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On contact CR-submanifolds of Kenmotsu manifolds

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Abstract. In this paper, we study the differential geometry of contact CR-submanifolds of a Kenmotsu manifold. Necessary and sufficient conditions are given for a submanifold to be a contact CR-submanifold in Kenmotsu manifolds. Finally, the induced structures on submanifolds are investigated, these structures are categorized and we discuss these results.

1 Introduction

In [4], K. Kenmotsu defined and studied a new class of almost contact manifolds called Kenmotsu manifolds. The study of the differential geometry of a contact CR-submanifolds, as a generalization of invariant and anti-invariant submanifolds, of an almost contact metric manifold was initiated by A. Bejancu [3] and was followed by several geometers. Several authors studied contact CR-submanifolds of different classes of almost contact metric manifolds given in the references of this paper.

The contact CR-submanifolds are rich and interesting subject. Therefore we continue to work in this subject matter.

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The purpose of this paper is to study the differential geometric theory of submanifolds immersed in Kenmotsu manifold. We obtain the new integrability conditions of the distributions of contact CR-submanifolds and prove some characterizations for the induced structure to be parallel.

2 Preliminaries

In this section, we give some notations used throughout this paper. We recall some necessary facts and formulas from the Kenmotsu manifolds. A $(2m+1)$-dimensional Riemannian manifold $(\bar{M}, g)$ is said to be a Kenmotsu manifold if there exist on $\bar{M}$ a $(1,1)$ tensor field φ, a vector field ξ(called the structure vector field) and 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$

and

$$(\bar{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad \bar{\nabla}_X \xi = X - \eta(X)\xi, \tag{2}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ is a Levi-Civita connection on $\bar{M}$ and $\Gamma(T\bar{M})$ denotes the set of all differentiable vectors on $\bar{M}$ [5].

A plane section π in $T_x\bar{M}$ is called a φ-section if it is spanned by X and φX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a φ-section is called a φ-holomorphic sectional curvature. A Kenmotsu manifold with constant φ-holomorphic sectional curvature c is said to be a Kenmotsu space form and it is denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of a $\bar{M}(c)$ is also given by

$$\begin{aligned} \bar{R}(X, Y)Z &= \left(\frac{c-3}{4}\right)\{g(Y, Z)X - g(X, Z)Y\} + \left(\frac{c+1}{4}\right)\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi + g(X, \varphi Z)\varphi Y \\ &- g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\}, \end{aligned} \tag{3}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$ [1].

Now, let M be an isometrically immersed submanifold in $\bar{M}$. In the rest of this paper, we assume the submanifold M of $\bar{M}$ is tangent to the structure vector field ξ. Then the formulas of Gauss and Weingarten for M in $\bar{M}$ are given, respectively, by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{4}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{5}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where $\bar{\nabla}$ and ∇ denote the Riemannian connections on $\bar{M}$ and M, respectively, h is the second fundamental form, $\nabla^{\perp}$ is the normal connection on the normal bundle $T^{\perp}M$ and A_V is the shape operator of M in $\bar{M}$. It is well known that the second fundamental form and the shape operator are related by formulae

$$g(A_V X, Y) = g((h(X, Y), V), \tag{6}$$

where, g denotes the Riemannian metric on $\bar{M}$ as well as M. For any submanifold M of a Riemannian manifold $\bar{M}$, the equation of Gauss is given by

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\bar{\nabla}_X h)(Y, Z) \\ &- (\bar{\nabla}_Y h)(X, Z), \end{aligned} \tag{7}$$

for any $X, Y, Z \in \Gamma(TM)$, where $\bar{R}$ and R denote the Riemannian curvature tensors of $\bar{M}$ and M, respectively. The covariant derivative $\bar{\nabla}h$ of h is defined by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y), \tag{8}$$

and the covariant derivative $\bar{\nabla}A$ is defined by

$$(\bar{\nabla}_X A)_V Y = \nabla_X (A_V Y) - A_{\nabla^{\perp}_X V} Y - A_V \nabla_X Y, \tag{9}$$

for any $X, Y, Z \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

The normal component of (7) is said to be the Codazzi equation and it is given by

$$(\bar{R}(X, Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \tag{10}$$

where $(\bar{R}(X, Y)Z)^{\perp}$ denotes the normal part of $\bar{R}(X, Y)Z$. If $(\bar{R}(X, Y)Z)^{\perp} = 0$, then M is said to be curvature-invariant submanifold of $\bar{M}$.

