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On expansive homeomorphism of uniform spaces

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Abstract. We study the notion of expansive homeomorphisms on uniform spaces. It is shown that if there exists a topologically expansive homeomorphism on a uniform space, then the space is always a Hausdorff space and hence a regular space. Further, we characterize orbit expansive homeomorphisms in terms of topologically expansive homeomorphisms and conclude that if there exist a topologically expansive homeomorphism on a compact uniform space then the space is always metrizable.

1 Introduction

A homeomorphism $h : X \longrightarrow X$ defined on metric space X is said to be an *expansive homeomorphism* provided there exists a real number $c > 0$ such that whenever $x, y \in X$ with $x \neq y$ then there exists an integer n (depending on x, y) satisfying $d(h^n(x), h^n(y)) > c$. Constant c is called an *expansive constant* for h. In 1950, Utz, [18], introduced the concept of expansive homeomorphisms

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with the name unstable homeomorphisms. The examples discussed in this paper on compact spaces were sub dynamics of shift maps, thus one can say that the theory of expansive homeomorphisms started based on symbolic dynamics but it quickly developed by itself.

Much attention has been paid to the existence / non–existence of expansive homeomorphisms on given spaces. Each compact metric space that admits an expansive homeomorphism is finite-dimensional [13]. The spaces admitting expansive homeomorphisms include the Cantor set, the real line/half-line, all open $n-$cells, $n \geq 2$ [12]. On the other hand, spaces not admitting expansive homeomorphisms includes any Peano continuum in the plane [9], the 2-sphere the projective plane and the Klein bottle [8].

Another important aspects of expansive homeomorphism is the study of its various generalizations and variations in different setting. The very first of such variation was given by Schwartzman, [16], in 1952 in terms of positively expansive maps, wherein the points gets separated by non–negative iterates of the continuous map. In 1970, Reddy, [14], studied point–wise expansive maps whereas $h-$expansivity was studied by R. Bowen, [4]. Kato defined and studied the notion of continuum–wise expansive homeomorphism [10]. Shah studied notion of positive expansivity of maps on metric $G-$spaces [17] whereas Barzanouni studied finite expansive homeomorphisms [2]. Tarun Das *et al.* [7] used the notion of expansive homeomorphism on topological space to prove the Spectral Decomposition Theorem on non–compact spaces. Achigar *et al.* studied the notion of orbit expansivity on non–Hausdorff space [1]. Authors in [3] studied expansivity for group actions. In this paper we study expansive homeomorphisms on uniform spaces.

In Section 2 we discuss preliminaries regarding uniform spaces and expansive homeomorphisms on metric /topological space required for the content of the paper. The notion of expansive homeomorphisms on topological spaces was first studied in [7] whereas on uniform spaces was first studied in [6] in the form of positively topological expansive maps. In Section 3 of this paper we define and study expansive homeomorphism on uniform spaces. Through examples it is justified that topologically expansive homeomorphism is weaker than metric expansive homeomorphism whereas stronger than expansive homeomorphism defined on topological space. Further, we show that if a uniform space admits a topologically expansive homeomorphism then the space is always a Hausdorff space and hence a regular space. The notion of orbit expansivity was first introduced in [1]. A characterization of orbit expansive homeomorphism on compact uniform spaces is obtained in terms of topologically expansive homeomorphism. As a consequence of this we conclude that if there is a topologically expansive homeomorphism on a compact uniform space then the space is always metrizable.

2 Preliminaries

In this Section we discuss basics required for the content of the paper.

2.1 Uniform spaces

Uniform spaces were introduced by A. Weil [19] as a generalization of metric spaces and topological groups. Recall, in a uniform space X, the closeness of a pair of points is not measured by a real number, like in a metric space, but by the fact that this pair of points belong or does not belong to certain subsets of the cartesian product, $X \times X$. These subsets are called the *entourages* of the uniform structure.

