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On certain subclasses of analytic functions associated with Poisson distribution series

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Abstract. In this paper, we find the necessary and sufficient conditions, inclusion relations for Poisson distribution series $\mathcal{K}(m,z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!}e^{-m}z^n$ to be in the subclasses $\mathcal{S}(k,\lambda)$ and $\mathcal{C}(k,\lambda)$ of analytic functions with negative coefficients. Further, we obtain necessary and sufficient conditions for the integral operator $\mathcal{G}(m,z) = \int_0^z \frac{\mathcal{F}(m,t)}{t}dt$ to be in the above classes.

1 Introduction and definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad z \in \mathcal{U}. \tag{2}$$

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A function f of the form (2) is in $\mathcal{S}(k,\lambda)$ if it satisfies the condition

$$\left|\frac{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-1}{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}+1}\right|<k,\quad (0<k\leq 1,\ 0\leq\lambda<1, z\in\mathcal{U})$$

and $f\in\mathcal{C}(k,\lambda)$ if and only if $zf'\in\mathcal{S}(k,\lambda)$. The class $\mathcal{S}(k,\lambda)$ was introduced by Frasin et al. [3].

We note that $\mathcal{S}(k,0)=\mathcal{S}(k)$ and $\mathcal{C}(k,0)=\mathcal{C}(k)$, where the classes $\mathcal{S}(k)$ and $\mathcal{C}(k)$ were introduced and studied by Padmanabhan [9] (see also, [5], [8]).

A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(A,B)$,$\tau\in\mathbb{C}\backslash\{0\}$, $-1\leq B<A\leq 1$, if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau-B[f'(z)-1]}\right|<1,\qquad z\in\mathcal{U}.$$

This class was introduced by Dixit and Pal [2].

A variable x is said to be Poisson distributed if it takes the values $0,1,2,3,\ldots$ with probabilities e^{-m}, $m\frac{e^{-m}}{1!}$, $m^2\frac{e^{-m}}{2!}$, $m^3\frac{e^{-m}}{3!},\ldots$ respectively, where m is called the parameter. Thus

$$P(x=r)=\frac{m^r e^{-m}}{r!},\ r=0,1,2,3,\ldots.$$

Very recently, Porwal [10] (see also, [6, 7]) introduce a power series whose coefficients are probabilities of Poisson distribution

$$\mathcal{K}(m,z)=z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n,\qquad z\in\mathcal{U},$$

where $m>0$. By ratio test the radius of convergence of above series is infinity. In [10], Porwal also defined the series

$$\mathcal{F}(m,z)=2z-\mathcal{K}(m,z)=z-\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n,\qquad z\in\mathcal{U}.$$

Using the Hadamard product, Porwal and Kumar [12] introduced a new linear operator $\mathcal{I}(m,z):\mathcal{A}\to\mathcal{A}$ defined by

$$\mathcal{I}(m,z)f=\mathcal{K}(m,z)*f(z)=z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}a_n z^n,\qquad z\in\mathcal{U},$$

where $*$ denote the convolution or Hadamard product of two series.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions by using hypergeometric functions (see [1, 4, 13, 14]) and by the recent investigations of Porwal ([10, 12, 11]), in the present paper we determine the necessary and sufficient conditions for $\mathcal{F}(m,z)$ to be in our new classes $\mathcal{S}(k,\lambda)$ and $\mathcal{C}(k,\lambda)$ and connections of these subclasses with $\mathcal{R}^{\tau}(A,B)$. Finally, we give conditions for the integral operator $\mathcal{G}(m,z)=\int_0^z \frac{\mathcal{F}(m,t)}{t}dt$ to be in the classes $\mathcal{S}(k,\lambda)$ and $\mathcal{C}(k,\lambda)$.

To establish our main results, we will require the following Lemmas.

Lemma 1 [3] *A function* f *of the form* (2) *is in* $\mathcal{S}(k,\lambda)$ *if and only if it satisfies*

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\,|a_n|\le 2k \tag{3}$$

where $0<k\le 1$ and $0\le\lambda<1$. The result is sharp.

Lemma 2 [3] *A function* f *of the form* (2) *is in* $\mathcal{C}(k,\lambda)$ *if and only if it satisfies*

$$\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\,|a_n|\le 2k \tag{4}$$

where $0<k\le 1$ *and* $0\le\lambda<1$. *The result is sharp.*

Lemma 3 [2] *If* $f\in\mathcal{R}^{\tau}(A,B)$ *is of the form, then*

$$|a_n|\le (A-B)\frac{|\tau|}{n},\qquad n\in\mathbb{N}-\{1\}.$$

The result is sharp.

