ has an open locally finite refinement. Now we begin with some known notions and definitions which will be used in this work.

**Definition 1** [5] *A subset $A$ of a space $(X, \tau)$ is called generalized closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.*

**Theorem 1** [4] *If $A$ is dense in $X$, then for every open $U \subseteq X$ we have $cl(U) = cl(U \cap A)$.*

**Definition 2** *Let $A, B, C$ and $Y$ be subsets of a space $(X, \tau)$. Then:*

i. *A cover $\mathcal{U}$ of $Y$ is called an $A-$open cover of $Y$ [1] if $Y \subseteq \underset{U \in \mathcal{U}}{\cup} U$ and $U$ is open in $(A, \tau_A)$ for every $U \in \mathcal{U}$.*

ii. *A collection $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ is called $A-$locally finite[1] if $\mathcal{U}$ is locally finite in $(A, \tau_A)$.*

iii. *If $\mathcal{U}$ and $\mathcal{V}$ are covers of $Y$, then $\mathcal{V}$ is called $A-$refinement of $\mathcal{U}$ [1] if for every $V \in \mathcal{V}$, $V \subseteq A$ and there exists $U \in \mathcal{U}$ such that $V \subseteq U$. If for every $V \subseteq \mathcal{V}$, $V$ is open in $(A, \tau_A)$ then $\mathcal{V}$ is called an $A$-open refinement of $\mathcal{U}$.*

iv. *$Y$ is called $\alpha-$countably paracompact of $(X, \tau)$ [3] if every countable open cover of $Y$ by members of $\tau$ has a locally finite open refinement by members of $\tau$.*

v. *$Y$ is called $\beta-$countably paracompact of $(X, \tau)$ [3] if $(Y, \tau_Y)$ is countably paracompact as a subspace.*

The proof of the following proposition is obvious.

**Proposition 1** *Let* $Y$ *be a subset of a topological space* $(X, \tau)$. *If a collection* $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ *is* $X-$*locally finite, then* $\mathcal{U}$ *is* $Y-$*locally finite.*

The following example shows that the converse of the above proposition is not true in general.

**Example 1** *Let* $X = \mathbb{R}$ *with the topology* $\tau = \{U : 0 \notin U\} \cup \{\mathbb{R}\}$. *Put* $Y = \mathbb{Q}^* = \mathbb{Q} - \{0\}$. *Then the collection* $\{\{y\} : y \in Y\}$ *is* $Y-$*locally finite but it is not* $X-$*locally finite.*

In the following proposition, we shall show when the converse of the above proposition is true.

**Proposition 2** *Let* $Y$ *be a closed subset of a topological space* $(X, \tau)$. *If* $\mathcal{U} = \{U_\alpha : \alpha \in I, U_\alpha \subseteq Y\}$ *is a* $Y-$*locally finite collection of subsets of* $Y$, *then* $\mathcal{U}$ *is* $X-$*locally finite.*

**Proof.** Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be $Y-$locally finite such that $U_\alpha \subseteq Y$ for each $\alpha \in I$. If $x \in X$, then either $x \in Y$ or $x \notin Y$. If $x \in Y$, then there exists an open set $W$ in $(Y, \tau_Y)$ such that $x \in W$ and $W$ intersects at most finitely many members of $\mathcal{U}$. Now $W = M \cap Y$ for some $M \in \tau$. As $\mathcal{U}$ is a collection of subsets of $Y$, so $M$ intersects at most finitely many members of $\mathcal{U}$. Now if $x \notin Y$, then $X - Y$ is open in $(X, \tau)$ containing $x$ which intersects no member of $\mathcal{U}$. $\square$

**Corollary 1** *Let* $Y$ *be a closed subset of a topological space* $(X, \tau)$. *The collection* $\{U_\alpha : \alpha \in I, U_\alpha \subseteq Y\}$ *is* $Y-$*locally finite iff* $\mathcal{U}$ *is* $X-$*locally finite.*

## 2 $\gamma-$countably paracompact sets

In this section we shall present the concept of $\gamma-$countably paracompact sets.