Let X be a non-empty set. A relation on X is a subset of $X \times X$. If U is a relation, then the *inverse* of U is denoted by U^{-1} and is a relation given by

$$U^{-1} = \{(y, x) : (x, y) \in U\}.$$

A relation U is said to be *symmetric* if $U = U^{-1}$. Note that $U \cap U^{-1}$ is always a symmetric set. If U and V are relations, then the *composite of* U and V is denoted by $U \circ V$ and is given by

$$U \circ V = \{(x, z) \in X \times X : \exists\, y \in X \text{ such that } (x, y) \in V \ \&\ (y, z) \in U\}.$$

The set, denoted by $\triangle_X$, given by $\triangle_X = \{(x, x) : x \in X\}$ is called the *identity relation* or *the diagonal of* X. For every subset A of X the set $U[A]$ is a subset of X and is given by $U[A] = \{y \in X : (x, y) \in U,$ for some $x \in A\}$. In case if $A = \{x\}$ then we denote it by $U[x]$ instead of $U[\{x\}]$. We now recall the definition of uniform space.

Definition 1 *A* uniform structure (or uniformity) *on a set* X *is a non–empty collection* $\mathcal{U}$ *of subsets of* $X \times X$ *satisfying the following properties:*

1. *If* $U \in \mathcal{U}$*, then* $\triangle_X \subset U$.
2. *If* $U \in \mathcal{U}$*, then* $U^{-1} \in \mathcal{U}$.
3. *If* $U \in \mathcal{U}$*, then* $V o V \subseteq U$*, for some* $V \in \mathcal{U}$.
4. *If* U *and* V *are elements of* $\mathcal{U}$*, then* $U \cap V \in \mathcal{U}$.
5. *If* $U \in \mathcal{U}$ *and* $U \subseteq V \subseteq X \times X$*, then* $V \in \mathcal{U}$.

The pair $(X,\mathcal{U})$ *(or simply* X*) is called as a* uniform space.

Obviously every metric on a set X induces a uniform structure on X and every uniform structure on a set X defines a topology on X. Further, if the uniform structure comes from a metric, the associated topology coincides with the topology obtained by the metric. Also, there may be several different uniformities on a set X. For instance, the largest uniformity on X is the collection of all subsets of $X\times X$ which contains $\triangle_X$ whereas the smallest uniformity on X contains only $X\times X$. For more details on uniform spaces one can refer to [11].

Example 1 *Consider* $\mathbb{R}$ *with usual metric* d*. For every* $\epsilon > 0$*, let*

$$U_\epsilon^d := \left\{(x,y)\in\mathbb{R}^2 : d(x,y) < \epsilon\right\}$$

Then the collection

$$\mathcal{U}_d = \left\{E\subseteq\mathbb{R}^2 : U_\epsilon^d \subseteq E, \quad \textit{for some } \epsilon > 0\right\}$$

is a uniformity on $\mathbb{R}$*. Further, let* ρ *be an another metric on* $\mathbb{R}$ *given by* $\rho(x,y) = |e^x - e^y|, \ \ x,y\in\mathbb{R}$*. If for* $\epsilon > 0$*,*

$$U_\epsilon^\rho := \left\{(x,y)\in\mathbb{R}^2 : \rho(x,y) < \epsilon\right\}$$

then the collection

$$\mathcal{U}_\rho = \left\{E\subseteq\mathbb{R}^2 : U_\epsilon^\rho \subseteq E \ \ \textit{for some } \epsilon > 0\right\}$$

is also a uniformity on $\mathbb{R}$*. Note that these two uniformities are distinct as the set* $\{(x,y) : |x-y| < 1\}$ *is in* $\mathcal{U}_d$ *but it is not in* $\mathcal{U}_\rho$*.*

