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New results on connected dominating structures in graphs

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Abstract. A set of vertices in a graph is a dominating set if every vertex not in the set is adjacent to at least one vertex in the set. A dominating structure is a subgraph induced by the dominating set. Connected domination is a type of domination where the dominating structure is connected. Clique domination is a type of domination in graphs where the dominating structure is a complete subgraph. The clique domination number of a graph G denoted by $\gamma_k(G)$ is the minimum cardinality among all the clique dominating sets of G. We present few properties of graphs admitting dominating cliques along with bounds on clique domination number in terms of order and size of the graph. A necessary and sufficient condition for the existence of dominating clique in strong product of graphs is presented. A forbidden subgraph condition necessary to imply the existence of a connected dominating set of size four also is found.

1 Introduction

The study of domination in graphs is to a great extent a result of the study of games and recreational mathematics. It began when C.F. De Jaenisch attempted to determine the minimum number of queens that can be placed on

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an $n \times n$ chess board so that all squares are either attacked by a queen or are occupied by a queen [10]. Domination in graph can be defined in a similar terms as finding a set of vertices in a graph such that every vertex in the graph is either adjacent to some vertex in the set or is in the set. Further development in domination was observed in late 1950s with Claude Berge [3] introducing coefficient of external stability which is now known as domination number. A set of vertices in a graph is a *dominating set* if every vertex in the graph which is not in the dominating set is adjacent to one or more vertices in the dominating set. The *domination number*, $\gamma(G)$, of a graph G is the minimum number of vertices in a dominating set. Over the course of time different types of domination in graphs such as total domination, connected domination and independent domination were developed by imposing conditions on the dominating set. For example a *connected dominating set* is a dominating set that induces a connected subgraph. [17, 9, 7, 12, 2, 16].

A dominating structure in a graph is a subgraph induced by its dominating set. Identification of graphs possessing specific types of dominating structures is a problem that caught the attention of several researchers. In this paper we are exploring graphs having complete graphs as a dominating structure. Every graph referred to in this article is finite, undirected, simple and connected. [5, 14, 4] A *clique dominating set* is a dominating set that induces a complete subgraph. A *clique dominated graph* is a graph that contains a clique as a dominating structure. Cozzens and Kelleher were the first to deal with dominating cliques. The *clique domination number*, $\gamma_k(G)$, of a graph G is the minimum number of vertices in a clique dominating set.

The concept of domination is very useful to model several real-world problems such as social networks , bus routing, land surveying, computer and communication networks . Facility allocation is another area wherein one finds one of the most important applications of domination; in particular connected domination and clique domination. It involves optimal placement of facilities in a given area.[6, 7, 8] For example let us consider the problem of effective allocation of airports and air routes of a country. The airports in important cities of a country are connected with each other, while every other airport is connected with that of at least one of the important cities. Another instance is a wireless sensor network which is comprised of autonomous sensor nodes where the connected dominating set enable faster communication by forming a virtual network backbone for information and control routing.[15, 11, 13, 1]

2 Related results

It is note worthy that every graph need not have a dominating clique. The smallest clique being K_1, the smallest dominating clique is a single vertex. It is clear that a graph with a dominating vertex has a star as spanning tree. Wolk [18] gave the necessary condition for the graphs to have dominating clique of size one and he called such a dominating clique a central vertex or a central point. Dominating clique of size two is an edge called dominating edge.

Theorem 1 (Wolk [18]) *If G is a finite connected graph with no induced P_4 or C_4, then G has a dominating vertex.*

Cozzens and Kelleher [5] extended the theorem to get a forbidden subgraph condition to establish the existence of a dominating clique, which is presented below

Theorem 2 (Cozzens and Kelleher [5]) *If G is a connected graph that has no induced P_5 or C_5 then G has a dominating clique.*

Although the above result ensures the existence of a dominating clique, it does not specify the size of the dominating clique. In the direction,Cozzens and Kelleher [5] have explored the problem of identifying graphs possessing connected dominating set of size 3.

The notation K_{n+p} [5] represents the complete graph K_n on n vertices along with n pendants, one at each vertex of the complete graph. For example K_{3+p} is the net graph. K_{3+p} and K_{4+p} are shown in the Figure 1.

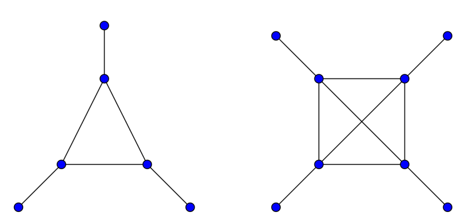

Figure 1: The graphs K_{3+p} and K_{4+p}

Note that connected dominating sets of size one and two respectively are defined uniquely whereas a connected dominating set of size three is either a P_3 or a K_3.

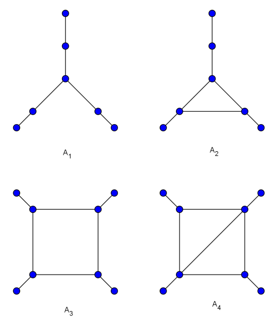

Figure 2: The graphs A_1, A_2, A_3 and A_4

The characterization obtained by Cozzens and Kelleher [5] was in terms of a class $\mathcal{A} = \{P_6, C_6, K_{4+p}, A_1, A_2, A_3, A_4\}$ of graphs where the graphs A_1, A_2, A_3 and A_4 are as shown Figure 2.