The Ricci equation is given by

$$g(\bar{R}(X, Y)V, U) = g(R^{\perp}(X, Y)V, U) + g([A_U, A_V]X, Y), \tag{11}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$, where $R^{\perp}$ denotes the Riemannian curvature tensor of the normal vector bundle $T^{\perp}M$ and if $R^{\perp}$=0, then the normal connection of M is called flat [6].

Taking into account (3) and (11), we have

$$\begin{aligned} g(R^{\perp}(X,Y)V,U) &= \left(\frac{c+1}{4}\right)\{g(X,\varphi V)g(U,\varphi Y)-g(Y,\varphi V)g(\varphi X,U) \\ &+ 2g(X,\varphi Y)g(\varphi V,U)+g([A_V,A_U]X,Y)\}, \end{aligned} \tag{12}$$

for any $X,Y\in\Gamma(TM)$ and $U,V\in\Gamma(T^{\perp}M)$.

By using (3) and (7), the Riemannian curvature tensor R of an immersed submanifold M of a Kenmotsu space form $\bar{M}(c)$ is given by

$$\begin{aligned} R(X,Y)Z &= \left(\frac{c-3}{4}\right)\{g(Y,Z)X-g(X,Z)Y\}+\left(\frac{c+1}{4}\right)\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X+\eta(Y)g(X,Z)\xi-\eta(X)g(Y,Z)\xi+g(X,\varphi Z)PY \\ &- g(Y,\varphi Z)PX+2g(X,\varphi Y)PZ\}+A_{h(Y,Z)}X-A_{h(X,Z)}Y. \end{aligned} \tag{13}$$

From (3) and (10), for a submanifold, the Codazzi equation is given by

$$\begin{aligned} (\bar{\nabla}_X h)(Y,Z)-(\bar{\nabla}_Y h)(X,Z) &= \left(\frac{c+1}{4}\right)\{g(X,\varphi Z)FY-g(Y,\varphi Z)FX \\ &+ 2g(X,\varphi Y)FZ\}. \end{aligned} \tag{14}$$

3 Contact CR-submanifolds of a Kenmotsu manifold

Now, let M be an isometrically immersed submanifold of a Kenmotsu manifold $\bar{M}$. For any vector X tangent to M, we set

$$\varphi X = PX + FX, \tag{15}$$

where PX and FX denote the tangent and normal parts of φX, respectively. Then P is an endomorphism of the TM and F is a normal-bundle valued 1-form of TM.

The covariant derivatives of P and F are, respectively, defined by

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y \tag{16}$$

and

$$(\nabla_X F)Y = \nabla_X^{\perp} FY - F\nabla_X Y. \tag{17}$$

In the same way, for any vector field V normal to M, φV can be written in the following way;

$$\varphi V = BV + CV, \tag{18}$$

where BV and CV denote the tangent and normal parts of φV, respectively. Also, B is an endomorphism of the normal bundle $T^{\perp}M$ of TM and C is an endomorphism of the subbundle of the normal bundle $T^{\perp}M$.

The covariant derivatives of B and C are also, respectively, defined by

$$(\nabla_X B)V = \nabla_X BV - B\nabla_X^{\perp}V \tag{19}$$

and

$$(\nabla_X C)V = \nabla_X^{\perp}CV - C\nabla_X^{\perp}V. \tag{20}$$

Furthermore, for any $X, Y \in \Gamma(TM)$, we have $g(PX, Y) = -g(X, PY)$ and $U, V \in \Gamma(T^{\perp}M)$, we get $g(U, CV) = -g(CU, V)$. These show that P and C are also skew-symmetric tensor fields. Moreover, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$ we have

$$g(FX, V) = -g(X, BV), \tag{21}$$

which gives the relation between F and B.

Definition 1 *Let M be an isometrically immersed submanifold of a Kenmotsu manifold $\bar{M}$. Then M is called a contact CR-submanifold of $\bar{M}$ if there is a differentiable distribution $D : p \longrightarrow D_p \subseteq T_p(M)$ on M satisfying the following conditions:*

i) $\xi \in D$,

ii) D is invariant with respect to φ, i.e., $\varphi D_x \subset T_p(M)$ for each $p \in M$, and

iii) the orthogonal complementary distribution $D^{\perp} : p \longrightarrow D_p^{\perp} \subseteq T_p(M)$ satisfies $\varphi D_p^{\perp} \subseteq T_p^{\perp}M$ for each $p \in M$.

