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Notes on functions preserving density

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Abstract. Let $d(A)$ denote the asymptotic density of the set of positive integers. Let $\mathcal{AD}$ denote the set of all sets A having asymptotic density, and let $\mathcal{D}_\delta$ denote the set of all sets A for which the difference between its upper and lower density is less than δ. In the paper are studied fuctions $f : \mathbb{N} \to \mathbb{N}$ (not necessary a one-to-one functions) such that $A \in \mathcal{AD}$ implies $f(A) \in \mathcal{AD}$ and fuctions $f : \mathbb{N} \to \mathbb{N}$ for that $A \in \mathcal{AD}$ implies $f(A) \in \mathcal{D}_\delta$. Our results generalize a theorem in [M. B. Nathanson, R. Parikh, *Density of sets of natural numbers and Lévy group*, J. Number Theory **124** (2007), 151–158.]

1 Introduction

Denote by $\mathbb{N}$ the set of all positive integers. For $A \subset \mathbb{N}$ let $A(n)$ denote the counting function of the set A. The lower asymptotic density of A is

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n},$$

the upper asymptotic density of A is

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}.$$

If $\overline{d}(A) = \underline{d}(A)$, we say that A has an asymptotic density and we denote it by $d(A)$. For more details on the asymptotic density we refer to the paper [1].

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Let the group $\mathcal{L}^\sharp$ consists of all permutations of positive integers f such that $A \in \mathcal{AD}$ if and only if $f(A) \in \mathcal{AD}$, and the Lévy group $\mathcal{L}^\star$ consists of all permutations $f \in \mathcal{L}^\sharp$ such that $d(f(A)) = d(A)$ for all $A \in \mathcal{AD}$. Nathanson and Parikh [3] proved that the groups $\mathcal{L}^\sharp$ and $\mathcal{L}^\star$ coincide. Remark, more complicated result in the same direction was proved in [4], but with different assumptions on the transformation f. Connection between the Lévy group and finitely additive measures on integers extending the asymptotic density was studied in [5].

The mentioned Natanson and Parikh's result follows from the following stronger theorem.

Theorem A [2, Theorem 2] *Let $f : \mathbb{N} \to \mathbb{N}$ be a one-to-one function such that if $A \in \mathcal{AD}$, then $f(A) \in \mathcal{AD}$, that is, if the set A of positive integers has asymptotic density, then the set $f(A)$ also has asymptotic density. Let $\lambda = d(f(\mathbb{N}))$. Then*

$$d(f(A)) = \lambda d(A)$$

for all $A \in \mathcal{AD}$.

We generalize this result showing that the condition for f to be one-to-one function is not necessary and we will consider the set of functions $\mathcal{D}_\delta$ instead of $\mathcal{AD}$.

2 Results

Theorem 1 *Let $h : \mathbb{N} \to \mathbb{N}$ be a function (not necessary a one-to-one) such that if the set A of positive integers has asymptotic density, then the set $h(A)$ also has asymptotic density. Let $\lambda = d(h(\mathbb{N}))$. Then*

$$d(h(A)) = \lambda d(A)$$

for all $A \in \mathcal{AD}$.

Proof. Let the symmetric differerence of the sets X and Y be denoted by $X \ominus Y$. We construct a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ such that

$$d(f(\mathbb{N}) \ominus h(\mathbb{N})) = 0.$$

Then the assertion follows immediately from the Theorem A.

First, we construct a function $g : \mathbb{N} \to \mathbb{N}$ for that $\mathbb{N} \setminus g(\mathbb{N})$ is infinite and the density of the symmetric differerence of the sets $h(\mathbb{N})$ and $g(\mathbb{N})$ is zero. It can be done easily using an infinite set

$$S = \{a_1, a_2, a_3, \dots\}$$

with the property $d(S) = 0$. Obviously, we may define the set S as the set of all squares or as the set of all primes,... Let us define

$$g(n) = \begin{cases} a_{2k}, & \text{if } h(n) = a_k \\ h(n), & \text{if } h(n) \notin S \end{cases}.$$

Let

$$B = \{a_1, a_3, a_5, \dots, a_{2k+1}, \dots\}.$$

We have $B \subset \mathbb{N} \setminus g(\mathbb{N})$ and $d(B) = 0$.

We construct the injective function f and a sequence of sets $B_1, B_2, \dots$ by induction.