Let X be a uniform space with uniformity $\mathcal{U}$. Then, the natural topology, $\tau_{\mathcal{U}}$, on X is the family of all subsets T of X such that for every x in T, there is $U\in\mathcal{U}$ for which $U[x]\subseteq T$. Therefore, for each $U\in\mathcal{U}$, $U[x]$ is a neighborhood of x. Further, the interior of a subset A of X consists of all those points y of X such that $U[y]\subseteq A$, for some $U\in\mathcal{U}$. For the proof of this, one can refer to [11, Theorem 4, P.178]. With the product topology on $X\times X$, it follows that every member of $\mathcal{U}$ is a neighborhood of Δ_X in $X\times X$. However, converse need not be true in general. For instance, in Example 1 every element of $\mathcal{U}_d$ is a neighborhood of $\Delta_\mathbb{R}$ in $\mathbb{R}^2$ but $\left\{(x,y) : |x-y| < \frac{1}{1+|y|}\right\}$ is a neighborhood of $\Delta_\mathbb{R}$ but not a member of $\mathcal{U}_d$. Also, it is known that if X is a compact uniform space, then $\mathcal{U}$ consists of all the neighborhoods of the diagonal Δ_X [11]. Therefore for compact Hausdorff spaces the topology generated by different uniformities is unique and hence the only uniformity on X in this case is the natural uniformity. Proof of the following Lemma can be found in [11].

Lemma 1 *Let X be a uniform space with uniformity $\mathcal{U}$. Then the following are equivalent:*

1. *X is a T_1-space.*
2. *X is a Hausdorff space.*
3. *$\bigcap\{U : U \in \mathcal{U}\} = \Delta_X$.*
4. *X is a regular space.*

2.2 Various kind of expansivity on metric/topological spaces

Let X be a metric space with metric d and let $f : X \longrightarrow X$ be a homeomorphism. For $x \in X$ and a positive real number c, set

$$\Gamma_c(x, f) = \{y : d(f^n(x), f^n(y)) \leq c, \forall n \in \mathbb{Z}\}.$$

$\Gamma_c(x, f)$ is known as *the dynamical ball of x of size c*. Note that for each c, $\Gamma_c(x, f)$ is always non–empty. We recall the definition expansive homeomorphism defined by Utz [18].

Definition 2 *Let X be a metric space with metric d and let $f : X \longrightarrow X$ be a homeomorphism. Then f is said to be a* metric expansive homeomorphism*, if there exists $c > 0$ such that $\Gamma_c(x, f) = \{x\}$, for all $x \in X$. Constant c is known as an expansive constant for f.*

In the following we give some known example of metric expansive homeomorphisms.

Example 2

1. *Consider the set of real numbers $\mathbb{R}$ with usual metric d. For $\alpha \in \mathbb{R} \setminus \{0, 1, -1\}$, define $f_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ by $f_\alpha(x) = \alpha x$. Then f_α is a metric expansive homeomorphism with any positive real number c as an expansive constant.*
2. *Consider $X = \left\{\pm\frac{1}{n}, \pm\left(1 - \frac{1}{n}\right)\right\}$ with the metric d given by $d(x, y) = |x - y|$. Let $f : X \longrightarrow X$ be a map which fixes $0, 1, -1$ and takes any*

element $x \in X \setminus \{0, 1, -1\}$ *to its immediate right element. Then* f *is a metric expansive homeomorphism with expansive constant* c*, where* $0 < c < \frac{1}{6}$.

The notion of metric expansive homeomorphism is independent of the choice of metric if the space is compact but not the expansive constant. If the space is non–compact, then the notion of metric expansivity depends on the choice of metric even if the topology induced by different metrics are equivalent. For instance, see Example 4. Different variants and generalizations of expansivity are studied. We study few of them in this section.

