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On A-energy and S-energy of certain class of graphs

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Abstract. Let A and S be the adjacency and the Seidel matrix of a graph G respectively. A-energy is the ordinary energy $E(G)$ of a graph G defined as the sum of the absolute values of eigenvalues of A. Analogously, S-energy is the Seidel energy $E_S(G)$ of a graph G defined to be the sum of the absolute values of eigenvalues of the Seidel matrix S. In this article, certain class of A-equienergetic and S-equienergetic graphs are presented. Also some linear relations on A-energies and S-energies are given.

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1 Introduction

Let G be a simple, finite and undirected graph of order n with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. The adjacency matrix $A = [a_{ij}]$ of G is a square matrix of order n whose (i, j)-th entry $a_{ij} = 1$ if v_i and v_j are adjacent and 0 otherwise. The eigenvalues $\theta_1, \theta_2, \dots, \theta_n$ of A are called the A-eigenvalues of G and their collection is called the spectrum or A-spectrum of G. If $\theta_1, \theta_2, \dots, \theta_k$ are the distinct A-eigenvalues of G of order n with respective multiplicities $m_1, m_2, \dots, m_k$, then the A-spectrum of G is denoted by

$$\mathrm{Spec}(G) = \begin{pmatrix} \theta_1 & \theta_2 & \cdots & \theta_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix}, \text{ where } \sum_{j=1}^{k} m_j = n.$$

In 1966 J. H. van Lint and J. J. Seidel introduced real symmetric $\{0, \pm 1\}$-matrix called the Seidel matrix S is defined as $S = J - I - 2A$, where J is the matrix of order n whose all entries are equal to 1 and I is the identity matrix of order n. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of S are called the Seidel eigenvalues or S-eigenvalues of G and their collection is called the Seidel spectrum or S-spectrum of G. If $\lambda_1, \lambda_2, \dots, \lambda_k$ are the distinct S-eigenvalues of G of order n with respective multiplicities $m_1, m_2, \dots, m_k$, then the Seidel spectrum or S-spectrum of G is denoted by

$$\mathrm{Spec}_S(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix}, \text{ where } \sum_{j=1}^{k} m_j = n.$$

The number of positive and negative A-eigenvalues of G are denoted by n^+ and n^- respectively. The complement of a graph G is denoted by $\overline{G}$. A graph G is an r-regular graph if all its vertices have same degree equal to r. The line graph of G, denoted by $L(G)$ is a graph whose vertex set has one-to-one correspondence with the edge set of G and two vertices are adjacent in $L(G)$ if the corresponding edges are adjacent in G. For $k = 1, 2, \dots,$ the k-th iterated line graph of G is defined as $L^k(G) = L(L^{k-1}(G))$, where $L^0(G) = G$ and $L^1(G) = L(G)$ [10]. Let K_n be the complete graph of order n and $K_{n_1, n_2, \dots, n_k}$ be the complete multipartite graph of order $n = \sum_{j=1}^{k} n_j$.

If $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ be the A-eigenvalues of G, then the energy or A-energy is defined as

$$E(G) = \sum_{j=1}^{n} |\theta_j| = 2 \sum_{j=1}^{n^+} \theta_j = -2 \sum_{j=1}^{n^-} \theta_{n-j+1}.$$

Several researchers have introduced many graph operations such as complement, disjoint union, join, graph products etc. The graph products Cartesian product, tensor product and strong product are known as the standard graph products and have been well studied in the graph theory. The energy of a graph introduced in 1978 [8] in connection with molecular chemistry and gained its own importance in the spectral graph theory.

Two graphs G_1 and G_2 of same order are said to be equienergetic or A-equienergetic if $E(G_1) = E(G_2)$. Similar to A-energy, the Seidel energy or S-energy $E_S(G)$ [9] of a graph G is defined as the sum of the absolute values of the eigenvalues of Seidel matrix S. Two graphs G_1 and G_2 of same order are said to be Seidel equienergetic or S-equienergetic if $E_S(G_1) = E_S(G_2)$. Numerous results dealing with the non-cospectral, A-equienergetic graphs have been appeared in the literature. Balakrishnan [2] and Stevanović [23] constructed A-equienergetic graphs using tensor product. Ramane and Walikar [20] and Liu and Liu [12] constructed A-equienergetic graphs by join of two graphs. Bonifácio et al. [3] and Ramane et al. [17] obtained some class of A-equienergetic graphs through Cartesian product, tensor product and strong product. Ramane et al. [21] obtained non-cospectral A-equienergetic iterated line graphs from regular graphs. For other results on A-equienergetic and S-equienergetic graphs one can see [1, 4, 7, 11, 12, 13, 14, 15, 18, 22, 24]. For other notation, terminology and results related to the spectra of graphs we follow [6]. One of the interesting and difficult problem in the study of energy of a graph in spectral graph theory is to find non-isomorphic graphs of same order with same energy. So for in the literature the linear relations on energies of two non isomorphic graphs are not well studied except A-equienergetic or S-equienergetic graphs. This motivates to find some class of graphs which satisfies the linear relations on energies of different graphs.

This article is organized as follows. In section 2, basic definitions, known results on A-eigenvalues of graph products, A-energy of a graph, S-eigenvalues and S-energy of a graph are presented. In section 3, certain class of A-equienergetic graphs are constructed and obtained some linear relations on the A-energies. In section 4, some class of the S-equienergetic graphs are constructed and obtained some linear relations on the S-energies.

2 Preliminaries

In this section, we shall list some known results which are needed in the next two sections.

Theorem 1 [6] *Let* G *be an* r*-regular graph of order* n *with the* A*-eigenvalues* $\theta_1 = r, \theta_2, \dots, \theta_n$*. Then the* A*-eigenvalues of* $\overline{G}$ *are* $n-r-1, -\theta_2-1, \dots, -\theta_n-1$*.*

Theorem 2 [6] *Let* G *be an* r*-regular graph on* n *vertices and* m *edges with the* A*-eigenvalues* $\theta_1 = r, \theta_2, \dots, \theta_n$*. Then the* A*-eigenvalues of* $L(G)$ *are* $\theta_i + r - 2$*,* $i = 1, 2, \dots, n$ *and* -2 *(*$m - n$ *times).*

Theorem 3 [5] *Let* G *is an* r*-regular graph of order* n *with the* A*-eigenvalues* $\theta_1 = r, \theta_2, \dots, \theta_n$*. Then the eigenvalues of* S *are* $n-2r-1, -1-2\theta_2, \dots, -1-2\theta_n$*.*

Theorem 4 [16] *Let* G_1 *and* G_2 *be two* r*-regular graphs of same order* n*,* $r \geq 3$*. Then for* $k \geq 2$*,* $L^k(G_1)$ *and* $L^k(G_2)$ *are* S*-equienergetic.*

The Cartesian product of two graphs G_1 and G_2 is the graph $G_1 \Box G_2$ with vertex set $V(G_1) \times V(G_2)$, in which the vertices (u_1, u_2) and (v_1, v_2) are adjacent if either u_1 is adjacent to v_1 in G_1 and u_2 is equal to v_2 or u_1 is equal to v_1 and u_2 is adjacent to v_2 in G_2.

The tensor product of two graphs G_1 and G_2 is the graph $G_1 \otimes G_2$ with vertex set $V(G_1) \times V(G_2)$, in which the vertices (u_1, u_2) and (v_1, v_2) are adjacent if u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2.

