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On Cusa-Huygens type trigonometric and hyperbolic inequalities

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Abstract. Recently a trigonometric inequality by N. Cusa and C. Huygens (see e.g. [1], [6]) has been discussed extensively in mathematical literature (see e.g. [4], [6, 7]). By using a unified method based on monotonicity or convexity of certain functions, we shall obtain new Cusa-Huygens type inequalities. Hyperbolic versions will be pointed out, too.

1 Introduction

In recent years the trigonometric inequality

$$\frac{\sin x}{x} < \frac{\cos x + 2}{3}, \quad 0 < x < \frac{\pi}{2} \tag{1}$$

among with other inequalities, has attracted attention of several researchers. This inequality is due to N. Cusa and C. Huygens (see [6] for more details regarding this result).

Recently, E. Neuman and J. Sándor [4] have shown that inequality (1) implies a result due to S. Wu and H. Srivastava [10], namely

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2, \quad 0 < x < \frac{\pi}{2} \tag{2}$$

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called as "the second Wilker inequality". Relation (2) implies in turn the classical and famous Wilker inequality (see [9]):

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{3}$$

For many papers, and refinements of (2) and (3), see [4] and the references therein.

A hyperbolic counterpart of (1) has been obtained in [4]:

$$\frac{\sinh x}{x} < \frac{\cosh x + 2}{3}, \quad x > 0. \tag{4}$$

We will call (4) as the hyperbolic Cusa-Huygens inequality, and remark that if (4) is true, then holds clearly also for $x < 0$.

In what follows, we will obtain new proofs of (1) and (4), as well as new inequalities or counterparts of these relations.

2 Main results

Theorem 1 *Let* $f(x) = \dfrac{x(2+\cos x)}{\sin x}$, $0 < x < \dfrac{\pi}{2}$. *Then* f *is a strictly increasing function. Particularly, one has*

$$\frac{2+\cos x}{\pi} < \frac{\sin x}{x} < \frac{2+\cos x}{3}, \quad 0 < x < \frac{\pi}{2}. \tag{5}$$

Theorem 2 *Let* $g(x) = \dfrac{x\left(\dfrac{4}{\pi}+\cos x\right)}{\sin x}$, $0 < x < \dfrac{\pi}{2}$. *Then* g *is a strictly decreasing function. Particularly, one has*

$$\frac{1+\cos x}{2} < \frac{\dfrac{4}{\pi}+\cos x}{\dfrac{4}{\pi}+1} < \frac{\sin x}{x} < \frac{\dfrac{4}{\pi}+\cos x}{2}. \tag{6}$$

Proof. We shall give a common proofs of Theorems 1 and 2. Let us define the application

$$f_a(x) = \frac{x(a+\cos x)}{\sin x}, \quad 0 < x < \frac{\pi}{2}.$$

Then, easy computations yield that

$$\sin^2 x \cdot f_a'(x) = a\sin x + \sin x\cos x - ax\cos x - x = h(x). \tag{7}$$

The function h is defined on $\left[0, \frac{\pi}{2}\right]$. We get

$$h'(x) = (\sin x)(ax - 2\sin x).$$

Therefore, one obtains that

(i) If

$$\frac{\sin x}{x} < \frac{a}{2},$$

then $h'(x) > 0$. Thus by (7) one has $h(x) > h(0) = 0$, implying $f_a'(x) > 0$, i.e. f_a is strictly increasing.

(ii) If

$$\frac{\sin x}{x} > \frac{a}{2},$$

then $h'(x) < 0$, implying as above that f_a is strictly decreasing.

Select now $a = 2$ in (i). Then $f_a(x) = f(x)$, and the function f in Theorem 1 will be strictly increasing. Selecting $a = \frac{4}{\pi}$ in (ii), by the famous Jordan inequality (see e.g. [3], [7], [8], [2])

$$\frac{\sin x}{x} > \frac{2}{\pi}, \tag{8}$$

so $f_a(x) = g(x)$ of Theorem 2 will be strictly decreasing.