Let (X, τ) be a topological space. For a subset $A \subseteq X$ and a cover $\mathcal{U}$ of X we write $A \prec \mathcal{U}$ if there exists $C \in \mathcal{U}$ such that $A \subseteq C$. If $\mathcal{V}$ is a family of subsets of X, then $\mathcal{V} \prec \mathcal{U}$ means that for each $A \in \mathcal{V}$, $A \prec \mathcal{U}$. If, in addition $\mathcal{V}$ is a cover of X, then $\mathcal{V}$ is said to be *refinement of* $\mathcal{U}$. Join of two covers $\mathcal{U}$ and $\mathcal{V}$ is a cover given by $\mathcal{U} \wedge \mathcal{V} = \{U \cap V | U \in \mathcal{U}, V \in \mathcal{V}\}$. Every open cover $\mathcal{U}$ of cardinality k can be refined by an open cover $\mathcal{V} = \bigwedge_{i=1}^{k} \mathcal{U}$ such that $\mathcal{V} \prec \mathcal{U}$ and $\mathcal{V} \bigwedge \mathcal{V} = \mathcal{V}$. The notion for orbit expansivity for homeomorphisms was first defined in [1]. We recall the definition.

Definition 3 *Let* $f : X \longrightarrow X$ *be a homeomorphism defined on a topological space* X*. Then* f *is said to be* an orbit expansive homeomorphism *if there is a finite open cover* $\mathcal{U}$ *of* X *such that if for each* $n \in \mathbb{Z}$*, the set* $\{f^n(x), f^n(y)\} \prec \mathcal{U}$*, then* $x = y$*. The cover* $\mathcal{U}$ *of* X *is called* an orbit expansive covering of f.

It can be observed that if f is an orbit expansive homeomorphism on a compact metric space and $\mathcal{U}$ is an orbit expansive covering of f, then $\mathcal{U}$ is a generator for f and therefore f is an expansive homeomorphism. Conversely, every expansive homeomorphism on a compact metric space has a generator $\mathcal{U}$, which is also an orbit expansive covering of f. Hence on compact metric space expansivity is equivalent to orbit expansivity. Another generalization of expansivity was defined and studied in [7]. We recall the definition.

Definition 4 *Let* X *be a topological space. Then a homeomorphism* $f : X \longrightarrow X$ *is said to be an* expansive homeomorphism *if there exists a closed neighborhood* N *of* Δ_X *such that for any two distinct* $x, y \in X$*, there is* $n \in \mathbb{Z}$ *satisfying* $(f^n(x), f^n(y)) \notin N$*. Neighborhood* N *is called* an expansive neighborhood *for* f.

Note that the term used in [7] is topologically expansive but we used the term expansive in above definition to differentiate it from our definition of expansivity on uniform spaces. Obviously, metric expansivity implies expansivity. Through examples it was justified in [7], that in general expansivity need not imply metric expansivity. Also, similar to proof of [15, Theorem 4], one can show that on a locally compact metric space X, if f is expansive with expansive neighborhood N, then for every $\epsilon > 0$ we can construct a metric d compatible with the topology of X such that f is a metric expansive with expansive constant $\epsilon > 0$.

3 Topologically expansive homeomorphism

In this section we study expansive homeomorphisms on uniform spaces. The notion was first defined in [6]. Let X be an uniform space with uniformity $\mathcal{U}$ and $f : X \longrightarrow X$ be a homeomorphism. For an entourage $D \in \mathcal{U}$ let

$$\Gamma_D(x, f) = \{y \ : (f^n(x), f^n(y)) \in D, \ \forall n \in \mathbb{Z}\}.$$

Definition 5 *Let X be an uniform space with uniformity $\mathcal{U}$. A homeomorphism $f : X \longrightarrow X$ is said to be a* topologically expansive homeomorphism, *if there exists an entourage $A \in \mathcal{U}$, such that for every $x \in X$,*

$$\Gamma_A(x, f) = \{x\}$$

Entourage A is called an expansive entourage.