The strong product of two graphs G_1 and G_2 is the graph $G_1 \boxtimes G_2$ with vertex set $V(G_1) \times V(G_2)$, in which the vertices (u_1, u_2) and (v_1, v_2) are adjacent whenever u_1 and v_1 are equal or adjacent in G_1, and u_2 and v_2 are equal or adjacent in G_2. If G_1 and G_2 are two regular graphs then $G_1 \Box G_2$, $G_1 \otimes G_2$ and $G_1 \boxtimes G_2$ are also regular graphs.

Lemma 5 [5] *If* $\mu_1, \mu_2, \dots, \mu_n$ *are the* A*-eigenvalues of a graph* G_1 *and* $\sigma_1, \sigma_2, \dots, \sigma_m$ *are the* A*-eigenvalues of a graph* G_2 *then*
(i) the A*-eigenvalues of* $G_1 \Box G_2$ *are* $\mu_i + \sigma_j$*,* $i = 1, 2, \dots, n$*;* $j = 1, 2, \dots, m$*,*
(ii) the A*-eigenvalues of* $G_1 \otimes G_2$ *are* $\mu_i \sigma_j$*,* $i = 1, 2, \dots, n$*;* $j = 1, 2, \dots, m$*,*
(iii) the A*-eigenvalues of* $G_1 \boxtimes G_2$ *are* $\mu_i \sigma_j + \mu_i + \sigma_j$*,* $i = 1, 2, \dots, n$*;* $j = 1, 2, \dots, m$*.*

Lemma 6 [17] *The* A*-spectrum of the line graph of a complete bipartite graph* $K_{p,q}$*, where* $p, q \geq 2$ *is*

$$\mathrm{Spec}(L(K_{p,q})) = \begin{pmatrix} p+q-2 & p-2 & q-2 & -2 \\ 1 & q-1 & p-1 & (p-1)(q-1) \end{pmatrix}.$$

3 A-equienergetic graphs and linear relations on A-energies of certain class of graphs

3.1 A-equienergetic graphs

Theorem 7 *Let* G_1 *and* G_2 *be two* r*-regular* A*-equienergetic graphs of order* n*. Then for* $p \geq r$, $E(G_1 \Box K_{\underbrace{p,p,\dots,p}_{k\ \text{times}}}) = E(G_2 \Box K_{\underbrace{p,p,\dots,p}_{k\ \text{times}}})$.

Proof. We have, $\operatorname{Spec}(K_{\underbrace{p,p,\dots,p}_{k\ \text{times}}}) = \begin{pmatrix} p(k-1) & 0 & -p \\ 1 & k(p-1) & k-1 \end{pmatrix}$.

Let $\operatorname{Spec}(G_1) = \begin{pmatrix} r & \theta_2 & \dots & \theta_k \\ 1 & m_2 & \dots & m_k \end{pmatrix}$, where $1 + \sum_{j=2}^{k} m_j = n$.

By (i) of Lemma 5,

$$\operatorname{Spec}(G_1 \Box K_{\underbrace{p,p,\dots,p}_{k\ \text{times}}}) = \left(\begin{matrix} r+p(k-1) & r & r-p & \theta_2+p(k-1) & \cdots \\ 1 & k(p-1) & k-1 & m_2 & \cdots \end{matrix} \right.$$

$$\left. \begin{matrix} \theta_k+p(k-1) & \theta_2 & \cdots & \theta_k & \theta_2-p & \cdots & \theta_k-p \\ m_k & km_2(p-1) & \cdots & km_k(p-1) & m_2(k-1) & \cdots & m_k(k-1) \end{matrix} \right).$$

Therefore

$E(G_1 \Box K_{\underbrace{p,p,\dots,p}_{k\ \text{times}}})$

$$\begin{aligned} &= |r+p(k-1)| + |r|k(p-1) + |r-p|(k-1) + \sum_{i=2}^{k} m_i|\theta_i + p(k-1)| \\ &\quad + \sum_{i=2}^{k} km_i(p-1)|\theta_i| + \sum_{i=2}^{k} m_i(k-1)|\theta_i - p| \\ &= r + p(k-1) + kr(p-1) + (p-r)(k-1) + \sum_{i=2}^{k} m_i(\theta_i + p(k-1)) \\ &\quad + k(p-1)\sum_{i=2}^{k} m_i|\theta_i| + \sum_{i=2}^{k} m_i(k-1)(p-\theta_i) \\ &\qquad \text{since } \theta_i + p(k-1) \geq -r + p \geq 0 \text{ and } \theta_i - p \leq r - p \leq 0 \\ &= 2np(k-1) + k(p-1)E(G_1). \end{aligned}$$

Since G_2 is also an r-regular graph of order n, we have

$$E(G_2 \square K_{\underbrace{p, p, \ldots, p}_{k \text{ times}}}) = 2np(k-1) + k(p-1)E(G_2).$$

If G_1 and G_2 are A-equienergetic then $E(G_1 \square K_{\underbrace{p, p, \ldots, p}_{k \text{ times}}}) = E(G_2 \square K_{\underbrace{p, p, \ldots, p}_{k \text{ times}}})$ which completes the proof. □

Remark 8 *Recently in [19] Ramane et al. proved that* $E(G_1 \square K_{p,p}) = E(G_2 \square K_{p,p})$. *It is noted that this result becomes particular case of Theorem 7.*

Theorem 9 *If* $p \geq n \geq 2$ *and* $k \geq 2$ *then* $E(K_{\underbrace{p, p, \ldots, p}_{k \text{ times}}} \square K_{n-1}) = E(K_{\underbrace{p-1, p-1, \ldots, p-1}_{k \text{ times}}} \square K_n)$ *if and only if* $p = n$.

Proof. We have

$$\operatorname{Spec}(K_{\underbrace{p, p, \ldots, p}_{k \text{ times}}}) = \begin{pmatrix} p(k-1) & 0 & -p \\ 1 & k(p-1) & k-1 \end{pmatrix}$$

and

$$\operatorname{Spec}(K_{n-1}) = \begin{pmatrix} n-2 & -1 \\ 1 & n-2 \end{pmatrix}.$$

By (i) of Lemma 5,

$$\operatorname{Spec}(K_{\underbrace{p, p, \ldots, p}_{k \text{ times}}} \square K_{n-1}) = \begin{pmatrix} pk-p+n-2 & pk-p-1 & n-2 \\ 1 & n-2 & k(p-1) \end{pmatrix.$$
$$\begin{matrix} -1 & n-p-2 & -p-1 \\ k(n-2)(p-1) & k-1 & (k-1)(n-2) \end{pmatrix}.$$

If $p \geq n \geq 2$ then $pk-p+n-2, pk-p-1$ and $n-2$ are only the positive A-eigenvalues of $K_{\underbrace{p, p, \ldots, p}_{k \text{ times}}} \square K_{n-1}$. Therefore from definition of A-energy, we get

$$\begin{aligned} &E(K_{\underbrace{p, p, \ldots, p}_{k \text{ times}}} \square K_{n-1}) \\ &= 2[pk-p+n-2+(n-2)(pk-p-1)+k(n-2)(p-1)] \\ &= 2[2npk-3pk-np+p-nk+2k]. \end{aligned}$$