**Definition 3** *Let* $A, B, C$ *and* $Y$ *be subsets of a space* $(X, \tau)$. *Then* $Y$ *is called* $ABC-$*countably paracompact set of* $(X, \tau)$, *if every countable* $A-$*open cover of* $Y$ *has a* $B-$*locally finite* $C-$*open refinement.*

Note that a subset $Y$ of a space $(X, \tau)$ is $\alpha-$countably paracompact iff it is $XXX-$countably paracompact and it is $\beta-$countably paracompact iff it is $YYY-$countably paracompact.

**Definition 4** *Let* $Y$ *be a subset of a topological space* $(X,\tau)$*. Then* $Y$ *is called* $\gamma-$*countably paracompact if it is* $XXY-$*countably paracompact.*

**Example 2** *Let* $X=\mathbb{R}$ *with the topology* $\tau=\{U:\mathbb{R}-\mathbb{Q}\subseteq U\}\cup\{\phi\}$*. Then* $Y=\mathbb{Q}$ *is* $\gamma-$*countably paracompact.*

**Proposition 3** *Let* $Y$ *be a subset of a topological space* $(X,\tau)$*. Then* $Y$ *is* $\gamma-$*countably paracompact iff for every countable* $X-$*open cover* $\mathcal{U}=\{U_n:n\in\mathbb{N}\}$ *of* $Y$ *there exists an* $X-$*locally finite* $Y-$*open cover* $\mathcal{V}=\{V_n:n\in\mathbb{N}\}$ *of* $Y$ *such that* $V_n\subseteq U_n$ *for* $n=1,2,....$

**Proof.** Let $Y$ be a $\gamma-$countably paracompact set. If $\mathcal{U}=\{U_n:n\in\mathbb{N}\}$ is a countable $X-$open cover of $Y$, then there exists an $X-$locally finite $Y-$open refinement of $\mathcal{U}$, say $\mathcal{W}$. So for every $W\in\mathcal{W}$ choose a natural number $n(W)$ such that $W\subseteq U_{n(W)}$. Then define $V_n=\underset{n\in\mathbb{N}}{\cup}\{W:n(W)=n\}$. Hence $\mathcal{V}=\{V_n:n\in\mathbb{N}\}$ is open in $Y$ and it is $X-$locally finite such that $V_n\subseteq U_n$ for $n=1,2,....$ □

**Proposition 4** *Let* $Y$ *be a subset of a topological space* $(X,\tau)$*. If* $Y$ *is* $\gamma-$*countably paracompact, then for every increasing countable* $X-$*open cover* $\mathcal{U}=\{U_n:n\in\mathbb{N}\}$ *of* $Y$ *there exists a* $Y-$*open cover* $\mathcal{V}=\{V_n:n\in\mathbb{N}\}$ *of* $Y$ *such that* $cl_Y(V_n)\subseteq U_n$ *for* $n=1,2,....$

**Proof.** Let $\mathcal{U}=\{U_n:n\in\mathbb{N}\}$ be an increasing countable $X-$open cover of $Y$. Then, by Proposition 3, there exists an $X-$locally finite $Y-$open cover $\mathcal{V}=\{V_n:n\in\mathbb{N}\}$ of $Y$ such that $V_n\subseteq U_n$. To show that $cl_Y(V_n)\subseteq U_n$, set $F_n=Y-\underset{m>n}{\cup}V_m$. Then $F_n$ is closed in $Y$ such that $V_n\subseteq F_n\subseteq\underset{m\leq n}{\cup}V_m\subseteq U_n$ and so $cl_Y(V_n)\subseteq U_n$. □

**Proposition 5** *Let* $Y$ *be a subset of a topological space* $(X,\tau)$ *and suppose that for every countable* $X-$*open cover* $\mathcal{U}=\{U_n:n\in\mathbb{N}\}$ *of* $Y$ *there exists an* $X-$*locally finite* $Y-$*open cover* $\mathcal{V}=\{V_n:n\in\mathbb{N}\}$ *of* $Y$ *such that* $V_n\subseteq U_n$ *for* $n=1,2,...$ *If* $W_1\subseteq W_2\subseteq...$ *is an increasing sequence of open sets in* $X$ *such that* $\underset{n\in\mathbb{N}}{\cup}W_n=Y$*, then there exists a sequence* $F_1\subseteq F_2\subseteq...$ *of closed subsets of* $Y$ *such that* $F_n\subseteq W_n$ *for* $n=1,2,...$ *and* $\underset{n\in\mathbb{N}}{\cup}int_Y(F_n)=Y$ *.*