Since every entourage $A \in \mathcal{U}$ contains some closed neighborhood F of Δ_X, it follows that every topologically expansive homeomorphism is an expansive homeomorphism. But in general converse need not be true as we can observe from the following Example:

Example 3 *Consider $\mathbb{R}$ with the uniformity $\mathcal{U}_d$ as given in Example 1. Then the translation T defined on $\mathbb{R}$ by $T(x) = x+1$ is an expansive homeomorphism with an expansive neighbourhood $N = \{(x, y) \in \mathbb{R}^2 : |x - y| \leq e^{-x}\}$. Note that $N \notin \mathcal{U}$. In fact, it is easy to observe that T is not topologically expansive.*

Example 4 *Consider $\mathbb{R}$ with uniformities $\mathcal{U}_\rho$ and $\mathcal{U}_d$ as given in Example 1. Define a homeomorphism $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = x + \ln 2$. Then it can be easily verified that f is topologically expansive for a closed entourage $A \in \mathcal{U}_\rho$ but not for any closed entourage $D \in \mathcal{U}_d$. Further, observe that f is metric expansivity with respect to metric ρ but is not metric expansive with respect to metric d.*

From Example 3, it can be observed that topologically expansivity is stronger than expansivity whereas from Example 4, it can be concluded that it is weaker than metric expansivity. Also, from Example 4, it can be concluded that the notion of topological expansivity depends on the choice of uniformity on the space and the notion of metric expansivity depends on the metric of the space. In spite of expansivity, in the following Proposition we show that if a uniform space admits a topologically expansive homeomorphism, the space is always Hausdorff space.

Proposition 1 *Let* X *be a uniform space with uniformity* $\mathcal{U}$ *and let* $f : X \longrightarrow X$ *be a topologically expansive homeomorphism. Then* X *is always a Hausdorff space.*

Proof. Let D be an expansive entourage of f. Since $\mathcal{U}$ is a uniformity on X there exists a symmetric set $E \in \mathcal{U}$, such that

$$E o E \subseteq D.$$

Given two distinct points x and y of X, by topological expansivity of f, there exists n in $\mathbb{Z}$, such that $(f^n(x), f^n(y)) \notin D$. But this implies

$$(f^n(x), f^n(y)) \notin E \circ E.$$

Let $U = f^{-n}(E[f^n(x)])$ and $V = f^{-n}(E[f^n(y)])$. Then $\mathrm{int}(U)$ and $\mathrm{int}(V)$ are open subsets of X with $x \in \mathrm{int}(U)$ and $y \in \mathrm{int}(V)$. Further, $U \cap V = \emptyset$. For, if $t \in U \cap V$, then $f^n(t) \in E[f^n(x)] \cap E[f^n(y)]$. But this implies that $(f^n(x), f^n(y)) \in E \circ E$, which is a contradiction. Hence X is a Hausdorff space. □

Following Corollary is a consequence of just Proposition 1 and Lemma 1.

Corollary 1 *If uniform space* X *admits a topological expansive homeomorphism then* X *is a regular space.*

Recall, for a compact Hausdorff space X, all uniformities generates a same topology on the space and therefore it is sufficient to work with the natural uniformity on X. Hence as consequence of Proposition 1, we can conclude the following:

Corollary 2 *Topological expansivity on a compact Hausdorff uniform space does not depend on choice of uniformity on the space.*

Since every compact metric space admits a unique uniform structure, it follows that on compact metric space: metric expansivity, topological expansivity and expansivity are equivalent.

Let X be a uniform space with uniformity $\mathcal{U}$. A cover $\mathcal{A}$ of a space X is a uniform cover if there is $U \in \mathcal{U}$ such that $U[x]$ is a subset of some member of the cover for every $x \in X$, equivalently, $\{U[x] : x \in X\} \prec \mathcal{A}$. It is known that every open cover of a compact uniform space is uniform cover. For instance, see Theorem 33 in [11].