Next, we have

$$\mathrm{Spec}(K_{\underbrace{p-1,p-1,\ldots,p-1}_{k\ \text{times}}}) = \begin{pmatrix} (p-1)(k-1) & 0 & -(p-1) \\ 1 & k(p-2) & k-1 \end{pmatrix}$$

and

$$\mathrm{Spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

By (i) of Lemma 5,
$\mathrm{Spec}(K_{\underbrace{p-1,p-1,\ldots,p-1}_{k\ \text{times}}} \square K_n)$

$$= \begin{pmatrix} pk-p+n-k & pk-p-k & n-1 & -1 & n-p & -p \\ 1 & n-1 & k(p-2) & k(n-1)(p-2) & k-1 & (k-1)(n-1) \end{pmatrix}.$$

If $p \geq n \geq 2$ then $pk-p+n-k, pk-p-k$ and $n-1$ are only the positive A-eigenvalues of $K_{\underbrace{p-1,p-1,\ldots,p-1}_{k\ \text{times}}} \square K_n$. Therefore from definition of A-energy, we get
$E(K_{\underbrace{p-1,p-1,\ldots,p-1}_{k\ \text{times}}} \square K_n)$

$$\begin{aligned} &= 2[pk-p-k+n+(n-1)(pk-p-k)+k(n-1)(p-2)] \\ &= 2[2npk-3nk-np+n-pk+2k]. \end{aligned}$$

The graphs $K_{\underbrace{p,p,\ldots,p}_{k\ \text{times}}} \square K_{n-1}$ and $K_{\underbrace{p-1,p-1,\ldots,p-1}_{k\ \text{times}}} \square K_n$ are A-equienergetic if and only if

$$E(K_{\underbrace{p,p,\ldots,p}_{k\ \text{times}}} \square K_{n-1}) = E(K_{\underbrace{p-1,p-1,\ldots,p-1}_{k\ \text{times}}} \square K_n).$$

That is, $2[2npk-3pk-np+p-nk+2k] = 2[2npk-3nk-np+n-pk+2k]$ or $p+2nk = n+2pk$, which implies $(n-p)(2k-1) = 0$. Since k is positive integer, we get $p = n$. This completes the proof. □

Theorem 10 *Let* G *be the Petersen graph and* $p, q \geq 4$. *Then* $E\left(G\square\overline{L(K_{p,q})}\right) = E\left(\overline{G}\square L(K_{p,q})\right)$.

Proof. The A-spectrum of Petersen graph is

$$\text{Spec}(G) = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 5 & 4 \end{pmatrix}.$$

From Lemma 6 and Theorem 1, A-spectrum of $\overline{L(K_{p,q})}$ is

$$\text{Spec}\left(\overline{L(K_{p,q})}\right) = \begin{pmatrix} pq-p-q+1 & 1-p & 1-q & 1 \\ 1 & q-1 & p-1 & (p-1)(q-1) \end{pmatrix}.$$

Now by (i) of Lemma 5, A-spectrum of $G\square\overline{L(K_{p,q})}$ is

$$\left(\begin{matrix} pq-p-q+4 & 4-p & 4-q & 4 & pq-p-q+2 & 2-p \\ 1 & q-1 & p-1 & pq-p-q+1 & 5 & 5q-5 \end{matrix}\right.$$
$$\begin{matrix} 2-q & 2 & pq-p-q-1 & -1-p & -1-q \\ 5p-5 & 5pq-5p-5q+5 & 4 & 4q-4 & 4p-4 \end{matrix}$$
$$\left.\begin{matrix} -1 \\ 4pq-4p-4q+4 \end{matrix}\right).$$

If $p, q \geq 4$ then $pq-p-q+4, 4, pq-p-q+2, 2$ and $pq-p-q-1$ are only the positive A-eigenvalues of $G\square\overline{L(K_{p,q})}$. Therefore from definition of A-energy

$$\begin{aligned} E\left(G\square\overline{L(K_{p,q})}\right) &= 2[pq-p-q+4+4(pq-p-q-1)+5(pq-p-q+2) \\ &\quad +2(5pq-5p-5q+5)+4(pq-p-q+1)] \\ &= 48(p-1)(q-1). \end{aligned}$$

Now

$$\text{Spec}(\overline{G}) = \begin{pmatrix} 6 & 1 & -2 \\ 1 & 4 & 5 \end{pmatrix}.$$

By using (i) of Lemma 5 and Lemma 6, $\overline{G}\square L(K_{p,q})$ has A-spectrum,

$$\left(\begin{matrix} p+q+4 & p+4 & q+4 & 4 & p+q-4 & p-4 & q-4 \\ 1 & q-1 & p-1 & pq-p-q+1 & 5 & 5q-5 & 5p-5 \end{matrix}\right.$$
$$\left.\begin{matrix} -4 & p+q-1 & p-1 & q-1 & -1 \\ 5pq-5p-5q+5 & 4 & 4q-4 & 4p-4 & 4pq-4p-4q+4 \end{matrix}\right).$$

If $p, q \geq 4$ then -4 and -1 are only the negative A-eigenvalues of $\overline{G}\square L(K_{p,q})$. Therefore from definition of A-energy

$$\begin{aligned} E(\overline{G}\square L(K_{p,q})) &= 2[4(5pq-5p-5q+5)+4pq-4p-4q+4] \\ &= 48(p-1)(q-1) \end{aligned}$$

which completes the proof. □

Recently in [17] Ramane et al. proved that $E(L(K_{p,q})) = E(\overline{L(K_{p,q})})$. In the following A-equienergetic graphs with the help of these graphs are given.

Theorem 11 *If* $p, q \geq 5$ *then* $E(L(K_{p,q})\square L(K_4)) = E(\overline{L(K_{p,q})}\square L(K_4))$.

Proof. The A-spectrum of K_4 is

$$\mathrm{Spec}(K_4) = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}.$$

From Theorem 2,

$$\mathrm{Spec}(L(K_4)) = \begin{pmatrix} 4 & 0 & -2 \\ 1 & 3 & 2 \end{pmatrix}.$$

By using (i) of Lemma 5 and Lemma 6, $L(K_{p,q})\square L(K_4)$ has A-spectrum

$$\begin{pmatrix} p+q+2 & p+q-2 & p+q-4 & p+2 & p-2 & p-4 & q+2 & q-2 \\ 1 & 3 & 2 & q-1 & 3q-3 & 2q-2 & p-1 & 3p-3 \\ q-4 & -4 & 2 & -2 \\ 2p-2 & 2pq-2p-2q+2 & pq-p-q+1 & 3pq-3p-3q+3 \end{pmatrix}.$$

If $p, q \geq 5$ then -4 and -2 are only the negative A-eigenvalues of $L(K_{p,q})\square L(K_4)$. Therefore from definition of A-energy

$$\begin{aligned} E(L(K_{p,q})\square L(K_4)) &= 2[4(2pq-2p-2q+2)+2(3pq-3p-3q+3)] \\ &= 28(p-1)(q-1). \end{aligned}$$