**Proof.** Let $W_1\subseteq W_2\subseteq...$ be an increasing sequence of $X-$open sets such that $\underset{n\in\mathbb{N}}{\cup}W_n=Y$. Then, there exists an $X-$locally finite $Y-$open cover $\mathcal{V}=\{V_n:$

$n \in \mathbb{N}\}$ of $Y$ such that $V_n \subseteq W_n$ for all $n$. Now, define $F_n = Y - \underset{j>n}{\cup} V_j$ which is closed in $Y$ and for $n \in \mathbb{N}$ we have $F_n \subseteq \underset{j\leq n}{\cup} V_j \subseteq \underset{j\leq n}{\cup} W_j = W_n$. To show that $\underset{n\in\mathbb{N}}{\cup} int_Y(F_n) = Y$, it is enough to show that $Y \subseteq \underset{n\in\mathbb{N}}{\cup} int_Y(F_n)$. Let $y \in Y$. Then there exists an open $O$ in $(X, \tau)$ such that $y \in O$ and $(O \cap Y) \cap \underset{m>n}{\cup} V_m = \phi$ for some $n \in \mathbb{N}$, so we have $y \in O \cap Y \subseteq Y - \underset{m>n}{\cup} V_m = F_n$. Hence $y \in \underset{n\in\mathbb{N}}{\cup} int_Y(F_n)$. □

**Proposition 6** *Let $Y$ be a closed subset of a topological space $(X, \tau)$. Then every countable $X-$open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of $Y$ has an $X-$locally finite $Y-$open cover $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ such that $V_n \subseteq U_n$ for all $n$ iff for every increasing sequence $W_1 \subseteq W_2 \subseteq ...$ of open sets in $X$ such that $Y = \underset{n\in\mathbb{N}}{\cup} W_n$ there exists a sequence $F_1 \subseteq F_2 \subseteq ...$ of closed subsets of $Y$ such that $F_n \subseteq W_n$ for $n = 1, 2, ...$, moreover $\underset{n\in\mathbb{N}}{\cup} int_Y(F_n) = Y$.*

**Proof.** We show the sufficiency part. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable $X-$open cover of $Y$. Set $W_n = \underset{j\leq n}{\cup} U_j$. Then $W_1 \subseteq W_2 \subseteq ...$, such that $\underset{n\in\mathbb{N}}{\cup} W_n = Y$. So there exists $F_1 \subseteq F_2 \subseteq ...$ of closed subsets of $Y$ such that $F_n \subseteq W_n$ for $n = 1, 2, ...$ and $\underset{n\in\mathbb{N}}{\cup} int_Y(F_n) = Y$. Define $V_n = (U_n \cap Y) - \underset{j<n}{\cup} F_j$. Then $V_n$ is open in $Y$ and $V_n \subseteq U_n$ for $n = 1, 2, ....$ To show that $\underset{n\in\mathbb{N}}{\cup} V_n = Y$, let $y \in Y$ and $j$ be the first index such that $y \in (U_j \cap Y)$. Therefore, $y \in V_j$. To complete the proof we show that $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ is $Y-$locally finite. Let $y \in Y$. Then there exists $j$ such that $x \in int_Y(F_j)$ and $int_Y(F_j) \cap V_n = \phi$ for $n > j$. Therefore, $\mathcal{V}$ is $Y-$locally finite and so by Proposition 2, $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ is $X-$locally finite. □

From above discussion we can get the following Theorem.