For a contact CR-submanifold M of a Kenmotsu manifold, for the structure vector field $\xi \in \Gamma(D) \subseteq \Gamma(TM)$, from (1), we have

$$\varphi\xi = P\xi + F\xi = 0,$$

which is equivalent to

$$P\xi = F\xi = 0. \tag{22}$$

Furthermore, applying φ to (15), by using (1), (18), we conclude that

$$P^2 + BF = -I + \eta \otimes \xi \text{ and } FP + CF = 0. \tag{23}$$

Similarly, applying φ to (18), making use of (1), (15), we have

$$C^2 + FB = -I \text{ and } PB + BC = 0. \tag{24}$$

Proposition 1 *Let* M *be a contact CR-submanifold of a Kenmotsu manifold* $\bar{M}$*. Then the invariant distribution* D *has an almost contact metric structure* (P, ξ, η, g) *and so* $\dim(D_p) = \text{odd}$ *for each* $p \in M$*.*

Now, we denote the orthogonal distribution of $\varphi(D^{\perp})$ in $T^{\perp}M$ by ν. Then we have the direct decomposition

$$T^{\perp}M = \varphi(D^{\perp}) \oplus \nu \text{ and } \varphi(D^{\perp}) \perp \nu. \tag{25}$$

Here we note that ν is an invariant subbundle with respect to φ and so $\dim(\nu) = \text{even}$.

Theorem 1 *Let* M *be an isometrically immersed submanifold of a Kenmotsu manifold* $\bar{M}$*. Then* M *is a contact CR-submanifold if and only if* $FP = 0$*.*

Proof. We assume that M is a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. We denote the orthogonal projections on D and $D^{\perp}$ by R and S, respectively. Then we have

$$R + S = I, \quad R^2 = R, \quad S^2 = S \text{ and } RS = SR = 0. \tag{26}$$

For any $X \in \Gamma(TM)$, we can write

$$X = RX + SX \text{ and } \varphi X = \varphi RX + \varphi SX = PRX + FRX + PSX + FSX. \tag{27}$$

Since D is invariant distribution, it is clear that

$$FR = 0 \text{ and } SPR = 0. \tag{28}$$

On the other hand, we can easily verify that

$$RP = P = PR.$$

From the second side of (23), we reach

$$FPR + CFR = 0. \tag{29}$$

Since $FR = 0$, (29) reduces to

$$FP = 0. \tag{30}$$

By virtue of (23) and (30), we arrive at

$$CF = 0. \tag{31}$$

Conversely, let M be a submanifold of a Kenmotsu manifold $\bar{M}$ such that (30) is satisfied. For any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, by direct calculations, we have

$$\begin{aligned} g(X, \varphi^2 V) &= g(\varphi^2 X, V) \\ g(X, \varphi BV) &= g(\varphi FX, V), \\ g(X, PBV) &= g(CFX, V) = 0. \end{aligned}$$

Thus we get

$$PB = 0. \tag{32}$$

Making use of the equations (23), (24) and (32), we have $P^3 + P = 0$ and $C^3 + C = 0$ which show that P and C are f-structures on TM and $T^{\perp}M$, respectively. Here if we put $R = -P^2 + \eta \otimes \xi$ and $S = I + P^2 - \eta \otimes \xi$, then we can easily see that

$$R + S = I, \quad R^2 = R, \quad S^2 = S \text{ and } RS = SR = 0, \tag{33}$$

that is, R and S are orthogonal projections and they define orthogonal complementary distributions such as D and $D^{\perp}$. Since $R = -P^2 + \eta \otimes \xi$ and $P^3 + P = 0$, we get $PR = P$ and $PS = 0$. Taking account of P being skew-symmetric and S being symmetric, we have

$$\begin{aligned} g(SPX, Y) &= g(PX, SY) \\ &= -g(X, PSY) = 0, \end{aligned}$$

for any $X, Y \in \Gamma(TM)$. Thus we have

$$SP = 0.$$

It implies that

$$SPR = 0.$$

Since $R = -P^2 + \eta \otimes \xi$, $P\xi = F\xi = 0$ and from (30), it is clear that

$$FR = 0. \tag{34}$$

(33) and (34) tell us that D and $D^{\perp}$ are invariant and anti-invariant distributions on M, respectively. Furthermore, from the definitions of R and S, we have