By using Theorem 1, (i) of Lemma 5 and Lemma 6, $\overline{L(K_{p,q})}\square L(K_4)$ has A-spectrum,

$$\begin{pmatrix} pq-p-q+5 & pq-p-q+1 & pq-p-q-1 & 5-p & 1-p & -1-p \\ 1 & 3 & 2 & q-1 & 3q-3 & 2q-2 \\ 5-q & 1-q & -1-q & 5 & 1 \\ p-1 & 3p-3 & 2p-2 & pq-p-q+1 & 3pq-3p-3q+3 \\ -1 \\ 2pq-2p-2q+2 \end{pmatrix}.$$

If $p, q \geq 5$ then $pq-p-q+5, pq-p-q+1, pq-p-q-1, 5$ and 1 are only the positive A-eigenvalues of $\overline{L(K_{p,q})}\square L(K_4)$. Therefore from definition of A-energy

$$\begin{aligned} E(\overline{L(K_{p,q})}\square L(K_4)) &= 2[pq-p-q+5+3(pq-p-q+1)+2(pq-p-q-1) \\ &\quad +5(pq-p-q+1)+3pq-3p-3q+3] \\ &= 28(p-1)(q-1) \end{aligned}$$

which completes the proof. $\square$

Theorem 12 *If* $3 \leq p, q \leq 6$ *then* $E(\overline{L(K_{p,q})\square L(K_4)}) = E(\overline{\overline{L(K_{p,q})}\square L(K_4)})$.

Proof. We have

$$\mathrm{Spec}(L(K_4)) = \begin{pmatrix} 4 & 0 & -2 \\ 1 & 3 & 2 \end{pmatrix}.$$

By using (i) of Lemma 5, Lemma 6 and Theorem 1, $\overline{L(K_{p,q})\square L(K_4)}$ has A-spectrum

$$\left(\begin{matrix} 6pq-p-q-3 & -1-p-q+2 & -1-p-q+4 & -1-p-2 & -1-p+2 \\ 1 & 3 & 2 & q-1 & 3q-3 \end{matrix}\right.$$
$$\begin{matrix} -1-p+4 & -1-q-2 & -1-q+2 & -1-q+4 & -3 \\ 2q-2 & p-1 & 3p-3 & 2p-2 & pq-p-q+1 \end{matrix}$$
$$\left.\begin{matrix} 1 & 3 \\ 3pq-3p-3q+3 & 2pq-2p-2q+2 \end{matrix}\right).$$

If $p, q \geq 3$ then $6pq-p-q-3, 1$ and 3 are only the positive A-eigenvalues of $\overline{L(K_{p,q})\square L(K_4)}$. Therefore from definition of A-energy, we have
$E(\overline{L(K_{p,q})\square L(K_4)})$

$$\begin{aligned} &= 2[6pq-p-q-3+1(3pq-3p-3q+3)+3(2pq-2p-2q+2)] \\ &= 2(15pq-10p-10q+6). \end{aligned}$$

By using (i) of Lemma 5, Lemma 6 and Theorem 1, $\overline{\overline{L(K_{p,q})}\square L(K_4)}$ has A-spectrum

$$\left(\begin{matrix} 5pq+p+q-6 & p+q-pq-2 & p+q-pq & p-6 & p-2 & p \\ 1 & 3 & 2 & q-1 & 3q-3 & 2q-2 \end{matrix}\right.$$
$$\begin{matrix} q-6 & q-2 & q & -6 & -2 \\ p-1 & 3p-3 & 2p-2 & pq-p-q+1 & 3pq-3p-3q+3 \end{matrix}$$
$$\left.\begin{matrix} 0 \\ 2pq-2p-2q+2 \end{matrix}\right).$$

If $3 \le p, q \le 6$ then $5pq + p + q - 6, p - 2, p, q - 2$ and q are only the positive A-eigenvalues of $\overline{\overline{L(K_{p,q})}\square L(K_4)}$. Therefore from definition of A-energy

$$\begin{aligned} E(\overline{\overline{L(K_{p,q})}\square L(K_4)}) &= 2[5pq + p + q - 6 + (p-2)(3q-3) + p(2q-2) \\ &\quad + (q-2)(3p-3) + q(2p-2)] \\ &= 2(15pq - 10p - 10q + 6) \end{aligned}$$

which completes the proof. □

Theorem 13 *If* $p, q \ge 3$ *then* $E(L(K_{p,q})\square\overline{L(K_4)}) = E(\overline{L(K_{p,q})}\square\overline{L(K_4)})$.

Proof. We have

$$\text{Spec}(L(K_4)) = \begin{pmatrix} 4 & 0 & -2 \\ 1 & 3 & 2 \end{pmatrix}$$

and from Theorem 1

$$\text{Spec}(\overline{L(K_4)}) = \begin{pmatrix} 1 & -1 \\ 3 & 3 \end{pmatrix}.$$

By using (i) of Lemma 5 and Lemma 6, $L(K_{p,q})\square\overline{L(K_4)}$ has A-spectrum

$$\begin{pmatrix} p+q-1 & p+q-3 & p-1 & p-3 & q-1 & q-3 \\ 3 & 3 & 3q-3 & 3q-3 & 3p-3 & 3p-3 \end{pmatrix.$$
$$\begin{matrix} -1 & -3 \\ 3pq-3p-3q+3 & 3pq-3p-3q+3 \end{pmatrix}.$$

If $p, q \ge 3$ then -1 and -3 are only the negative A-eigenvalues of $L(K_{p,q})\square\overline{L(K_4)}$. Therefore from definition of A-energy

$$\begin{aligned} E(L(K_{p,q})\square\overline{L(K_4)}) &= 2[3(p-1)(q-1) + 9(p-1)(q-1)] \\ &= 24(p-1)(q-1). \end{aligned}$$

By using Lemma 6, Theorem 1 and (i) of Lemma 5, $\overline{L(K_{p,q})}\square\overline{L(K_4)}$ has A-spectrum

$$\begin{pmatrix} pq-p-q+2 & pq-p-q & 2-p & -p & 2-q & -q \\ 3 & 3 & 3q-3 & 3q-3 & 3p-3 & 3p-3 \end{matrix}$$
$$\begin{matrix} 2 & 0 \\ 3pq-3p-3q+3 & 3pq-3p-3q+3 \end{pmatrix}.$$

If $p,q \geq 3$ then $pq-p-q+2, pq-p-q$ and 2 are only the positive A-eigenvalues of $\overline{L(K_{p,q})}\square\overline{L(K_4)}$. Therefore from definition of A-energy

$$\begin{aligned} E(\overline{L(K_{p,q})}\square\overline{L(K_4)}) &= 2[3(pq-p-q+2)+3(pq-p-q) \\ &\quad +2(3pq-3p-3q+3)] \\ &= 24(p-1)(q-1) \end{aligned}$$

which completes the proof. □

Theorem 14 *If* $p,q \geq 4$ *then* $E(\overline{L(K_{p,q})}\square C_6) = E\left(L(K_{p,q})\square\overline{C_6}\right)$ *where* C_6 *is the cycle of order* 6*.*

Proof. We have

$$\mathrm{Spec}(C_6) = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \end{pmatrix}.$$