Let X be a topological space and $f : X \longrightarrow X$ be an orbit expansive homeomorphism with an orbit expansive covering $\mathcal{A}$. Equivalently, f is orbit expansive if for every subset B of X, $f^n(B) \prec \mathcal{A}$ for all $n \in \mathbb{Z}$, then B is singleton. In the following we show that on compact uniform space, topological expansivity is equivalent to orbit expansivity:

Theorem 1 *Let X be a compact uniform space with uniformity $\mathcal{U}$. Then $f : X \longrightarrow X$ is a topologically expansive homeomorphism if and only if it is an orbit expansive homeomorphism.*

Proof. Let f be a topologically expansive homeomorphism with an expansive entourage D, $D \in \mathcal{U}$. Choose $E \in \mathcal{U}$ such that $EoE \subseteq D$. Now, $E \in \mathcal{U}$ and $\mathcal{U}$ is a uniformity. Therefore E contains diagonal and hence the collection $\{E[x] : x \in X\}$ is a cover of X by neighbourhoods. But X is compact. Let $\mathcal{A}$ be a finite subcover of $\{E[x] : x \in X\}$. We show that $\mathcal{A}$ is an orbit expansive covering for f. For $x, y \in X$ suppose that for each $n \in \mathbb{Z}$, $\{f^n(x), f^n(y)\} \prec \mathcal{A}$. But this implies that for each $n \in \mathbb{Z}$,

$$(f^n(x), f^n(y)) \in EoE \subseteq D.$$

Since D is expansive entourage, it follows that $x = y$. Hence $\mathcal{A}$ is an orbit expansive covering.

Conversely, let $\mathcal{A}$ be an orbit expansive covering of f. Since X is a compact uniform space, $\mathcal{A}$ is a uniform cover. Therefore there exists $U \in \mathcal{U}$ such that $\{U[x] : x \in X\} \prec \mathcal{A}$. Since the family of closed members of a uniformity $\mathcal{U}$ is a basis of $\mathcal{U}$, there is a closed member $D \in \mathcal{U}$ such that $D \subseteq U$. We claim that D is an expansive entourage of f. For $x, y \in X$ and for all $n \in \mathbb{Z}$, suppose

$$(f^n(x), f^n(y)) \in D.$$

Therefore, for each $n \in \mathbb{Z}$,

$$\{f^n(x), f^n(y)\} \subseteq U[f^n(x)].$$

This further implies that

$$\{f^n(x), f^n(y)\} \prec \{U[t] : t \in X\} \prec \mathcal{A}.$$

But $\mathcal{A}$ is an orbit expansive covering of f and therefore $x = y$. Hence f is topologically expansive with expansive entourage D. □

In [1, Theorem 2.7] authors showed that if a compact Hausdorff topological space admits an orbit expansive homeomorphism then it is metrizable. Therefore by Proposition 1 and Proposition 1, we have:

Corollary 3 *If a compact uniform space admits a topologically expansive homeomorphism, then it is always metrizable.*

Again as a consequence of Corollary 3, it follows that topological expansivity is equivalent with metric expansivity and it does not depend uniformity. However the following example shows that Corollary 3, is false for non-compact Hausdorff uniform spaces.

Example 5 *Consider* $\mathbb{R}$ *with the topology* $\tau_{\mathbb{R}}$ *whose base consists of all intervals* $[x, r)$*, where* x *is a real number,* r *is a rational number and* $x < r$*. Then* $\mathbb{R}$ *with topology* $\tau_{\mathbb{R}}$ *is a non–compact, paracompact, Hausdorff and not metrizable space. Also, it is known that every paracompact Hausdorff space, admits the uniform structure* $\mathcal{U}$*, consisting of all neighborhood of the diagonal. For instance, see* [11, Page 208]*. Hence if*

$$D = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x - y| < 1\},$$

then $D \in \mathcal{U}$*. Define* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *by* $f(x) = 3x$*. Then it is easy to see that* f *is topologically expansive with expansive entourage* D*. Note that* $\mathbb{R}$ *with uniformity* $\mathcal{U}$ *is a non-compact Hausdorff space.*

In the following Remark, we observe certain results related to topological expansivity as a consequence of expansivity.