$$R\xi = \xi \text{ and } S\xi = 0,$$

that is, the distribution D contains ξ. On the other hand, setting

$$R = -P^2 \ \text{ and } \ S = I + P^2,$$

we can easily see that projections R and S define orthogonal distributions such as D and $D^{\perp}$, respectively. Thus we have

$$PR = P, \ \ SP = 0, \ \ FR = 0 \ \text{ and } \ PS = 0,$$

that is, D is an invariant distribution, $D^{\perp}$ is an anti-invariant distribution and

$$R\xi = 0 \ \text{ and } \ S\xi = \xi.$$

This tell us that ξ belongs to $D^{\perp}$. Hence the proof is complete. □

Now, let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. Then for any $X, Y \in \Gamma(TM)$, by using (2), (4), (5), (15) and (18), we have

$$\begin{aligned} (\bar{\nabla}_X\varphi)Y &= \bar{\nabla}_X\varphi Y - \varphi\bar{\nabla}_X Y \\ g(\varphi X, Y)\xi - \eta(Y)\varphi X &= \bar{\nabla}_X PY + \bar{\nabla}_X FY - \varphi\nabla_X Y - \varphi h(X, Y). \end{aligned}$$

From the tangent and normal components of this last equations, respectively, we have

$$(\nabla_X P)Y = A_{FY}X + Bh(X, Y) + g(\varphi X, Y)\xi - \eta(Y)PX \tag{35}$$

and

$$(\nabla_X F)Y = Ch(X, Y) - h(X, PY) - \eta(Y)FX. \tag{36}$$

In the same way, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we have

$$\begin{aligned} (\bar{\nabla}_X\varphi)V &= \bar{\nabla}_X\varphi V - \varphi\bar{\nabla}_X V \\ g(\varphi X, V)\xi &= (\nabla_X B)V + (\nabla_X C)V + h(X, BV) - A_{CV}X + PA_V X \\ &+ FA_V X. \end{aligned} \tag{37}$$

From the normal and tangent components of (37), respectively, we have

$$(\nabla_X C)V = -h(X, BV) - FA_V X, \tag{38}$$

and

$$(\nabla_X B)V = g(FX, V)\xi + A_{CV}X - PA_V X. \tag{39}$$

On the other hand, since M is tangent to ξ, making use of (2) and (6) we obtain

$$A_V\xi = h(X, \xi) = 0 \tag{40}$$

for all $V \in \Gamma(T^{\perp}M)$ and $X \in \Gamma(TM)$. It is well-known that $Bh = 0$ plays an important role in the geometry of submanifolds. This means that the induced structure P is a Kenmotsu structure on M. Then (35) reduces to

$$(\nabla_X P)Y = g(PX, Y)\xi - \eta(Y)PX, \tag{41}$$

for any $X, Y \in \Gamma(D)$. This means that the induced structure P is a Kenmotsu structure on M. Moreover, for any $Z, W \in \Gamma(D^{\perp})$ and $U \in \Gamma(TM)$, also by using (2) and (6), we have

$$\begin{aligned} g(A_{FZ}W - A_{FW}Z, U) &= g(h(W, U), FZ) - g(h(Z, U), FW) \\ &= g(\bar{\nabla}_U W, \varphi Z) - g(\bar{\nabla}_U Z, \varphi W) \\ &= g(\varphi\bar{\nabla}_U Z, W) - g(\bar{\nabla}_U \varphi Z, W) = -g((\bar{\nabla}_U \varphi)Z, W) \\ &= g(\varphi Z, U)\eta(W) - g(\varphi W, U)\eta(Z) = 0. \end{aligned}$$

It follows that

$$A_{FZ}W = A_{FW}Z, \tag{42}$$

for any $Z, W \in \Gamma(D^{\perp})$.

Hence we have the following theorem.

Theorem 2 *Let* M *be a contact CR-submanifold of a Kenmotsu manifold* $\bar{M}$. *Then the anti-invariant distribution* $D^{\perp}$ *is completely integrable and its maximal integral submanifold is an anti-invariant submanifold of* $\bar{M}$.