By using Lemma 6, Theorem 1 and (i) of Lemma 5, A-spectrum of $\overline{L(K_{p,q})}\square C_6$ is

$$\begin{pmatrix} pq-p-q+3 & pq-p-q+2 & pq-p-q & pq-p-q-1 & 3-p \\ 1 & 2 & 2 & 1 & q-1 \end{pmatrix.$$
$$\begin{matrix} 2-p & -p & -1-p & 3-q & 2-q & -q & -1-q & 3 \\ 2q-2 & 2q-2 & q-1 & p-1 & 2p-2 & 2p-2 & p-1 & pq-p-q+1 \end{matrix}$$
$$\left.\begin{matrix} 2 & 0 & -1 \\ 2pq-2p-2q+2 & 2pq-2p-2q+2 & pq-p-q+1 \end{matrix}\right).$$

If $p,q \geq 4$ then $pq-p-q+3, pq-p-q+2, pq-p-q, pq-p-q-1, 3$ and 2 are only the positive A-eigenvalues of $\overline{L(K_{p,q})}\square C_6$. Therefore from definition of A-energy
$E\left(\overline{L(K_{p,q})}\square C_6\right)$

$$\begin{aligned} &= 2[pq-p-q+3+2(pq-p-q+2)+2(pq-p-q) \\ &\quad +(pq-p-q-1)+3(pq-p-q+1)+2(2pq-2p-2q+2)] \\ &= 26(p-1)(q-1). \end{aligned}$$

Now by Theorem 1

$$\mathrm{Spec}(\overline{C_6}) = \begin{pmatrix} 3 & 1 & 0 & -2 \\ 1 & 1 & 2 & 2 \end{pmatrix}.$$

By using Lemma 6 and (i) of Lemma 5 $L(K_{p,q})\Box\overline{C_6}$ has A-spectrum

$$\begin{pmatrix} p+q+1 & p+q-1 & p+q-2 & p+q-4 & p+1 & p-1 & p-2 & p-4 \\ 1 & 1 & 2 & 2 & q-1 & q-1 & 2q-2 & 2q-2 \end{pmatrix.$$
$$\begin{matrix} q+1 & q-1 & q-2 & q-4 & 1 & -1 \\ p-1 & p-1 & 2p-2 & 2p-2 & pq-p-q+1 & pq-p-q+1 \end{matrix}$$
$$\left.\begin{matrix} -2 & -4 \\ 2pq-2p-2q+2 & 2pq-2p-2q+2 \end{matrix}\right).$$

If $p, q \geq 4$ then $-1, -2$ and -4 are only the negative A-eigenvalues of $L(K_{p,q})\Box\overline{C_6}$. Therefore from definition of A-energy

$$\begin{aligned} E\left(L(K_{p,q})\Box\overline{C_6}\right) &= 2[pq-p-q+1+2(2pq-2p-2q+2) \\ &\quad +4(2pq-2p-2q+2)] \\ &= 26(p-1)(q-1) \end{aligned}$$

which completes the proof. □

Theorem 15 *If* $p, q \geq 3$ *then* $E\left(L(K_{p,q})\Box\overline{W_5}\right) = E(\overline{L(K_{p,q})}\Box\overline{W_5})$ *where* W_5 *is the wheel of order* 5.

Proof. Since A-spectrum of $\overline{W_5}$ is

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix}.$$

Now by (i) of Lemma 5, A-spectrum of $L(K_{p,q})\Box\overline{W_5}$ is

$$\left(\begin{matrix} p+q-1 & p+q-2 & p+q-3 & p-1 & p-2 & p-3 & q-1 & q-2 \\ 2 & 1 & 2 & 2q-2 & q-1 & 2q-2 & 2p-2 & p-1 \end{matrix}\right.$$
$$\left.\begin{matrix} q-3 & -1 & -2 & -3 \\ 2p-2 & 2pq-2p-2q+2 & pq-p-q+1 & 2pq-2p-2q+2 \end{matrix}\right).$$

If $p, q \geq 3$ then $-1, -2$ and -3 are only the negative A-eigenvalues of $L(K_{p,q})\Box\overline{W_5}$. Therefore from definition of A-energy

$$\begin{aligned} E\left(L(K_{p,q})\Box\overline{W_5}\right) &= 2[2pq-2p-2q+2+2(pq-p-q+1) \\ &\quad +3(2pq-2p-2q+2)] \\ &= 20(p-1)(q-1). \end{aligned}$$

By using Lemma 6, Theorem 1 and (i) of Lemma 5, A-spectrum of $\overline{L(K_{p,q})}\Box\overline{W_5}$ is

$$\begin{pmatrix} pq-p-q+2 & pq-p-q+1 & pq-p-q & 2-p & 1-p & -p & 2-q \\ 2 & 1 & 2 & 2q-2 & q-1 & 2q-2 & 2p-2 \end{pmatrix.}$$
$$\begin{matrix} 1-q & -q & 2 & 1 & 0 \\ p-1 & 2p-2 & 2pq-2p-2q+2 & pq-p-q+1 & 2pq-2p-2q+2 \end{matrix}\Bigg).$$

If $p,q \geq 3$ then $pq-p-q+2, pq-p-q+1, pq-p-q, 2$ and 1 are only the positive A-eigenvalues of $\overline{L(K_{p,q})}\Box\overline{W_5}$. Therefore from definition of A-energy

$$\begin{aligned} E\left(\overline{L(K_{p,q})}\Box\overline{W_5}\right) &= 2[2(pq-p-q+2)+pq-p-q+1+2(pq-p-q) \\ &\quad +2(2pq-2p-2q+2)+(pq-p-q+1)] \\ &= 20(p-1)(q-1) \end{aligned}$$

which completes the proof. □

Theorem 16 *If* $n \geq 3$ *then* $E(\overline{K_{n,n}\otimes K_{n-1}}) = E(\overline{K_{n-1,n-1}\otimes K_n})$.

Proof. We have

$$\mathrm{Spec}(K_{n,n}) = \begin{pmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{pmatrix}$$

and

$$\mathrm{Spec}(K_{n-1}) = \begin{pmatrix} n-2 & -1 \\ 1 & n-2 \end{pmatrix}.$$

By using (ii) of Lemma 5 and Theorem 1,

$$\mathrm{Spec}(\overline{K_{n,n}\otimes K_{n-1}}) = \begin{pmatrix} n^2-1 & -1-n & -1 & n(n-2)-1 & n-1 \\ 1 & n-2 & (n-1)(2n-2) & 1 & n-2 \end{pmatrix}.$$

If $n \geq 3$ then $n^2-1, n(n-2)-1$ and $n-1$ are only the positive A-eigenvalues of $\overline{K_{n,n}\otimes K_{n-1}}$ and from definition of A-energy

$$\begin{aligned} E\left(\overline{K_{n,n}\otimes K_{n-1}}\right) &= 2[n^2-1-1+n(n-2)+(n-1)(n-2)] \\ &= 6n^2-10n. \end{aligned}$$

Now

$$\mathrm{Spec}(K_{n-1,n-1}) = \begin{pmatrix} n-1 & 0 & -(n-1) \\ 1 & 2n-4 & 1 \end{pmatrix}$$

and

$$\mathrm{Spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

By using (ii) of Lemma 5 and Theorem 1,
$\mathrm{Spec}(\overline{K_{n-1,n-1} \otimes K_n}) =$
$$\begin{pmatrix} n^2-2 & -1+(n-1) & -1 & -1 & (n-1)^2-1 & -1-(n-1) \\ 1 & n-1 & 2n-4 & (2n-4)(n-1) & 1 & n-1 \end{pmatrix}.$$
If $n \geq 3$ then $n^2-2, n-2$ and $(n-1)^2-1$ are only the positive A-eigenvalues of $\overline{K_{n-1,n-1} \otimes K_n}$. Therefore from definition of A-energy

$$\begin{aligned} E\left(\overline{K_{n-1,n-1} \otimes K_n}\right) &= 2[n^2-2+(n-1)(n-2)+(n-1)^2-1] \\ &= 6n^2-10n \end{aligned}$$

which completes the proof. □

3.2 Linear relations on energies of graphs

Linear relations on energies of different graphs are not yet well studied in the study of graph energies except equienergetic graphs, that is $E(G_1)-E(G_2)=0$. In the following we present some linear relations on energies of different graphs of same order of the type $aE(G_1)+bE(G_2)=c$, where a, b and c are real numbers.