**Theorem 2** *Let $Y$ be a closed subset of a topological space $(X, \tau)$. Then the following are equivalent:*

i. $Y$ *is $\gamma-$countably paracompact.*

ii. *For every countable $X-$open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of $Y$, there exists an $X-$locally finite $Y-$open cover $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ of $Y$ such $V_n \subseteq U_n$ for all $n$.*

iii. *For every increasing sequence* $W_1 \subseteq W_2 \subseteq ...$ *of open sets in* $X$ *such that* $\underset{n\in\mathbb{N}}{\cup} W_n = Y$*, there exists* $F_1 \subseteq F_2, ...$ *of closed subsets of* $Y$ *such that* $F_n \subseteq W_n$ *for* $n = 1, 2, ...,$ *moreover* $\underset{n\in\mathbb{N}}{\cup} \mathrm{int}_Y(F_n) = Y$.

iv. *For every increasing countable* $X-$*open cover* $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ *of* $Y$*, there exists a* $Y-$*open cover* $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ *of* $Y$ *such that* $\mathrm{cl}_Y(V_n) \subseteq U_n$ *for* $n = 1, 2, ....$

v. *For every decreasing* $X-$*closed collection* $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$ *such that* $(\underset{n\in\mathbb{N}}{\cap} F_n) \cap Y = \phi$*, there exists a* $Y-$*open cover* $\mathcal{O} = \{O_n : n \in \mathbb{N}\}$ *of* $Y$ *such that* $\mathrm{cl}_Y(O_n) \cap F_n = \phi$ *for* $n = 1, 2, ....$

**Proof.** Only we prove $(iv \to i)$. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable $X-$open cover of $Y$. Define $W_n = \underset{j\leq n}{\cup} U_j$. Then the collection $\{W_n : n \in \mathbb{N}\}$ is an increasing countable $X-$open cover of $Y$, by $(iv)$, there exists a $Y-$open cover $\{V_n : n \in \mathbb{N}\}$ of $Y$ such that $\mathrm{cl}_Y(V_n) \subseteq W_n$. Define $O_n = (U_n \cap Y) - \underset{j<n}{\cup} \mathrm{cl}_Y(V_j)$. Then $\{O_n : n \in \mathbb{N}\}$ is an $X-$locally finite $Y-$open refinement of $\mathcal{U}$. □

To identify more characterization of $\gamma-$countably paracompact we need the following theorem.

**Theorem 3** *Let* $Y$ *be a* $\gamma-$*countably paracompact set in a space* $(X, \tau)$*. If* $F$ *is a generalized closed subset of* $(X, \tau)$ *such that* $F \subseteq Y$*, then* $F$ *is* $\gamma-$*countably paracompact set in* $(X, \tau)$.

**Proof.** Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable $X-$open cover of $F$. Then $F \subseteq \underset{n\in N}{\cup} \mathcal{U} = U$. Since $F$ is a generalized closed subset in $(X, \tau)$ and $U$ is open in $X$, then $\mathrm{cl}(F) \subseteq U$. Therefore, the collection $(X - \mathrm{cl}(F)) \cup \{U_n : n \in \mathbb{N}\}$ is an $X-$open cover of the $\gamma-$countably paracompact set $Y$ and so it has an $X-$locally finite open refinement, say $\mathcal{V}^*$. Put $\mathcal{V} = \{V \in \mathcal{V}^* : \exists\ U_V \in \mathcal{U}$ such that $V \subseteq U_V\}$. Finally, define $\mathcal{W} = \{V \cap F : V \in \mathcal{V}\}$. Then, it is clear that $\mathcal{W}$ is $X-$locally finite and it is $F-$open refinement of $\mathcal{U}$ since for each $V \in \mathcal{V}$ there exists an open $O_V$ in $(X, \tau)$ such that $V = O_V \cap Y$ and so $V \cap F = O_V \cap Y \cap F = O_V \cap F$, which is open in $F$. □

**Corollary 2** *If* $F \subseteq Y \subseteq X$ *such that* $Y$ *is a* $\gamma-$*countably paracompact set and* $F$ *is a closed set in* $(X, \tau)$*. Then* $F$ *is* $\gamma-$*countably paracompact set in* $(X, \tau)$.

**Corollary 3** *A closed subset of a countably paracompact space is* $\gamma-$*countably paracompact set.*

Let $\{(X_\alpha, \tau_\alpha) : \alpha \in I\}$ be a collection of topological spaces such that $X_\alpha \cap X_\beta = \phi$ for each $\alpha \neq \beta$. Let $X = \underset{\alpha \in I}{\cup} X_\alpha$ be topologized by $\tau_s = \{G \subseteq X : G \cap X_\alpha \in \tau_\alpha$ for each $\alpha \in I\}$. Then $(X, \tau_s)$ is called the sum of the spaces $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ and we write $X = \underset{\alpha \in I}{\oplus} X_\alpha$.