Now remarking that $f(0) < f(x) < f\left(\frac{\pi}{2}\right)$ and $g(0) > g(x) > g\left(\frac{\pi}{2}\right)$, after some elementary transformations, we obtain relations (5) and (6). □

Remarks. 1. The right side of (5) is the Cusa-Huygens inequality (1), while the left side seems to be new.

2. The first inequality of (6) follows by an easy computation, based on $0 < \cos x < 1$. The inequality

$$\frac{1 + \cos x}{2} < \frac{\sin x}{x} \tag{9}$$

appeared in paper [5], and rediscovered by other authors (see e.g. [2]).

3. It is easy to see that inequalities (5) and (6) are not comparable, i.e. none of these inequalities implies the other one for all $0 < x < \pi/2$.

Before turning to the hyperbolic case, the following auxiliary result will be proved:

Lemma 1 *For all $x \geq 0$ one has the inequalities*

$$\cos x \cosh x \leq 1 \tag{10}$$

and

$$\sin x \sinh x \leq x^2. \tag{11}$$

Proof. Let $m(x) = \cos x \cosh x - 1$, $x \geq 0$. Then

$$m'(x) = -\sin x \cosh x + \cosh x \sinh x,$$

$$m''(x) = -2 \sin x \sinh x < 0.$$

Thus $m'(x) < m'(0) = 0$ and $m(x) < m(0) = 0$ for $x > 0$, implying (10), with equality only for $x = 0$.

For the proof of (11), let

$$n(x) = x^2 - \sin x \sinh x.$$

Then

$$n'(x) = 2x - \cos x \sinh x - \sin x \cosh x,$$

$$n''(x) = 2(1 - \cos x \cosh x) < 0$$

by (10), for $x > 0$. This easily implies (11). □

Theorem 3 *Let*

$$F(x) = \frac{x(2 + \cosh x)}{\sinh x}, \quad x > 0.$$

Then F is a strictly increasing function. Particularly, one has inequality (4). On the other hand,

$$\frac{2 + \cosh x}{k^*} < \frac{\sinh x}{x} < \frac{2 + \cosh x}{3}, \quad 0 < x < \frac{\pi}{2} \tag{12}$$

where $k^* = \frac{\pi}{2}(2 + \cosh \pi/2)/\sinh(\pi/2)$.

Theorem 4 *Let*

$$G(x) = \frac{x(\pi + \cosh x)}{\sinh x}, \quad x > 0.$$

Then G is a strictly decreasing function for $0 < x < \pi/2$. Particularly, one has

$$\frac{\pi + \cosh x}{\pi + 1} < \frac{\sinh x}{x} < \frac{\pi + \cosh x}{k}, \quad 0 < x < \frac{\pi}{2} \tag{13}$$

where $k = \frac{\pi}{2}(\pi + \cosh \pi/2)/\sinh(\pi/2)$.

Proof. We shall deduce common proofs to Theorem 3 and 4. Put

$$F_a(x) = \frac{x(a + \cosh x)}{\sinh x}, \quad x > 0.$$

An easy computation gives

$$(\sinh x)^2 F_a'(x) = g_a(x) = a \sinh x + \cosh x \sinh x - ax \cosh x - x.$$

The function g_a is defined for $x \geq 0$. As

$$g_a'(x) = (\sinh x)(2 \sinh x - ax),$$

we get that:

(i) If

$$\frac{\sinh x}{x} > \frac{a}{2},$$

then $g_a'(x) > 0$. This in turn will imply $F_a'(x) > 0$ for $x > 0$.

(ii) If

$$\frac{\sinh x}{x} < \frac{a}{2},$$

then $F_a'(x) < 0$ for $x > 0$.