2 The necessary and sufficient conditions

Theorem 1 *If* $m>0$, $0<k\le 1$ *and* $0\le\lambda<1$, *then* $\mathcal{F}(m,z)$ *is in* $\mathcal{S}(k,\lambda)$ *if and only if*

$$((1-\lambda)+k(1+\lambda))me^{m}\le 2k. \tag{5}$$

Proof. Since

$$\mathcal{F}(m,z)=z-\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n \tag{6}$$

according to (3) of Lemma 1, we must show that

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}\leq 2ke^{m}. \tag{7}$$

Writing $n=(n-1)+1$, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}\\
&\quad=\sum_{n=2}^{\infty}[(n-1)((1-\lambda)+k(1+\lambda))+2k]\frac{m^{n-1}}{(n-1)!}\\
&\quad=[(1-\lambda)+k(1+\lambda)]\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}+2k\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\\
&\quad=((1-\lambda)+k(1+\lambda))me^{m}+2k(e^{m}-1).
\end{aligned} \tag{8}$$

But this last expression is bounded above by $2ke^{m}$ if and only if (5) holds. □

Theorem 2 *If* $m>0$, $0<k\leq 1$ *and* $0\leq\lambda<1$, *then* $\mathcal{F}(m,z)$ *is in* $\mathcal{C}(k,\lambda)$ *if and only if*

$$((1-\lambda)+k(1+\lambda))m^{2}e^{m}+2(1+2k+k\lambda-\lambda)me^{m}\leq 2k. \tag{9}$$

Proof. In view of Lemma 2, it suffices to show that

$$\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}\leq 2ke^{m}.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}\\
&\quad=\sum_{n=2}^{\infty}n^{2}((1-\lambda)+k(1+\lambda))+n(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}.
\end{aligned} \tag{10}$$

Writing $n=(n-1)+1$ and $n^{2}=(n-1)(n-2)+3(n-1)+1$, in (10) we see that

$$\sum_{n=2}^{\infty}n^{2}((1-\lambda)+k(1+\lambda))+n(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}$$

$$= \sum_{n=2}^{\infty}(n-1)(n-2)((1-\lambda)+k(1+\lambda))\frac{m^{n-1}}{(n-1)!}$$

$$+\sum_{n=2}^{\infty}(n-1)[3((1-\lambda)+k(1+\lambda)+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}+\sum_{n=2}^{\infty}2k\frac{m^{n-1}}{(n-1)!}$$

$$= ((1-\lambda)+k(1+\lambda))\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-3)!} \;+2(1+2k+k\lambda-\lambda)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}$$

$$+2k\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}$$

$$= ((1-\lambda)+k(1+\lambda))m^2e^m+2(1+2k+k\lambda-\lambda)me^m+2k\,(e^m-1).$$

But this last expression is bounded above by $2ke^m$ if and only if (9) holds. □

By specializing the parameter $\lambda = 0$ in Theorems 1 and 2 , we have the following corollaries.

Corollary 1 *If* $m > 0$ *and* $0 < k \leq 1$, *then* $\mathcal{F}(m,z)$ *is in* $\mathcal{S}(k)$ *if and only if*

$$(1+k)me^m \leq 2k. \tag{11}$$

Corollary 2 *If* $m > 0$ *and* $0 < k \leq 1$, *then* $\mathcal{F}(m,z)$ *is in* $\mathcal{C}(k)$ *if and only if*

$$(1+k)m^2e^m+2(1+2k)me^m \leq 2k. \tag{12}$$

3 Inclusion properties

Theorem 3 *Let* $m > 0$, $0 < k \leq 1$ *and* $0 \leq \lambda < 1$. *If* $f \in \mathcal{R}^{\tau}(A,B)$, *then* $\mathcal{I}(m,z)f$ *is in* $\mathcal{S}(k,\lambda)$ *if and only if*

$$\begin{aligned}(A-B)\,|\tau|\,\Big[&((1-\lambda)+k(1+\lambda))(1-e^{-m})\\ &+\frac{(1-\lambda)(k-1)}{m}(1-e^{-m}(1+m))\Big] \leq 2k.\end{aligned} \tag{13}$$

Proof. In view of Lemma 1, it suffices to show that

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}\,|a_n| \leq 2ke^m.$$

Since $f \in \mathcal{R}^{\tau}(A,B)$, then by Lemma 3, we get

$$|a_n| \leq \frac{(A-B)\,|\tau|}{n}. \tag{14}$$

Thus, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}|a_n| \\
&\leq (A-B)|\tau|\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{n!} \\
&= (A-B)|\tau|\left[((1-\lambda)+k(1+\lambda))\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}+\frac{(1-\lambda)(k-1)}{m}\sum_{n=2}^{\infty}\frac{m^n}{n!}\right] \\
&= (A-B)|\tau|\left[((1-\lambda)+k(1+\lambda))(e^m-1)+\frac{(1-\lambda)(k-1)}{m}(e^m-1-m)\right].
\end{aligned}$$