**Theorem 4** *Let $A_\alpha \subseteq X$ for all $\alpha \in I$ and $A = \bigcup_{\alpha \in I} A_\alpha$. Then $A$ is $\gamma-$countable paracompact set in $X$ iff $A_\alpha$ is $\gamma-$countable paracompact set in $X_\alpha$ for all $\alpha \in I$.*

**Proof.** Let $\alpha \in I$ and $\mathcal{U}$ be a countable $X_\alpha-$open cover of $A_\alpha$. Then the collection $\{U : U \in \mathcal{U}\} \cup (\underset{\beta \neq \alpha}{\cup} X_\beta)$ is a countable $X-$open cover of the $\gamma-$countable paracompact set $A$ and so it has an $X-$locally finite $A-$open refinement, say $\mathcal{V}$. Put $\mathcal{V}_\mathcal{U} = \{V \cap A_\alpha : V \in \mathcal{V}$ and $V \subseteq U$ for some $U \in \mathcal{U}\}$. It is clear that $\mathcal{V}_\mathcal{U}$ is $X_\alpha-$ locally finite $A_\alpha-$open collection such that $\mathcal{V}_\mathcal{U} < \mathcal{U}$. To show that $\mathcal{V}_\mathcal{U}$ is a cover for $A_\alpha$. Let $x_\alpha \in A_\alpha$, then there exists $V \in \mathcal{V}$ such that $x_\alpha \in V$. Since $x_\alpha \notin X_\beta$ for all $\beta \neq \alpha$, then $V \subseteq U$ for some $U \in \mathcal{U}$ and $x_\alpha \in V \cap A_\alpha$. Conversely, Let $\mathcal{U}$ be a countable $X-$open cover of $A$. For all $\alpha \in I$, the collection $\mathcal{U}_\alpha = \{U \cap X_\alpha : U \in \mathcal{U}\}$ is a countable $X_\alpha-$open cover of the $\gamma-$countable paracompact set $A_\alpha$ in $X_\alpha$, so it has an $X_\alpha-$locally finite $A_\alpha-$open refinement, say $\mathcal{W}_\alpha$. For all $W \in \mathcal{W}_\alpha$, there exists an open set $H_{\alpha(W)}$ in $X_\alpha$ such that $W = A_\alpha \cap H_{\alpha(W)} = A \cap H_{\alpha(W)}$. Put $\mathcal{H} = \{W : W \in \mathcal{W}_\alpha, \alpha \in I\}$. Then, it is clear that $\mathcal{H}$ is an $A-$open refinement of $\mathcal{U}$. To show that $\mathcal{H}$ is $X-$locally finite, let $x \in X$. Then there exists $\alpha_\circ \in I$ such that $x \in X_{\alpha_\circ}$ and $x \notin X_\beta$ for all $\beta \neq \alpha$. Since $\mathcal{W}_{\alpha_\circ}$is $X_{\alpha_\circ}-$locally finite, then there exists an open set $K$ in $X_{\alpha_\circ}$(and so in $X$) such that $K$ is intersect at most finitely many numbers of $\mathcal{W}_{\alpha_\circ}$ and $K \cap W = \phi$ for all $W \in \mathcal{W}_\alpha, \alpha \neq \alpha_\circ$. Therefore, $\mathcal{H}$ is $X-$locally finite and so $A$ is $\gamma-$countable paracompact set in $X$. □

**Theorem 5** *Let $f : X \to Y$ be a perfect onto function and let $B$ be a $\gamma-$countably paracompact set in the space $(Y, \sigma)$. Then $f^{-1}(B)$ is $\gamma-$countably paracompact set in $(X, \tau)$.*