Let $f(1) = g(1)$ and $B_1 = B$. For $n \geq 1$

$$\begin{array}{lll} \text{if } \; g(n+1) \notin g(\mathbb{N} \cap [1, n]) & \text{let} & f(n+1) = g(n+1) \text{ and} \\ & & \quad B_{n+1} = B_n \\ \text{if } \; g(n+1) \in g(\mathbb{N} \cap [1, n]) & \text{let} & f(n+1) = \min B_n \text{ and} \\ & & \quad B_{n+1} = B_n \setminus \{f(n+1)\} \end{array}.$$

From the above construction follows that for any $A \subset \mathbb{N}$ the set $h(A)$ has asymptotic density if and only if $f(A)$ has asymptotic density and moreover $d(f(A)) = d(h(A))$ for arbitrary $A \in \mathcal{AD}$, so the assertion follows. □

By the above proved theorem the property that $A \in \mathcal{AD}$ implies $f(A) \in \mathcal{AD}$ is strong enough to ensure that in sense of asymptotic density large irregularities in the image set $f(\mathbb{N})$ cannot occur.

The main idea of the paper [3] was to show that if for a function f the density of the set A implies the density of the set $f(A)$ then the asymptotic density of $f(A)$ depends only on $d(A)$. Equivalently, if $A, B \in \mathcal{AD}$ and $d(A) = d(B)$, then $d(f(A)) = d(f(B))$.

In what follows we consider the question: Having a function $f : \mathbb{N} \to \mathbb{N}$ such that $A \in \mathcal{AD}$ implies $f(A) \in \mathcal{D}_f$, in the case $d(A) = d(B)$ what can we say about the upper and lower densities of the image sets $f(A)$ and $f(B)$?

In our studies the following "intertwinning lemma" will be fundamental.

Lemma 1 [**3**] *Let* A *and* B *be sets of positive integers such that* $d(A) = d(B) = \gamma$*. Then for a sufficiently fast growing sequence* (p_i) *if*

$$C = \bigcup_{i=1}^{\infty} A \cap (p_{2i-1}, p_{2i}] \cup \bigcup_{i=1}^{\infty} B \cap (p_{2i}, p_{2i+1}]$$

then

$$d(C) = \gamma.$$

Theorem 2 *Let* $\delta > 0$ *and let* $f : \mathbb{N} \to \mathbb{N}$ *be a one-to-one function such that if* $A \in \mathcal{AD}$ *then* $f(A) \in \mathcal{D}_\delta$*. Let* A, B *are arbitrary sets of positive integers with the property* $d(A) = d(B) = \gamma$*. Then*

$$\overline{d}(B) - \underline{d}(A) \leq \delta.$$

Proof. Let $\underline{d}(A) = \alpha$ and $\overline{d}(B) = \beta$. Suppose, contrary to our claim that

$$\beta > \alpha + \delta.$$

We will construct a set C for that $d(C) = \gamma$ but the set $f(C) \notin \mathcal{D}_\delta$. We will define the sequence (p_i) by induction and using this define the set C

$$C = \bigcup_{i=1}^{\infty} A \cap (p_{2i-1}, p_{2i}] \cup \bigcup_{i=1}^{\infty} B \cap (p_{2i}, p_{2i+1}]. \tag{1}$$

Induction hypothesis:

Suppose we have constructed sequences $p_1, \ldots, p_{2k+1}$, further $m_1, \ldots, m_{2k}$ and $n_2, \ldots, n_{2k+1}$ such that

$$\frac{|[m_{2i-1}, n_{2i}] \cap f(A)|}{n_{2i}} < \alpha + \frac{1}{i}, \tag{2}$$

$$\frac{|[m_{2i}, n_{2i+1}] \cap f(B)|}{n_{2i+1}} > \beta - \frac{1}{i} \tag{3}$$

for $i = 1, \ldots, k$ and

$$f(\mathbb{N} \setminus [p_j, p_{j+1}]) \cap [m_j, n_{j+1}] = \emptyset, \tag{4}$$

for $j = 1, \ldots 2k$.

Induction step: Let

$$m_{2k+1} = 1 + \max f(\mathbb{N} \cap [1, p_{2k+1}]).$$

From the fact that $\underline{d}(f(A)) = \alpha$ we get that for sufficiently large n_{2k+2} we have

$$\frac{|[m_{2k+1}, n_{2k+2}] \cap f(A)|}{n_{2k+2}} < \alpha + \frac{1}{k+1}$$

and moreover let $n_{2k+2} > (k+2).m_{2k+1}$.