Proof. For any $Z, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$, by using (2) and (42) we have

$$\begin{aligned} g([Z, W], X) &= g(\bar{\nabla}_Z W, X) - g(\bar{\nabla}_W Z, X) \\ &= g(\bar{\nabla}_W X, Z) - g(\bar{\nabla}_Z X, W) = g(\varphi\bar{\nabla}_W X, \varphi Z) - g(\varphi\bar{\nabla}_Z X, \varphi W) \\ &= g(\bar{\nabla}_W \varphi X - (\bar{\nabla}_W \varphi)X, \varphi Z) - g(\bar{\nabla}_Z \varphi X - (\bar{\nabla}_Z \varphi)X, \varphi W) \\ &= g(h(\varphi X, W), \varphi Z) - g(h(\varphi X, Z), \varphi W) - g(g(\varphi W, X)\xi \\ &- \eta(X)\varphi W, \varphi Z) + g(g(\varphi Z, X)\xi - \eta(X)\varphi Z, \varphi W) \\ &= g(A_{\varphi Z}W - A_{\varphi W}Z, \varphi X) = 0. \end{aligned}$$

Thus $[Z, W] \in \Gamma(D^{\perp})$ for any $Z, W \in \Gamma(D^{\perp})$, that is, $D^{\perp}$ is integrable. Thus the proof is complete. □

Theorem 3 *Let* M *be a contact CR-submanifold of a Kenmotsu manifold* $\bar{M}$. *Then the invariant distribution* D *is completely integrable and its maximal* *integral submanifold is an invariant submanifold of* $\bar{M}$ *if and only if the shape operator* A_V *of* M *satisfies*

$$A_V P + P A_V = 0, \tag{43}$$

for any $V \in \Gamma(T^{\perp}M)$.

Proof. In [1], it was proved that D is integrable if and only if the second fundamental form h of M satisfies the condition $h(X, PY) = h(PX, Y)$, for any $X, Y \in \Gamma(D)$. We can easily verify that this condition is equivalent to (43). So we omit the proof. □

Theorem 4 *Let* M *be a contact CR-submanifold of a Kenmotsu manifold* $\bar{M}$. *If the invariant distribution* D *is integrable, then* M *is* D*-minimal submanifold in* $\bar{M}$.

Proof. Let $\{e_1, e_2, \ldots, e_p, \varphi e_1, \varphi e_2, \ldots, \varphi e_p, \xi\}$ be an orthonormal frame of $\Gamma(D)$ and we denote the second fundamental form of M in $\bar{M}$ by h. Then the mean curvature tensor H of M can be written as

$$H = \frac{1}{2p+1}\left\{\sum_{i=1}^{p}\{h(e_i, e_i) + h(\varphi e_i, \varphi e_i)\} + h(\xi, \xi)\right\}. \tag{44}$$

By using (2) we mean that $h(\xi, \xi) = 0$. Since D is integrable, we have

$$\begin{aligned} H &= \frac{1}{2p+1}\left\{\sum_{i=1}^{p}\{h(e_i, e_i) + h(P^2 e_i, e_i)\right\} \\ &= \frac{1}{2p+1}\left\{\sum_{i=1}^{p}\{h(e_i, e_i) + h(-e_i + \eta(e_i)\xi, e_i)\right\} \\ &= \frac{1}{2p+1}\left\{\sum_{i=1}^{p}\{h(e_i, e_i) - h(e_i, e_i)\right\} = 0. \end{aligned}$$

This proves our assertion. □

Theorem 5 *Let* M *be a contact CR-submanifold of a Kenmotsu manifold* $\bar{M}$. *If the second fundamental form of the contact CR-submanifold* M *is parallel, then* M *is a totally geodesic submanifold.*

Proof. If the second fundamental form h of M is parallel, then by using (8), we have

$$\nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y) = 0,$$

for any $X, Y, Z \in \Gamma(TM)$. Here, choosing $Y = \xi$ and taking into account (2) and (40), we conclude that $h(X,Z) = 0$. This proves our assertion. □

Theorem 6 *Let M be a submanifold of a Kenmotsu manifold $\bar{M}$. Then M is a contact CR-submanifold if and only if the endomorphism C defines an f-structure on ν, that is, $C^3 + C = 0$.*

Proof. If M is a contact CR-submanifold, then from Theorem 1, we know that C is an f-structure on ν.