Theorem 17 *If* $n \geq 3$ *then* $E\left(\overline{L(K_4)\Box K_n}\right) - E\left(L(K_4)\Box K_n\right) = 2.$

Proof. We have

$$\mathrm{Spec}(L(K_4)) = \begin{pmatrix} 4 & 0 & -2 \\ 1 & 3 & 2 \end{pmatrix}$$

and

$$\mathrm{Spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

By using (i) of Lemma 5,

$$\mathrm{Spec}(L(K_4)\Box K_n) = \begin{pmatrix} n+3 & n-1 & n-3 & 3 & -1 & -3 \\ 1 & 3 & 2 & n-1 & 3n-3 & 2n-2 \end{pmatrix}.$$

If $n \geq 3$ then -1 and -3 are only the negative A-eigenvalues of $L(K_4)\Box K_n$. Therefore from definition of A-energy

$$\begin{aligned} E\left(L(K_4)\Box K_n\right) &= 2[(3n-3)+3(2n-2)] \\ &= 18n-18. \end{aligned} \tag{1}$$

By using Theorem 1

$$\mathrm{Spec}(\overline{L(K_4)\square K_n}) = \begin{pmatrix} 5n-4 & -n & 2-n & -4 & 0 & 2 \\ 1 & 3 & 2 & n-1 & 3n-3 & 2n-2 \end{pmatrix}.$$

If $n \geq 3$ then $-n, 2-n$ and -4 are only the negative A-eigenvalues of $\overline{L(K_4)\square K_n}$. Therefore from definition of A-energy

$$\begin{aligned} E\left(\overline{L(K_4)\square K_n}\right) &= 2[3n+2n-4+4n-4] \\ &= 18n-16. \end{aligned} \tag{2}$$

From (1) and (2) the result follows. □

Theorem 18 *If* $n \geq 3$ *then* $E(K_{n-1,n-1} \otimes K_n) - E(K_{n,n} \otimes K_{n-1}) = 4$.

Proof. We have

$$\mathrm{Spec}(K_{n,n}) = \begin{pmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{pmatrix}$$

and

$$\mathrm{Spec}(K_{n-1}) = \begin{pmatrix} n-2 & -1 \\ 1 & n-2 \end{pmatrix}.$$

By using (ii) of Lemma 5,
$\mathrm{Spec}(K_{n,n} \otimes K_{n-1}) =$

$$\begin{pmatrix} n(n-2) & -n & 0 & 0 & -n(n-2) & n \\ 1 & n-2 & 2n-2 & (n-2)(2n-2) & 1 & n-2 \end{pmatrix}.$$

If $n \geq 3$ then $n(n-2)$ and n are only the positive A-eigenvalues of $K_{n,n} \otimes K_{n-1}$. Therefore from definition of A-energy

$$\begin{aligned} E\left(K_{n,n} \otimes K_{n-1}\right) &= 2[n(n-2)+n(n-2)] \\ &= 4n^2-8n. \end{aligned} \tag{3}$$

Now

$$\mathrm{Spec}(K_{n-1,n-1}) = \begin{pmatrix} n-1 & 0 & -(n-1) \\ 1 & 2n-4 & 1 \end{pmatrix}$$

and

$$\mathrm{Spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

By using (ii) of Lemma 5,
$\mathrm{Spec}(K_{n-1,n-1} \otimes K_n) =$

$$\begin{pmatrix} (n-1)^2 & -(n-1) & 0 & 0 & -(n-1)^2 & n-1 \\ 1 & n-1 & 2n-4 & (n-2)(2n-2) & 1 & n-1 \end{pmatrix}.$$

If $n \geq 3$ then $(n-1)^2$ and $n-1$ are only the positive A-eigenvalues of $K_{n-1,n-1} \otimes K_n$. Therefore from definition of A-energy

$$\begin{aligned} E\left(K_{n-1,n-1} \otimes K_n\right) &= 2[(n-1)^2 + (n-1)^2] \\ &= 4n^2 - 8n + 4. \end{aligned} \tag{4}$$

From (3) and (4) result follows. □

Theorem 19 *If* $n \geq 3$ *then*

$$E(K_{\underbrace{n-1, n-1, \ldots, n-1}_{k \ \textit{times}}} \boxtimes K_n) - E(K_{\underbrace{n, n, \ldots, n}_{k \ \textit{times}}} \boxtimes K_{n-1}) = 2.$$

Proof. We have

$$\mathrm{Spec}(K_{\underbrace{n, n, \ldots, n}_{k \text{ times}}}) = \begin{pmatrix} n(k-1) & 0 & -n \\ 1 & k(n-1) & k-1 \end{pmatrix}$$

and

$$\mathrm{Spec}(K_{n-1}) = \begin{pmatrix} n-2 & -1 \\ 1 & n-2 \end{pmatrix}.$$

By using (iii) of Lemma 5,
$\mathrm{Spec}(K_{\underbrace{n, n, \ldots, n}_{k \text{ times}}} \boxtimes K_{n-1}) =$

$$\begin{pmatrix} n^2k - n^2 - nk + 2n - 2 & n-2 & -1 & -n^2 + 2n - 2 \\ 1 & k(n-1) & kn(n-2) & k-1 \end{pmatrix}.$$

If $n \geq 3$ then $n^2k - n^2 - nk + 2n - 2$ and $n-2$ are only the positive A-eigenvalues of $K_{\underbrace{n, n, \ldots, n}_{k \text{ times}}} \boxtimes K_{n-1}$. Therefore from definition of A-energy

$$\begin{aligned} E(K_{\underbrace{n, n, \ldots, n}_{k \text{ times}}} \boxtimes K_{n-1}) &= 2[n^2k - n^2 - nk + 2n - 2 + k(n-2)(n-1)] \\ &= 4n^2k - 2n^2 - 8nk + 4n + 4k - 4. \end{aligned} \tag{5}$$

Now

$$\mathrm{Spec}(K_{\underbrace{n-1, n-1, \ldots, n-1}_{k \text{ times}}}) = \begin{pmatrix} (n-1)(k-1) & 0 & -(n-1) \\ 1 & k(n-2) & k-1 \end{pmatrix}$$

and

$$\mathrm{Spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

By using (iii) of Lemma 5,

$$\mathrm{Spec}(K_{\underbrace{n-1, n-1, \dots, n-1}_{k \text{ times}}} \boxtimes K_n) =$$

$$\begin{pmatrix} n^2k - n^2 - nk + 2n - 1 & n-1 & -1 & -n^2+2n-1 \\ 1 & k(n-2) & k(n-1)^2 & k-1 \end{pmatrix}.$$

If $n \geq 3$ then $n^2k - n^2 - nk + 2n - 1$ and $n-1$ are only the positive A-eigenvalues of $K_{\underbrace{n-1, n-1, \dots, n-1}_{k \text{ times}}} \boxtimes K_n$. Therefore from definition of A-energy

$$\begin{aligned} E(K_{\underbrace{n-1, n-1, \dots, n-1}_{k \text{ times}}} \boxtimes K_n) &= 2[n^2k - n^2 - nk + 2n - 1 + k(n-2)(n-1)] \\ &= 4n^2k - 2n^2 - 8nk + 4n + 4k - 2. \end{aligned} \tag{6}$$

From (5) and (6) the result follows. □

Theorem 20 *If* $n \geq 3$ *then* $E(\overline{K_{n,n} \boxtimes K_{n-1}}) - E(\overline{K_{n-1,n-1} \boxtimes K_n}) = 4$.