Define p_{2k+2} as the least positive integer t satisfying

$$\min f([t, \infty) \cap \mathbb{N}) > n_{2k+2}.$$

From the definition of the numbers $m_{2k+1}, n_{2k+2}, p_{2k+2}$ follows that

$$f(\mathbb{N} \setminus [p_{2k+1}, p_{2k+2}]) \cap [m_{2k+1}, n_{2k+2}] = \emptyset.$$

Similarly, let

$$m_{2k+2} = 1 + \max f(\mathbb{N} \cap [1, p_{2k+2}]).$$

From $\overline{d}(f(B)) = \beta$ we have that for sufficiently large n_{2k+3} we have

$$\frac{|[m_{2k+2}, n_{2k+3}] \cap f(B)|}{n_{2k+3}} > \beta - \frac{1}{k+1}.$$

Define p_{2k+3} as the least positive integer t for that

$$\min f([t, \infty) \cap \mathbb{N}) > n_{2k+3}.$$

Analogously, from the definition of the numbers $m_{2k+2}, n_{2k+3}, p_{2k+3}$ we have

$$f(\mathbb{N} \setminus [p_{2k+2}, p_{2k+3}]) \cap [m_{2k+2}, n_{2k+3}] = \emptyset.$$

After completing induction the relations (2)-(4) hold for every $k \in \mathbb{N}$.

We estimate the upper and lower density of the constructed set C. Using (1) together with (2) and (4) we have

$$\begin{aligned} \liminf_{n\to\infty} \frac{f(C)(n)}{n} &\leq \liminf_{k\to\infty} \frac{f(C)(n_{2k})}{n_{2k}} \leq \liminf_{k\to\infty} \frac{m_{2k-1} + |[m_{2k-1}, n_{2k}] \cap f(A)|}{n_{2k}} \\ &\leq \liminf_{k\to\infty} \left(\frac{1}{k+1} + \alpha + \frac{1}{k} \right) = \alpha. \end{aligned}$$

On the other hand, by (1), (3) and (4)

$$\limsup_{n\to\infty} \frac{f(C)(n)}{n} \geq \limsup_{k\to\infty} \frac{f(C)(n_{2k+1})}{n_{2k+1}} \geq$$

$$\geq \limsup_{k\to\infty} \frac{|[m_{2k}, n_{2k+1}] \cap f(B)|}{n_{2k+1}} \geq \limsup_{k\to\infty} \left(\beta - \frac{1}{k}\right) = \beta. \qquad (5)$$

By Lemma 1 the set $C \in \mathcal{AD}$ but (5) and (5) yield to the fact that

$$\overline{d}(f(C)) - \underline{d}(f(C)) > \beta - \alpha > \delta$$

and therefore $f(C) \notin \mathcal{D}_\delta$. This contradiction completes the proof. □

Remarks. It is worth pointing out that

$$\bigcap_{n=1}^{\infty} \left\{ f : \mathbb{N} \to \mathbb{N};\ \text{if } A \in \mathcal{AD} \text{ then } f(A) \in \mathcal{D}_{\frac{1}{n}} \right\} =$$

$$= \{ f : \mathbb{N} \to \mathbb{N};\ \text{if } A \in \mathcal{AD} \text{ then } f(A) \in \mathcal{AD} \}.$$

In Theorem 2 the condition for the function f to be an injection is not necessary. It can be shown by the same way as in Theorem 1.

We have proved that for given $f : \mathbb{N} \to \mathbb{N}$ (if $A \in \mathcal{AD}$ then $f(A) \in \mathcal{D}_\delta$) the upper bound for $\overline{d}(f(A))$ and the lower bound for $\underline{d}(f(A))$ depends only on the asymptotic density of A. Clearly, for any dense set A and for any $\theta \in [0, 1]$ there is a set $B \subset A$ such that $d(B) = \theta.d(A)$ (see e.g. [2], Proposition 1), but using this fact we can only deduce, that these bounds are nondecreasing.

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References

[1] G. Grekos, On various definitions of density (survey), *Tatra Mt. Math. Publ.*, **31** (2005), 17–27.

[2] G. Grekos, L. Mišík, J. T. Tóth, Density sets of positive integers, *J. Number Theory*, **130** (2010), 1399–1407.

[3] M. B. Nathanson, R. Parikh, Density of sets of natural numbers and Lévy group, *J. Number Theory*, **124** (2007), 151–158.

[4] R. Giuliano Antonini, M. Paštéka, A comparison theorem for matrix limitation methods with applications, *Unif. Distrib. Theory,* **1** (2006), 87–109.

[5] M. Sleziak, M. Ziman, Lévy group and density measures, *J. Number Theory*, **128** (2008), 3005–3012.

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