By letting $a = 2$, by the known inequality $\sinh x > x$, we obtain the monotonicity if $F_2(x) = F(x)$ of Theorem 3. Since $F(0) = \lim_{x \to 0+} F(x) = 3$, inequality (4), and the right side of (12) follows. Now, the left side of (12) follows by $F(x) < F(\pi/2)$ for $x < \pi/2$.

By letting $a = \pi$ in (ii) we can deduce the results of Theorem 4. Indeed, by relation (11) of the Lemma 1 one can write $\frac{\sinh x}{x} < \frac{x}{\sin x}$ and by Jordan's inequality (8), we get $\frac{\sinh x}{x} < \frac{\pi}{2}$ thus $a = \pi$ may be selected. Remarking that $g(0) > g(x) > g\left(\frac{\pi}{2}\right)$, inequalities (13) will follow. □

Remark. By combining (12) and (13), we can deduce that:

$$3 < k^* < k < \pi + 1. \tag{14}$$

Now, the following convexity result will be used:

Lemma 2 *Let* $k(x) = \frac{1}{\tanh x} - \frac{1}{x}$, $x > 0$. *Then* k *is a strictly increasing, concave function.*

Proof. Simple computations give

$$k'(x) = \frac{1}{x^2} - \frac{1}{(\sinh x)^2} > 0$$

and

$$k''(x) = \frac{2[x^3 \cosh x - (\sinh x)^3]}{x^3 (\sinh x)^3} < 0,$$

since by a result of I. Lazarević (see e.g. [3], [4]) one has

$$\frac{\sinh x}{x} > (\cosh x)^{1/3}. \tag{15}$$

This proves Lemma 2. □

Theorem 5 *Let the function* $k(x)$ *be defined as in Lemma 2. Then one has*

$$\frac{1 + x^2 \cdot \dfrac{k(r)}{r}}{\cosh x} \le \frac{x}{\sinh x} \text{ for any } 0 < x \le r \tag{16}$$

and

$$\frac{x}{\sinh x} \le \frac{1 + k(r)x + k'(r)x(x-r)}{\cosh x} \text{ for any } 0 < x, r. \tag{17}$$

In both inequalities (16) and (17) there is equality only for $x = r$.

Proof. Remark that $k(0+) = \lim_{x \to 0+} k(x) = 0$, and that by the concavity of k, the graph of function k is above the line segment joining the points $A(0,0)$ and $B(r, k(r))$. Thus $k(x) \ge \frac{k(r)}{r} \cdot x$ for any $x \in (0, r]$. By multiplying with x this inequality, after some transformations, we obtain (16).

For the proof of (17), write the tangent line to the graph of function k at the point $B(r, k(r))$. Since the equation of this line is $y = k(r) + k'(r)(x - r)$ and writing that $y \le k(x)$ for any $x > 0$, $r > 0$, after elementary transformations, we get relation (17).

□

For example, when $r = 1$ we get:

$$\left[x^2 \left(\frac{2}{e^2 - 1}\right) + 1\right] / \cosh x \le \frac{x}{\sinh x} \text{ for all } 0 < x \le 1 \tag{18}$$

and

$$\frac{x}{\sinh x} \le \left[1 + \left(\frac{2}{e^2-1}\right)x + \left(\frac{e^4-6e^2+1}{e^4-2e^2+1}\right)x(x-1)\right] / \cosh x \qquad (19)$$

for any $x > 0$.

In both inequalities (18) and (19) there is equality only for $x = 1$.

In what follows a convexity result will be proved:

Lemma 3 *Let* $j(x) = 3x - 2\sinh x - \sinh x \cos x$, $0 < x < \frac{\pi}{2}$. *Then* j *is a strictly convex function.*

Proof. Since $j''(x) = 2(\cosh x \sin x - \sinh x) > 0$ is equivalent to

$$\sin x > \tanh x, \quad 0 < x < \frac{\pi}{2} \qquad (20)$$

we will show that inequality (20) holds true for any $x \in \left(0, \frac{\pi}{2}\right)$. We note that in [2] it is shown that (20) holds for $x \in (0, 1)$, but here we shall prove with another method the stronger result (20).