**Proof.** Let $\mathcal{U}$ =$\{U_n : n \in \mathbb{N}\}$ be a countable $X-$open cover of $f^{-1}(B)$. For each $y \in B$, $\mathcal{U}$ is an $X-$open cover of the compact set $f^{-1}(y)$, so there exists a finite subset $\{U_1, U_2, ... U_n\}$ of $\mathcal{U}$ such that $f^{-1}(y) \subseteq \overset{n}{\underset{i=1}{\cup}} U_i = U_y$ and $U_y$ is open in $(X, \tau)$. Put $V_y = Y - f(X - U_y)$. Since $f$ is closed then the collection $\mathcal{V} = \{V_y : y \in B\}$ is a countable $Y-$open cover of $B$, and so it has a $Y-$locally finite $B-$open refinement, say $\mathcal{W} = \{W_j : j = 1, 2, ...\}$. Since $f$ is continuous, the family $f^{-1}(\mathcal{W}) = \{f^{-1}(W_j) : j = 1, 2, ...\}$ is an $X-$locally finite $f^{-1}(B)-$open cover of $f^{-1}(B)$ such that for each $j = 1, 2, ...$ $f^{-1}(W_j) \subseteq U_{y_j}$ for some $y_j \in B$. Finally, the collection $\{f^{-1}(W_j) \cap U_i : j = 1, 2, ... , i \in i_{y_j}\}$ is an $X-$locally finite $f^{-1}(B)-$open refinement of $\mathcal{U}$. Therefore, $f^{-1}(B)$ is $\gamma-$countably paracompact. □

An $E_1$ space [2] is a topological space such that every point is the intersection of a countable number of closed neighborhoods. Note that in [2] show that every $E_1$ space is $T_2$.

**Theorem 6** *Every $\gamma-$countable paracompact subset of $E_1$ space is closed.*

**Proof.** Let $Y$ be a $\gamma-$countably paracompact subset of an $E_1$ space $(X, \tau)$ and let $x \notin Y$. Let $\{C_n : n \in \mathbb{N}\}$ be a countable family of closed neighborhoods of $x$ such that $\{x\} = \cap C_n$. Now, $\{X - C_n : n \in \mathbb{N}\}$ is a countable $X-$open cover of $Y$ and $x \notin cl(X - C_n)$ for any $n$. Hence there is an $X-$locally finite $Y-$open refinement of $\{X - C_n : n \in \mathbb{N}\}$, say $\mathcal{W}$. Put $H = \cup\{W : W \in \mathcal{W}\}$, then $cl(H) = \cup\{cl(W) : W \in \mathcal{W}\}$. Finally, put $H^* = X - cl(H)$. So $H^*$ is open in $(X, \tau)$ such that $x \in H^*$ and $H^* \cap Y = \phi$. Therefore, $x \notin cl(Y)$ and $Y$ is closed. □

## 3 The relationship between $\alpha-$countably paracompact, $\beta-$countably paracompact and $\gamma-$countably paracompact sets

In this section we study the relationship between $\alpha-$countably paracompact, $\beta-$countably paracompact and $\gamma-$countably paracompact sets.

It follows from the definition that every $\alpha-$countably paracompact set is $\gamma-$countably paracompact and every $\gamma-$countably paracompact set is $\beta-$countably paracompact. The following two examples show that the converse are not true in general.

**Example 3** *Let $X = \mathbb{R}$ with the topology $\tau = \{U : \mathbb{R} - \mathbb{Q} \subseteq U\} \cup \{\phi\}$. Put $Y = \mathbb{Q}$. Then $Y$ is $\gamma-$countably paracompact, note that if $\mathcal{U}$ is a countably $X-$open cover of $Y$, then the collection $\{\{y\} : y \in Y\}$ is an $X-$locally finite $Y-$open refinement of $\mathcal{U}$. Now, to show $Y$ is not $\alpha-$countably paracompact, let $\mathcal{U} = \{(\mathbb{R} - \mathbb{Q}) \cup \{y\} : y \in Y\}$. Then $\mathcal{U}$ is a countable $X-$open cover of $Y$. If $\mathcal{V}$ is an $X-$locally finite $X-$open refinement of $\mathcal{U}$, then for every $y \in Y$ there exists $y \in V \in \mathcal{V}$ such that $y \in V \subseteq (\mathbb{R} - \mathbb{Q}) \cup \{y\}$. Thus, $V = (\mathbb{R} - \mathbb{Q}) \cup \{y\}$ which means $\mathcal{V}$ is not $X-$locally finite.*

**Example 4** *Let* $X = \mathbb{R}$ *with the topology* $\tau = \{U : 0 \notin U\} \cup \{\mathbb{R}\}$. *Then* $Y = \mathbb{Q}^* = \mathbb{Q} - \{0\}$ *is* $\beta-$*countably paracompact, since* $\tau_Y = \tau_{dis}$. *On the other hand,* $Y$ *is not* $\gamma-$*countably paracompact, since* $\mathcal{U} = \{\{y\} : y \in Y\}$ *is a countable* $X-$*open cover of* $Y$ *by members of* $\tau$ *and it is not* $X-$*locally finite.*

So what are the additional conditions that make the reversal of previous relationships true? This is what will be shown in the following Theorem.