Conversely, if C is an f-structure on ν, from (24) we can derive $CFB = 0$. So for any $V \in \Gamma(T^{\perp}M)$, by using (21), we have

$$\begin{aligned} g(BCV, BCV) &= g(\varphi CV, BCV) = -g(CV, FBCV) \\ &= g(V, CFBCV) = 0. \end{aligned}$$

This implies that $BC = 0$ which is equivalent to $PB = 0$. Also, from Theorem 3.1 we conclude that M is a contact CR-submanifold. □

Theorem 7 *Let M be a submanifold of a Kenmotsu manifold $\bar{M}$. If the endomorphism P on M is parallel, then M is anti-invariant submanifold in $\bar{M}$.*

Proof. If P is parallel, from (35) and (40), we have

$$\begin{aligned} 0 &= g(\varphi X, Y) + g(A_{FY}X, \xi) + g(Bh(X,Y), \xi) \\ &= g(\varphi X, Y) + g(h(X,\xi), FY) \\ &= g(\varphi X, Y), \end{aligned}$$

for any $X, Y \in \Gamma(TM)$. This implies that M is anti-invariant submanifold. □

Theorem 8 *Let M be a submanifold of a Kenmotsu manifold $\bar{M}$. If the endomorphism F is parallel, then M is invariant submanifold in $\bar{M}$.*

Proof. If F is parallel, then from (36), we have

$$Ch(X,Y) - h(X,PY) - \eta(Y)FX = 0,$$

for any $X, Y \in \Gamma(TM)$. Here, choosing $Y = \xi$ and taking into account that $h(X,\xi) = 0$, we conclude that $FX = 0$. This proves our assertion. □

Theorem 9 *Let* M *be a submanifold of a Kenmotsu manifold* $\bar{M}$. *Then the structure* F *is parallel if and only if the structure* B *is parallel.*

Proof. Making use of (36) and (39), we have

$$\begin{aligned} g((\nabla_X F)Y, V) &= g(Ch(X,Y), V) - g(h(X,PY), V) - \eta(Y)g(FX, V) \\ &= -g(h(X,Y), CV) - g(A_V X, PY) - g(FX, V)\eta(Y) \\ &= -g(A_{CV}Y, X) + g(PA_V X, Y) - g(FX, V)\eta(Y) \\ &= -g((\nabla_X B)V, Y), \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. This proves our assertion. □

From Theorem 8 and Theorem 9, we have the following corollary.

Corollary 1 *Let* M *be a submanifold of a Kenmotsu manifold* $\bar{M}$. *If the structure* B *is parallel, then* M *is invariant submanifold.*

For a contact CR-submanifold M, if the invariant distribution D and anti-invariant distribution $D^{\perp}$ are totally geodesic in M, then M is called contact CR-product. The following theorems characterize contact CR-products in Kenmotsu manifolds.

Theorem 10 *Let* M *be a contact CR-submanifold of a Kenmotsu manifold* $\bar{M}$. *Then* M *is a contact CR-product if and only if the shape operator* A *of* M *satisfies the condition*

$$A_{\varphi W}\varphi X + \eta(X)W = 0, \tag{45}$$

for all $X \in \Gamma(D)$ *and* $W \in (D^{\perp})$.

Proof. Let us assume that M is a contact CR-submanifold of $\bar{M}$. Then by using (2) and (4), we obtain

$$\begin{aligned} g(A_{\varphi W}\varphi X + \eta(X)W, Y) &= g(h(\varphi X, Y), \varphi W) = g(\bar{\nabla}_Y \varphi X, \varphi W) \\ &= g((\bar{\nabla}_Y \varphi)X + \varphi \bar{\nabla}_Y X, \varphi W) \\ &= g(g(\varphi Y, X)\xi - \eta(X)\varphi Y, \varphi W) + g(\nabla_Y X, W) \\ &= g(\nabla_Y X, W) \end{aligned}$$

and

$$\begin{aligned} g(A_{\varphi W}\varphi X + \eta(X)W, Z) &= g(h(\varphi X, Z), \varphi W) + \eta(X)g(Z, W) \\ &= g(\bar{\nabla}_Z \varphi X, \varphi W) + \eta(X)g(Z, W) \\ &= g((\bar{\nabla}_Z \varphi)X + \varphi \bar{\nabla}_Z X, \varphi W) \\ &= g(g(\varphi Z, X)\xi - \eta(X)\varphi Z, \varphi W) + g(\bar{\nabla}_Z X, W) \\ &+ \eta(X)g(Z, W) = -g(\nabla_Z W, X), \end{aligned}$$

for all $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$. So $\nabla_X Y \in \Gamma(D)$ and $\nabla_Z W \in \Gamma(D^{\perp})$ if and only if (45) is satisfied. This proves our assertion. □

Theorem 11 *Let* M *be a contact CR-submanifold of a Kenmotsu manifold* $\bar{M}$. *Then* M *is contact CR-product if and only if*

$$Bh(X, U) = 0, \tag{46}$$

for all $U \in \Gamma(TM)$ *and* $X \in \Gamma(D)$.