Inequality (20) may be written also as

$$p(x) = (e^x + e^{-x}) \sin x - (e^x - e^{-x}) > 0.$$

Since $p''(x) = (e^x - e^{-x})(2\cos x - 1)$ and $e^x - e^{-x} > 0$, the sign of $p''(x)$ depends on the sign of $2\cos x - 1$. Let $x_0 \in \left(0, \frac{\pi}{2}\right)$ be the unique number such that $2\cos x_0 - 1 = 0$. Here $x_0 = \arccos\left(\frac{1}{2}\right) \approx 1.0471$. Thus, $\cos x$ being a decreasing function, for all $x < x_0$ one has $\cos x > \frac{1}{2}$, i.e. $p''(x) > 0$ in $(0, x_0)$. This implies $p'(x) > p'(0) = 0$, where

$$p'(x) = (e^x - e^{-x}) \sin x + (e^x + e^{-x}) \cos x - (e^x + e^{-x}).$$

This in turn gives $p(x) > p(0) = 0$.

Let now $x_0 < x < \pi/2$. Then, as $p'(x_0) > 0$ and $p'\left(\frac{\pi}{2}\right) < 0$ and p' being continuous and decreasing, there exists a single $x_0 < x_1 < \pi/2$ such that $p'(x_1) = 0$. Then p' will be positive on (x_0, x_1) and negative on $\left(x_1, \frac{\pi}{2}\right)$. Thus p will be strictly decreasing on $\left(x_1, \frac{\pi}{2}\right)$, i.e. $p(x) > p\left(\frac{\pi}{2}\right) > 0$. This means that, for any $x \in \left(0, \frac{\pi}{2}\right)$ one has $p(x) > 0$, completing the proof of (20).

□

Now, via inequality (1), the following improvement of relation (11) will be proved:

Theorem 6 *For any* $x \in \left(0, \frac{\pi}{2}\right)$ *one has*

$$\frac{\sin x}{x} < \frac{\cos x + 2}{3} < \frac{x}{\sinh x}. \tag{21}$$

Proof. The first inequality of (21) is the Cusa-Huygens inequality (1). The second inequality of (21) may be written as $j(x) > 0$, where j is the function defined in Lemma 3. As $j'(0) = 0$ and $j'(x)$ is strictly increasing, $j'(x) > 0$, implying $j(x) > j(0) = 0$. This finishes the proof of (21).

□

Finally, we will prove a counterpart of inequality (20):

Theorem 7 *For any* $x \in \left(0, \frac{\pi}{2}\right)$ *one has*

$$\sin x \cos x < \frac{(\sin x)(1 + \cos x)}{2} < \frac{(x + \sin x \cos x)}{2} < \tanh x < \sin x. \tag{22}$$

Proof. The first two inequalities are consequences of $0 < \cos x < 1$ and $0 < \sin x < x$, respectively. The last relation is inequality (20), so we have to prove the third inequality. For this purpose, consider the application

$$u(x) = \tanh x - \frac{(x + \sin x \cos x)}{2},$$

where $x \in \left[0, \frac{\pi}{2}\right]$. An easy computation implies $(\cosh x)^2 . (u'(x)) = 1 - (\cos x \cosh x)^2 \leq 0$ by relation (10) of Lemma 1. Therefore, since $u(0) = 0$, and $u(x) \leq u(0)$, the inequality follows.

□

Remark. As a corollary, we get the following nontrivial relations: For all $x \in \left(0, \frac{\pi}{2}\right)$, we have:

$$x + \sin x \cos x < 2 \sin x \tag{23}$$

and

$$\sin x \cos x < \tanh x < \sin x. \tag{24}$$

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