Proof. We have

$$\mathrm{Spec}(K_{n,n}) = \begin{pmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{pmatrix}$$

and

$$\mathrm{Spec}(K_{n-1}) = \begin{pmatrix} n-2 & -1 \\ 1 & n-2 \end{pmatrix}.$$

By using (iii) of Lemma 5 and Theorem 1,

$$\mathrm{Spec}(\overline{K_{n,n} \boxtimes K_{n-1}}) = \begin{pmatrix} n^2 - 2n + 1 & 0 & -n+1 \\ 2 & 2n^2 - 2n & 2n-2 \end{pmatrix}.$$

If $n \geq 3$ then $n^2 - 2n + 1$ is only the positive A-eigenvalues of $\overline{K_{n,n} \boxtimes K_{n-1}}$. Therefore from definition of A-energy

$$\begin{aligned} E\left(\overline{K_{n,n} \boxtimes K_{n-1}}\right) &= 2[2n^2 - 4n + 2] \\ &= 4n^2 - 8n + 4. \end{aligned} \tag{7}$$

Now

$$\mathrm{Spec}(K_{n-1,n-1}) = \begin{pmatrix} n-1 & 0 & -(n-1) \\ 1 & 2n-4 & 1 \end{pmatrix}$$

and

$$\mathrm{Spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

By using (iii) of Lemma 5 and Theorem 1,

$$\mathrm{Spec}(\overline{K_{n-1,n-1} \boxtimes K_n}) = \begin{pmatrix} n^2-2n & 0 & -n \\ 2 & 2n^2-4n+2 & 2n-4 \end{pmatrix}.$$

If $n \geq 3$ then $n^2 - 2n$ is only the positive A-eigenvalues of $\overline{K_{n-1,n-1} \boxtimes K_n}$. Therefore from definition of A-energy

$$\begin{aligned} E\left(\overline{K_{n-1,n-1} \boxtimes K_n}\right) &= 2[2n^2 - 4n] \\ &= 4n^2 - 8n. \end{aligned} \tag{8}$$

From (7) and (8) the result follows. □

4 S-equienergetic graphs and linear relations on S-energies of certain class of graphs

4.1 S-equienergetic graphs

In [16] Ramane et al. studied S-energy of $L^2(G)$ for an r-regular graph G, $r \geq 3$ and constructed a large class of S-equienergetic graphs. The following result provides S-equienergetic graphs with the help of iterated line graphs $L^k(G)$ even for $k \geq 1$, where $L^0(G) = G$.

Theorem 21 *If* $n \geq 5$, $k \geq 0$ *then* $E_S(L^k(K_{n,n} \Box K_{n-1})) = E_S(L^k(K_{n-1,n-1} \Box K_n))$.

Proof. As $K_{n,n} \Box K_{n-1}$ and $K_{n-1,n-1} \Box K_n$ are both regular graphs of same order and of same degree, by Theorems 3 and 4, the result is true for $k \geq 2$. Now, it is enough to prove for $k = 0, 1$.
When $k = 0$.

$$\mathrm{Spec}(K_{n,n}) = \begin{pmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{pmatrix}$$

and

$$\mathrm{Spec}(K_{n-1}) = \begin{pmatrix} n-2 & -1 \\ 1 & n-2 \end{pmatrix}.$$

Therefore by (i) of Lemma 5,

$$\text{Spec}(K_{n,n}\square K_{n-1}) = \begin{pmatrix} 2n-2 & n-1 & n-2 & -1 & -2 & -n-1 \\ 1 & n-2 & 2n-2 & (2n-2)(n-2) & 1 & n-2 \end{pmatrix}$$

and

$$\text{Spec}(K_{n-1,n-1}\square K_{n}) = \begin{pmatrix} 2n-2 & n-2 & n-1 & -1 & 0 & -n \\ 1 & n-1 & 2n-4 & (2n-4)(n-1) & 1 & n-1 \end{pmatrix}.$$

Therefore by Theorem 3,
$\text{Spec}_S(K_{n,n}\square K_{n-1}) =$

$$\begin{pmatrix} 2n^2-6n+3 & 1-2n & 3-2n & 1 & 3 & 2n+1 \\ 1 & n-2 & 2n-2 & (2n-2)(n-2) & 1 & n-2 \end{pmatrix}$$

and $\text{Spec}_S(K_{n-1,n-1}\square K_{n}) =$

$$\begin{pmatrix} 2n^2-6n+3 & 3-2n & 1-2n & 1 & -1 & 2n-1 \\ 1 & n-1 & 2n-4 & (2n-4)(n-1) & 1 & n-1 \end{pmatrix}.$$

If $n \geq 5$ then $2n^2-6n+3, 1, 3$ and $2n+1$ are only the positive S-eigenvalues of $K_{n,n}\square K_{n-1}$. Therefore from definition of S-energy

$$\begin{aligned} E_S(K_{n,n}\square K_{n-1}) &= 2[2n^2-6n+3+(2n-2)(n-2)+3+(n-2)(2n+1)] \\ &= 12n^2-26n+16. \end{aligned}$$

If $n \geq 5$ then $2n^2-6n+3, 1$ and $2n-1$ are only the positive S-eigenvalues of $K_{n-1,n-1}\square K_{n}$. Therefore from definition of S-energy

$$\begin{aligned} E_S(K_{n-1,n-1}\square K_{n}) &= 2[2n^2-6n+3+(2n-4)(n-1)+(n-1)(2n-1)] \\ &= 12n^2-26n+16. \end{aligned}$$

Hence $E_S(K_{n,n}\square K_{n-1}) = E_S(K_{n-1,n-1}\square K_{n})$.