Remark 1 *Let* X *be a uniform space with uniformity* $\mathcal{U}$ *and let* $f : X \longrightarrow X$ *be a homeomorphism.*

1. *Suppose* X *is a locally compact, paracompact uniform space. Since every topologically expansive homeomorphism is an expansive homeomorphism, it follows from Lemma 9 of [7], that there is a proper expansive neighborhood for* f*. Note that this neighborhood need not be an entourage. Recall, a set* $M \subseteq X \times X$ *is proper if for every compact subset* A *of* X*, the set* $M[A]$ *is compact.*

2. *Let* f *be topologically expansive homeomorphism. Then by Proposition 13 of* [7], *it follows that for each* $n \in \mathbb{N}$, f^n *is expansive. Note that this* f^n *need not be in general topologically expansive. For instance, let* $\mathcal{U}$ *be the usual uniformity on* $[0, \infty)$ *and* $f : [0, \infty) \longrightarrow [0, \infty)$ *be as homeomorphism constructed by Bryant and Coleman in* [5]. *Then it is easy to verify that* f *is topologically expansive but* f^n *is not topologically expansive, for any* $n > 1$.

3. *Let* X *be a uniform space with uniformity* $\mathcal{U}$ *and* Y *be a uniform space with uniformity* $\mathcal{V}$. *Suppose* $f : X \longrightarrow X$ *is topologically expansive and* $h : X \longrightarrow Y$ *is a homeomorphism. Then by Proposition 13 of* [7], *it follows that* $h \circ f \circ h^{-1}$ *is expansive on* Y. *However, the homeomorphism* $h \circ f \circ h^{-1}$ *need not be topologically expansive. For instance, let* $\mathcal{U}_\rho$ *and* $\mathcal{U}_d$ *be uniformities on* $\mathbb{R}$ *as defined in Example 1. Consider the identity homeomorphism* $h : \mathbb{R} \longrightarrow \mathbb{R}$, *where the domain* $\mathbb{R}$ *is considered with uniformity* $\mathcal{U}_\rho$ *whereas co-domain* $\mathbb{R}$ *is considered with the uniformity* $\mathcal{U}_d$. *Then as observed in Example 4,* $f(x) = x + \ln(2)$ *is topologically expansive with respect to* $\mathcal{U}_\rho$ *but* $h \circ f \circ h^{-1}$ *is not topologically expansive with respect to* $\mathcal{U}_d$.

Observe here that in each of the above Example, f is not uniformly continuous. In the following we show that Remarks above are true if the maps are uniformly continuous. Recall, a map $f : X \longrightarrow X$ is uniformly continuous relative to the uniformity $\mathcal{U}$ if for every entourage $V \in \mathcal{U}$, $(f \times f)^{-1}(V) \in \mathcal{U}$.

Proposition 2 1. *Let* X *be a uniform space with uniformity* $\mathcal{U}$. *Suppose both* f *and* f^{-1} *are uniformly continuous relative to* $\mathcal{U}$. *Then* f *is topologically expansive if and only if* f^n *is topologically expansive, for all* $n \in \mathbb{Z} \backslash \{0\}$.

2. *Let* X *be a uniform space with uniformity* $\mathcal{U}$ *and* Y *be a uniform space with uniformity* $\mathcal{V}$. *Suppose* $h : X \longrightarrow Y$ *is a homeomorphism such that both* h *and* h^{-1} *are uniformly continuous. Then* f *is topologically expansive on* X *if and only if* $h \circ f \circ h^{-1}$ *is topologically expansive on* Y.

Since the proof of the Proposition 2 is similar to the proof of Proposition 13 in [7], we omit the proof.

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