Proof. For a contact CR-product M in [1], it was proved that $A_{\varphi W}X = 0$, for all $X \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$. This condition implies (46).

Conversely, we suppose that (46) is satisfied. Then we have

$$\begin{aligned} g(\nabla_X Y, W) &= g(\varphi\bar{\nabla}_X Y, \varphi W) = g(\bar{\nabla}_X \varphi Y, \varphi W) - g((\bar{\nabla}_X \varphi)Y, \varphi W) \\ &= g(h(X, PY), \varphi W) - g(g(\varphi X, Y)\xi - \eta(Y)\varphi X, \varphi W) \\ &= -g(Bh(X, PY), W) \end{aligned}$$

and

$$\begin{aligned} g(\nabla_Z W, \varphi X) &= -g(\bar{\nabla}_Z \varphi X, W) = -g((\bar{\nabla}_Z \varphi)X + \varphi\bar{\nabla}_Z X, W) \\ &= -g(g(\varphi Z, X)\xi - \eta(X)\varphi Z, W) + g(\bar{\nabla}_Z X, \varphi W) \\ &= -g(Bh(X, Z), W), \end{aligned}$$

for all $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$. This proves our assertion □

Theorem 12 *Let* M *be a contact CR-submanifold of a Kenmotsu manifold* $\bar{M}$. *The structure* C *is parallel if and only if the shape operator* A_V *of* M *satisfies the condition*

$$A_V BU = A_U BV, \tag{47}$$

for all $U, V \in \Gamma(T^{\perp}M)$.

Proof. From (21) and (38), we have

$$\begin{aligned} g((\nabla_X C)V, U) &= -g(h(X, BV), U) - g(FA_V X, U) = -g(A_U BV) + g(A_V X, BU) \\ &= g(A_V BU - A_U BV, X), \end{aligned}$$

for all $X \in \Gamma(TM)$. The proof is complete. □

Theorem 13 *Let* M *be a contact CR-submanifold of a Kenmotsu manifold* $\bar{M}$*. If* C *is parallel, then* M *is totally geodesic submanifold of* $\bar{M}$*.*

Proof. If C is parallel, from (38), we have

$$\varphi A_V X + h(X, BV) = 0, \tag{48}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. Applying φ to (48) and taking into account (2) and (40), we obtain

$$-A_V X + Bh(X, BV) = 0. \tag{49}$$

On the other hand, also by using (24) and (47), we conclude that

$$g(Bh(X, BV), Z) = -g(h(X, BV), FZ) = -g(A_{FZ}BV, X) = -g(A_V BFZ, X) = 0,$$

for all $Z \in \Gamma(D^{\perp})$. So arrive at $A_V = 0$, that is, M is totally geodesic in $\bar{M}$. □

4 Contact CR-submanifolds in Kenmotsu space forms

Theorem 14 *Let* M *be a contact CR-submanifold of a Kenmotsu space form* $\bar{M}(c)$ *such that* $c \neq -1$*. If* M *is a curvature-invariant contact CR-submanifold, then* M *is invariant or anti-invariant submanifold.*

Proof. We suppose that M is a curvature-invariant contact CR-submanifold of a Kenmotsu space form $\bar{M}(c)$ such that $c \neq -1$. Then from (14) we have

$$g(X, PZ)FY - g(Y, PZ)FX + 2g(X, PY)FZ = 0, \tag{50}$$

for any $X, Y, Z \in \Gamma(TM)$. Taking $Z = X$ in equation (50), we have

$$3g(PY, X)FX = 0.$$

This implies that $F = 0$ or $P = 0$, that is, M is invariant or anti-invariant submanifold. Thus the proof is complete. □

Thus we have the following corollary.

Corollary 2 *There isn't any curvature-invariant proper contact CR- submanifold of a Kenmotsu space form* $\bar{M}(c)$ *such that* $c \neq -1$*.*

Theorem 15 *Let* M *be a contact CR-submanifold of a Kenmotsu space form* $\bar{M}(c)$ *with flat normal connection such that* $c \neq -1$*. If* $PA_V = A_VP$ *for any vector* V *normal to* M*, then* M *is an anti-invariant or generic submanifold of* $\bar{M}(c)$*.*