**Theorem 7** [3] *Let* $Y$ *be a closed* $\beta-$*countably paracompact set in a normal space. Then* $Y$ *is* $\alpha-$*countably paracompact*

**Theorem 8** *Let* $Y$ *be a* $\gamma-$*countably paracompact set in a space* $(X, \tau)$. *Then* $Y$ *is* $\alpha-$*countably paracompact if one of the following holds:*

i. $Y$ *is closed in the normal space* $(X, \tau)$.

ii. $Y$ *is open set in the space* $(X, \tau)$.

**Proof.** The proof of (ii) is clear. The proof of (i) follows by Theorem 7 and from the fact that every $\gamma-$countably paracompact set is $\beta-$countably paracompact. □

**Theorem 9** *Let* $Y$ *be a closed* $\beta-$*countably paracompact set in a space* $(X, \tau)$. *Then* $Y$ *is* $\gamma-$*countably paracompact.*

**Proof.** Let $Y$ be a closed $\beta-$countably paracompact subset of $(X, \tau)$ and let $\mathcal{U}$ be a countable $X-$open cover of $Y$. Then the collection $\mathcal{W} = \{U \cap Y : U \in \mathcal{U}\}$ is a countable $Y-$open cover of $Y$ and so it has a $Y-$locally finite $Y-$open refinement say $\mathcal{V}$. Since $Y$ is closed set, by Proposition 2, $\mathcal{V}$ is $X-$locally finite. Also as for every $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subseteq U \cap Y \subseteq U \in \mathcal{U}$, so $\mathcal{V}$ is $X-$locally finite $Y-$open refinement of $\mathcal{U}$. Hence $Y$ is $\gamma-$countably paracompact. □

**Corollary 4** *Let* $Y$ *be closed in a normal space* $(X, \tau)$. *The following are equivalent:*

i. $Y$ *is* $\gamma-$*countably paracompact.*

ii. $Y$ *is* $\alpha-$*countably paracompact.*

iii. $Y$ *is* $\beta-$*countably paracompact.*

# 4 Locally $\gamma-$countably paracompact spaces

In this section we introduce locally $\gamma-$countably paracompact spaces and we study their properties.

**Definition 5** *A space $(X, \tau)$ is called locally $\gamma-$ countably paracompact if each point $x \in X$ has an open neighborhood $U$ in $(X, \tau)$ such that $cl(U)$ is $\gamma-$countably paracompact in $(X, \tau)$.*

The following result follow immediately from Theorem 9.

**Proposition 7** *Let $(X, \tau)$ be a space. Then $(X, \tau)$ is locally $\gamma-$ countably paracompact iff for all $x \in X$ there exists an open neighborhood $U$ in $(X, \tau)$ such that $cl(U)$ is $\beta-$countably paracompact.*

**Theorem 10** *Every closed subspace of a locally $\gamma-$countably paracompact space is locally $\gamma-$countably paracompact.*

**Proof.** Let $F$ be a closed subspace of a locally $\gamma-$countably paracompact space $(X, \tau)$. For every $x \in F$, there exists an open neighborhood $U$ of the point $x$ in the space $(X, \tau)$ such that $cl(U)$ is $\gamma-$countably paracompact space. The intersection $F \cap U$ is an open neighborhood of the point $x$ in the subspace $F$ and, by Corollary 3, $cl_F(F \cap U) = cl(F \cap U) \cap F = cl(F \cap U)$ is $\gamma-$countably paracompact, being a closed subset of the $\gamma-$countably paracompact set $cl(U)$, by Theorem 3. □