When $k = 1$ both $K_{n,n}\square K_{n-1}$ and $K_{n-1,n-1}\square K_{n}$ are regular graphs of same order $2n(n-1)$ and of same degree $2n-2$. Hence by (i) of Lemma 5,
$\text{Spec}(L(K_{n,n}\square K_{n-1})) =$

$$\begin{pmatrix} 4n-6 & 3n-5 & 3n-6 & 2n-5 & 2n-6 & n-5 & -2 \\ 1 & n-2 & 2n-2 & (2n-2)(n-2) & 1 & n-2 & 2n(n-1)(n-2) \end{pmatrix}$$

and Spec$(L(K_{n-1,n-1}\Box K_n))=$

$$\begin{pmatrix} 4n-6 & 3n-6 & 3n-5 & 2n-5 & 2n-4 & n-4 & -2 \\ 1 & n-1 & 2n-4 & (2n-4)(n-1) & 1 & n-1 & 2n(n-1)(n-2) \end{pmatrix}.$$

Therefore from Theorem 3,
Spec$_S(L(K_{n,n}\Box K_{n-1}))=$

$$\begin{pmatrix} 2n(n-1)^2-8n+11 & 9-6n & 11-6n & 9-4n & 11-4n & 9-2n & 3 \\ 1 & n-2 & 2n-2 & (2n-2)(n-2) & 1 & n-2 & 2n(n-1)(n-2) \end{pmatrix}$$

and Spec$_S(L(K_{n-1,n-1}\Box K_n))=$

$$\begin{pmatrix} 2n(n-1)^2-8n+11 & 11-6n & 9-6n & 9-4n & 7-4n & 7-2n & 3 \\ 1 & n-1 & 2n-4 & (2n-4)(n-1) & 1 & n-1 & 2n(n-1)(n-2) \end{pmatrix}.$$

If $n\geq 5$ then $2n(n-1)^2-8n+11$ and 3 are only the positive S-eigenvalues of $L(K_{n,n}\Box K_{n-1})$. Therefore from definition of S-energy

$$\begin{aligned} E_S(L(K_{n,n}\Box K_{n-1})) &= 2[2n(n-1)^2-8n+11+6n(n^2-3n+2)] \\ &= 16n^3-44n^2+18n+22. \end{aligned}$$

If $n\geq 5$ then $2n(n-1)^2-8n+11$ and 3 are only the positive S-eigenvalues of $L(K_{n,n}\Box K_{n-1})$. Therefore from definition of S-energy

$$\begin{aligned} E_S(L(K_{n,n}\Box K_{n-1})) &= 2[2n(n-1)^2-8n+11+6n(n^2-3n+2)] \\ &= 16n^3-44n^2+18n+22. \end{aligned}$$

Hence $E_S(L(K_{n,n}\Box K_{n-1}))=E_S(L(K_{n-1,n-1}\Box K_n))$. □

Theorem 22 *If* $n\geq 3$ *then* $E_S(K_{n,n}\boxtimes K_{n-1})=E_S(K_{n-1,n-1}\boxtimes K_n)$.

Proof. We have

$$\text{Spec}(K_{n,n})=\begin{pmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{pmatrix}$$

and

$$\mathrm{Spec}(K_{n-1}) = \begin{pmatrix} n-2 & -1 \\ 1 & n-2 \end{pmatrix}.$$

By using (iii) of Lemma 5 and Theorem 3,

$$\mathrm{Spec}_S(K_{n,n} \boxtimes K_{n-1}) = \begin{pmatrix} -2n+3 & 1 & 2n^2-4n+3 & -2n+3 \\ 1 & 2n^2-2n & 1 & 2n-2 \end{pmatrix}.$$

If $n \geq 3$ then $2n^2 - 4n + 3$ and 1 are only the positive S-eigenvalues of $K_{n,n} \boxtimes K_{n-1}$. Therefore from definition of S-energy

$$\begin{aligned} E_S\left(K_{n,n} \boxtimes K_{n-1}\right) &= 2[2n^2 - 2n + 2n^2 - 4n + 3] \\ &= 8n^2 - 16n + 6. \end{aligned}$$

Now

$$\mathrm{Spec}(K_{n-1,n-1}) = \begin{pmatrix} n-1 & 0 & -(n-1) \\ 1 & 2n-4 & 1 \end{pmatrix}$$

and

$$\mathrm{Spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

By using (iii) of Lemma 5 and Theorem 3,

$$\mathrm{Spec}_S(K_{n-1,n-1} \boxtimes K_n) = \begin{pmatrix} -2n+1 & 1 & 2n^2-4n+1 & -2n+1 \\ 1 & (n-1)(2n-2) & 1 & 2n-4 \end{pmatrix}.$$

If $n \geq 5$ then $2n^2 - 4n + 1$ and 1 are only the positive S-eigenvalues of $K_{n-1,n-1} \boxtimes K_n$. Therefore from definition of S-energy

$$\begin{aligned} E_S\left(K_{n-1,n-1} \boxtimes K_n\right) &= 2[(n-1)(2n-2) + 2n^2 - 4n + 1] \\ &= 8n^2 - 16n + 6 \end{aligned}$$

which completes the proof. □

4.2 Linear relations on S-energies of graphs

Linear relations on S-energies of different graphs are not yet well studied in the study of S-energies. In the following we present some linear relations on S-energies of different graphs of same order of the type $aE_S(G_1) + bE_S(G_2) = c$, where a, b and c are real numbers.

Theorem 23 *If* $n \geq 3$ *then* $E_S(K_{n-1,n-1} \otimes K_n) - E_S(K_{n,n} \otimes K_{n-1}) = 2$.

Proof. We have

$$\text{Spec}(K_{n,n}) = \begin{pmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{pmatrix}$$

and

$$\text{Spec}(K_{n-1}) = \begin{pmatrix} n-2 & -1 \\ 1 & n-2 \end{pmatrix}.$$

By using (ii) of Lemma 5 and Theorem 3,

$$\text{Spec}_S(K_{n,n} \otimes K_{n-1}) = \begin{pmatrix} 2n-1 & -1-2n & -1 & 2n(n-2)-1 \\ n-1 & n-2 & (n-1)(2n-2) & 1 \end{pmatrix}.$$

If $n \geq 5$ then $2n-1$ and $2n(n-2)-1$ are only the positive S-eigenvalues of $K_{n,n} \otimes K_{n-1}$. Therefore from definition of S-energy

$$\begin{aligned} E_S\left(K_{n,n} \otimes K_{n-1}\right) &= 2[(2n-1)(n-1) + 2n(n-2) - 1] \\ &= 8n^2 - 14n. \end{aligned} \tag{9}$$

Now

$$\text{Spec}(K_{n-1,n-1}) = \begin{pmatrix} n-1 & 0 & -(n-1) \\ 1 & 2n-4 & 1 \end{pmatrix}$$

and

$$\text{Spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

By using (ii) of Lemma 5 and Theorem 3,

$$\text{Spec}_S(K_{n-1,n-1} \otimes K_n) = \begin{pmatrix} 2n-3 & 1-2n & -1 & 2n^2-4n+1 \\ n & n-1 & n(2n-4) & 1 \end{pmatrix}.$$

If $n \geq 5$ then $2n-3$ and $2n^2-4n+1$ are only the positive S-eigenvalues of $K_{n-1,n-1} \otimes K_n$. Therefore from definition of S-energy

$$\begin{aligned} E_S\left(K_{n-1,n-1} \otimes K_n\right) &= 2[n(2n-3) + 2n^2 - 4n + 1] \\ &= 8n^2 - 14n + 2. \end{aligned} \tag{10}$$

From (9) and (10) the result follows. □

Conclusion

In this article we have obtained several classes of A-equienergetic and S-equienergetic graphs by using Cartesian product, tensor product and strong product. The results can be further extended to the other class of graphs. Also some linear relations between A-energies and S-energies of graphs has been established which shows a possible new direction in the study of relation between energies of different graphs of same order.

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