But this last expression is bounded above by $2ke^m$ if and only if (13) holds. □

Theorem 4 *Let* $m > 0$, $0 < k \leq 1$ *and* $0 \leq \lambda < 1$. *If* $f \in \mathcal{R}^{\tau}(A, B)$, *then* $\mathcal{F}(m, z)f$ *is in* $\mathcal{C}(k, \lambda)$ *if and only if*

$$(A-B)|\tau|\,[((1-\lambda)+k(1+\lambda))m+2k(1-e^{-m})] \leq 2k. \tag{15}$$

Proof. In view of Lemma 2, it suffices to show that

$$\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}|a_n| \leq 2ke^m.$$

Using (14), we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}|a_n| \\
&\leq \sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}\frac{(A-B)|\tau|}{n} \\
&= (A-B)|\tau|\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!} \\
&= (A-B)|\tau|\sum_{n=2}^{\infty}[(n-1)((1-\lambda)+k(1+\lambda))+2k]\frac{m^{n-1}}{(n-1)!} \\
&= (A-B)|\tau|\left[((1-\lambda)+k(1+\lambda))\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}+2k\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\right]
\end{aligned}$$

$$= (A-B)\,|\tau|\,[((1-\lambda)+k(1+\lambda))me^{m}+2k(e^{m}-1)].$$

But this last expression is bounded above by $2ke^{m}$ if and only if (15) holds. □

By taking $\lambda = 0$ in Theorems 3 and 4, we obtain the following corollaries.

Corollary 3 *Let* $m > 0$ *and* $0 < k \leq 1$. *If* $f \in \mathcal{R}^{\tau}(A,B)$, *then* $\mathcal{I}(m,z)f$ *is in* $\mathcal{S}(k)$ *if and only if*

$$(A-B)\,|\tau|\left[(1+k)(1-e^{-m})+\frac{(k-1)}{m}(1-e^{-m}(1+m))\right] \leq 2k. \tag{16}$$

Corollary 4 *Let* $m > 0$ *and* $0 < k \leq 1$. *If* $f \in \mathcal{R}^{\tau}(A,B)$, *then* $\mathcal{I}(m,z)f$ *is in* $\mathcal{C}(k)$ *if and only if*

$$(A-B)\,|\tau|\,[(1+k)m+2k(1-e^{-m})] \leq 2k. \tag{17}$$

4 An integral operator

In this section, we obtain the necessary and sufficient conditions for the integral operator $\mathcal{G}(m,z)$ defined by

$$\mathcal{G}(m,z) = \int_0^z \frac{\mathcal{F}(m,t)}{t}\,dt \tag{18}$$

to be in the class $\mathcal{C}(k,\lambda)$.

Theorem 5 *If* $m > 0$, $0 < k \leq 1$ *and* $0 \leq \lambda < 1$, *then the integral operator* $\mathcal{G}(m,z)$ *defined by* (18) *is in* $\mathcal{C}(k,\lambda)$ *if and only if* (5) *is satisfied.*

Proof. Since

$$\mathcal{G}(m,z) = z - \sum_{n=2}^{\infty} \frac{e^{-m}m^{n-1}}{n!} z^{n}$$

then by Lemma 2, we need only to show that

$$\sum_{n=2}^{\infty} n[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{n!} \leq 2ke^{m}.$$

or, equivalently

$$\sum_{n=2}^{\infty} [n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!} \leq 2ke^{m}.$$

From (8) it follows that

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}$$
$$= ((1-\lambda)+k(1+\lambda))me^{m}+2k(e^{m}-1)$$

and this last expression is bounded above by $2ke^{m}$ if and only if (5) holds. □

The proof of Theorem 6 (below) is much similar to that of Theorem 5 and so the details are omitted.

Theorem 6 *If* $m>0$, $0<k\le 1$ *and* $0\le\lambda<1$, *then the integral operator* $\mathcal{G}(m,z)$ *defined by* (18) *is in* $\mathcal{S}(k,\lambda)$ *if and only if*

$$((1-\lambda)+k(1+\lambda))(1-e^{-m})+\frac{(1-\lambda)(k-1)}{m}(1-e^{-m}-me^{-m})\le 2k.$$

By taking $\lambda=0$ in Theorems 5 and 6, we obtain the following corollaries.

Corollary 5 *If* $m>0$ *and* $0<k\le 1$, *then the integral operator defined by* (18) *is in* $\mathcal{C}(k)$ *if and only if* (11) *is satisfied.*

Corollary 6 *If* $m>0$ *and* $0<k\le 1$,*then the integral operator defined by* (18) *is in* $\mathcal{S}(k)$ *if and only if*

$$(1+k)(1-e^{-m})+\frac{(k-1)}{m}(1-e^{-m}-me^{-m})\le 2k.$$

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