**Theorem 11** *Every locally $\gamma-$countably paracompact $E_1$ space is $T_3$.*

**Proof.** Let $F$ be a closed subset of a locally $\gamma-$countably paracompact space $(X, \tau)$ and $x \notin F$. Let $cl(P_x)$ be the $\gamma-$countably paracompact such that $P_x$ is neighborhood of $x$ and let $\{C_n : n \in N\}$ be a countable family of closed neighborhood of $x$ such that $\{x\} = \cap C_n$. Put $H = cl(P_x) \cap F$. Then, by Theorem 3, $H$ is $\gamma-$countably paracompact set such that $x \notin H$.Thus the collection $\{X - C_n : n \in N\}$ is a countable $X-$open cover of $H$ and so it has an $X-$locally finite $H-$open refinement, say $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$. Since $H$ is closed in $X$, then $V = (\cup U_\alpha) \cup (X - cl(P_x))$ is an open set containing $F$ such that $x \notin cl(V)$. Hence $(X, \tau)$ is regular. □

**Example 5** *Let the Hausdroff neighborhoods of a point $p$ in the Euclidean plane consist of open circles with $p$ at the center excluding the points on the* *vertical diameters except* $p$ *itself. Since the resulting topology is a strengthening of the usual topology of the Euclidean plane it is an* $E_1$ *topology* ([2], Example 2)*. Since this is a* $T_2$*space which is not* $T_3$*, it can not be locally* $\gamma-$*countably paracompact.*

**Lemma 1** *Let* $Y$ *be an* $\alpha-$*countably paracompact Lindelöf subset of a regular locally* $\gamma-$*countably paracomact space* $X$*. If* $W$ *is an open set in* $(X, \tau)$ *such that* $Y \subseteq W$*, then there is an* $X-$*locally finite collection* $\{F_n : n \in \mathbb{N}\}$ *of closed* $\gamma-$*countably paracompact sets such that* $Y \subseteq \bigcup_{n\in\mathbb{N}} int(F_n) \subseteq \bigcup_{n\in\mathbb{N}} F_n \subseteq W$*.*

**Proof.** By the regularity of the space $X$, then for every $x \in Y$, there exists an open set $U_x$ in $X$ such that $x \in U_x \subseteq cl(U_x) \subseteq W$. On the other hand, $X$ is locally $\gamma-$countably paracompact space and so there exists an open set $H_x$ in $X$ such that $cl(H_x)$ is $\gamma-$countably paracompact set. Put $V_x = cl(H_x) \cap cl(U_x)$. Then, by Theorem 3, $V_x$ is a closed $\gamma-$countably paracompact set such that $x \in int(V_x) \subseteq W$. Therefore, the collection $\mathcal{V} = \{int(V_x) : x \in Y\}$ is an $X-$open cover of the Lindelöf set $Y$, so it has a countable subcover, say $\mathcal{V}^*$. Since $Y$ is $\gamma-$countably paracompact set, then $\mathcal{V}^*$ has an $X-$locally finite $X-$open refinement $\mathcal{H}$ which cover $Y$. Now, for every $H \in \mathcal{H}$, $cl(H)$ is a closed set in $X$ such that $cl(H) \subseteq V_x$ for some $x \in Y$ and so $cl(H)$, by Theorem 3, is $\gamma-$countably paracompact set. Thus, the collection $\{cl(H) : H \in \mathcal{H}\}$ is the required collection. □

**Theorem 12** *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a perfect function from a space* $(X, \tau)$ *onto a locally* $\gamma-$ *countably paracompact space* $(Y, \sigma)$*. Then* $(X, \tau)$ *is locally* $\gamma-$*countably paracompact.*

**Proof.** Let $x \in X$. Then there exists an open set $V$ in $(Y, \sigma)$ such that $f(x) \in V$ and $cl(V)$ is $\gamma-$countably paracompact in $(Y, \sigma)$. Now, by Theorem 5, $f^{-1}(cl(V))$ is $\gamma-$countably paracompact subset of $X$. Since $cl(f^{-1}(V)) \subseteq f^{-1}(cl(V))$, then by Theorem 3, $cl(f^{-1}(V))$ is $\gamma-$countably paracompact subset of $X$. □

**Corollary 5** *The product of a compact space* $(X, \tau)$ *and a locally* $\gamma-$*paracompact space* $(Y, \sigma)$ *is locally* $\gamma-$*countably paracompact.*

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