Proof. If the normal connection of M is flat, then from (12) we have

$$\begin{aligned} g([A_U, A_V]X, Y) &= \left(\frac{c+1}{4}\right)\{g(X, \varphi V)g(\varphi Y, U) - g(Y, \varphi V)g(\varphi X, U) \\ &+ 2g(X, \varphi Y)g(\varphi V, U)\}, \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$. Here, choosing $X = PY$ and $V = CU$, by direct calculations, we conclude that

$$g(A_UA_{CU}PY - A_{CU}A_UPY, Y) = (\frac{c+1}{2})\{g(P^2Y, Y)g(CU, CU)\}.$$

If $PA_U = A_UP$, then we can easily see that $(c+1)Tr(P^2)g(CU, CU) = 0$. This tells us that $P = 0$ (that is, M is anti-invariant submanifold) or $CU = 0$(that is, M is generic submanifold). □

Theorem 16 *Let* M *be a proper contact CR-submanifold of a Kenmotsu space form* $\bar{M}(c)$*. If the invariant distribution* D *is integrable, then* $c < -1$*.*

Proof. If the invariant distribution D is integrable, the from (43), we have

$$PA_VY + A_VPY = 0. \tag{51}$$

It follows that

$$g(A_VPY, BU) = 0, \tag{52}$$

for any $Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$. By differentiating the covariant derivative in the direction of $X \in \Gamma(T\bar{M})$ of (52), and by using (9), (19), we get

$$\begin{aligned} 0 &= g(\bar{\nabla}_X A_VPY, BU) + g(A_VPY, \bar{\nabla}_X BU) \\ &= g((\nabla_X A)_V PY + A_{\nabla^{\perp}_X V}PY + A_V(\nabla_X PY), BU) \\ &+ g((\nabla_X B)U + B\nabla^{\perp}_X U, A_VPY). \end{aligned}$$

Again, by using (35), (39) and taking into account (51), we obtain

$$\begin{aligned} -((\nabla_X A)_V PY, BU) &= -g((\nabla_X h)(PY, BU), V) \\ &= g(A_V\{A_{FY}X + Bh(X,Y) + g(\varphi X, Y)\xi - \eta(Y)PX\}, BU) \\ &+ g(g(FX,U)\xi + A_{CU}X - PA_U X, A_V PY) \\ &= g(A_V A_{FY}X + A_V Bh(X,Y), BU) + g(A_{CU}X, A_V PY) \\ &+ g(A_U PX, A_V PY) \\ -g((\nabla_X h)(PY, BU), V) &= g(A_{FY}X, A_V BU) + g(A_V BU, Bh(X,Y)) \\ +g(A_{CU}X, A_V PY) & \\ &+ g(A_U PX, A_V PY). \end{aligned}$$

Here, if PX is taken instead of X in this last equation, we have

$$\begin{aligned} -g((\bar{\nabla}_{PX} h)(PY, BU), V) &= g(A_{FY}PX, A_V BU) + g(A_V BU, Bh(PX,Y)) \\ &+ g(A_{CU}PX, A_V PY) + g(A_U P^2 X, A_V PY). \end{aligned}$$

Also, by using (51) and taking into account that M is a contact CR-submanifold in $\bar{M}(c)$, by direct calculations we have

$$\begin{aligned} g((\bar{\nabla}_{PY} h)(PX, BU) &- (\bar{\nabla}_{PX} h)(PY, BU), V) = g(A_{CU}A_V PY, PX) \\ &- g(A_{CU}A_V PX, PY) - g(A_U P^3 X, A_V Y) \\ &- g(A_U Y, A_V P^3 X) \\ &= g(A_U PX, A_V Y) + g(A_U Y, A_V PX) \\ &+ g(A_{CU}PX, A_V PY) - g(A_{CU}PY, A_V PX). \end{aligned} \tag{53}$$

Also, from (14), we get

$$\begin{aligned} \left(\frac{c+1}{2}\right) g(PY, X) g(BU, BV) &= g((\bar{\nabla}_{PY} h)(PX, BU) \\ &- (\bar{\nabla}_{PX} h)(PY, BU), V). \end{aligned} \tag{54}$$

Substituting (53) into (54), we obtain

$$\begin{aligned} \left(\frac{c+1}{4}\right) g(PY, X) g(BU, BV) &= g(A_U PX, A_V Y) + g(A_U Y, A_V PX) \\ &+ g(A_{CU}PX, A_U PY) - g(A_{CU}PY, A_V PX), \end{aligned}$$

which implies that

$$\left(\frac{c+1}{4}\right) g(PY, PY) g(U, U) = -g(A_U PY, A_U PY).$$

This proves our assertion. □

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