Theorem 3 (Cozzens and Kelleher [5]) *If G is a finite, connected graph with three or more vertices that has none of the graphs in Class $\mathcal{A}$ as an induced subgraph, then G has a connected dominating set of size three.*

We have settled the problem of obtaining a necessary condition for graphs to have a connected dominating set of size of 4 and the result is presented section 5. First we will explore the bounds for clique domination number γ_k.

3 Bounds for clique domination number

Recall that a private neighbour of a vertex v with respect to the set K is a vertex adjacent to only v from the set K. First we present the following proposition.

Proposition 4 *If K is a minimal dominating clique of a graph G, then every vertex in K has a private neighbour.*

Proof. On the contrary, assume that there is a vertex $v \in K$ having no private neighbor. Then v is adjacent to every vertex in K and v will have no private neighbour. This implies that $K - \{v\}$ is a smaller dominating clique contained in K, which contradicts the minimality of K. □

The bound obtained by Ore [10] for domination number, is true for clique domination as given below.

Proposition 5 *If a connected graph* G *of order* n *has a dominating clique, then* $\gamma_k(G) \leq n/2$.

Proof. Assume that $\gamma_k(G) > n/2$. Then γ_k-set of G being a minimal dominating clique, it is clear that there exists a vertex $v \in \gamma_k$-set of G which does not have a private neighbor which is a contradiction to the proposition 3.1. □

Remark 6 *The bound obtained in Proposition 3.2 is sharp and* K_{n+p} *is a class of graph that attains the bound. There are* $2n$ *vertices in* K_{n+p} *and the minimum dominating set is of size* n.

It is obvious that the domination number serves as a lower bound for clique domination number. Then the following inequality follows immediately.

Proposition 7 *If the graph* G *has a dominating clique, then* $\gamma(G) \leq \gamma_k(G) \leq \omega(G)$ *where* $\omega(G)$ *is the clique number of the graph.*

Remark 8 *Let* G *and* H *be two graphs. The corona product* $G \circ H$, *is the graph obtained by taking one copy of* G *and* $|V(G)|$ *copies of* H *and by joining each vertex of the* i*-th copy of* H *to the* i*-th vertex of* G, *where* $1 \leq i \leq |V(G)|$. *The graph* $K_r \circ K_s$, *where* $r \geq s$ *has* $\gamma(K_r \circ K_s) = \gamma_k(K_r \circ K_s) = \omega(K_r \circ K_s) = r$.

The next theorem gives a bound for the clique domination number of a graph in terms of its size.

Theorem 9 *If G is a graph with* m *edges possessing a dominating clique, then*

$$\gamma_k(G) \leq \frac{\sqrt{1+8m}-1}{2}.$$

Proof. We know that a clique of size γ_k has $\frac{\gamma_k(\gamma_k-1)}{2}$ edges. Therefore, $m \geq \frac{\gamma_k(\gamma_k-1)}{2} + \gamma_k$ so that $m \geq \frac{\gamma_k(\gamma_k+1)}{2}$. By solving which we will get $\gamma_k(G) \leq \frac{\sqrt{1+8m}-1}{2}$ or $\gamma_k(G) \leq \frac{-\sqrt{1+8m}-1}{2}$. Latter being impossible can be neglected. Hence $\gamma_k(G) \leq \frac{\sqrt{1+8m}-1}{2}$. □

Remark 10 *We can observe that the bound is sharp and* K_{n+p} *is a class of graph that attains the bound. Figure 1 shows the graphs* K_{3+p} *and* K_{4+p}.

Obviously a graph G with maximum degree $\Delta = n-1$ has a dominating vertex as the vertex with degree $n-1$ itself is a dominating vertex. We now consider graphs with $\Delta = n-2$ and obtain the following theorem.

Theorem 11 *If G is a connected graph with maximum degree $\Delta = n-2$, then G has a dominating edge.*

Proof. Since $\Delta = n-2$, there exists a vertex, say v in G which is adjacent to all but one vertex (obviously excluding v), say w, of the graph. But G is a connected graph and w is not adjacent to v which implies that w is adjacent to a neighbor of v say u, so that uv is a dominating edge of G. □

4 Clique domination in product of graphs

Clique domination problem for two types of graph product namely Lexicographical product and Cartesian product has been studied[4]. We now extend for clique domination in tensor products strong products of graphs.

The tensor product $G \times H$ of graphs G and H is a graph such that the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$; and any two vertices (u, v) and (u', v') are adjacent in $G \times H$ if and only if u is adjacent with u' in G and v is adjacent with v' in H.[17, 9]

We can understand by the definition of tensor product of graphs that any vertex (u', v') is not adjacent to any other vertex (u', v_i) and (u_j, v'), $\forall v_i \in V(H)$ and $\forall u_j \in V(G)$. For any two graphs G and H, $\gamma(G \times H) \geq 2$. For any graph G of order n, $G \times K_1$ is $\overline{K_n}$. And $G \times K_2$ is a bipartite graph.

Proposition 12 *For complete graphs K_r and K_s, $\gamma_k(K_r \times K_s) = 3$, if r,s ≥ 3*

Proof. We can observe that the tensor product of two complete graphs K_n and K_m is a graph with any vertex (u_i, v_j) is adjacent to all vertices (u_k, v_l), $\forall k \neq i$ and $\forall l \neq j$ Therefore, by choosing three vertices (u_{i_1}, v_{j_1}), (u_{i_2}, v_{j_2}) and (u_{i_3}, v_{j_3}) where $i_1 \neq i_2 \neq i_3$ and $j_1 \neq j_2 \neq j_3$ we obtain a dominating clique, thus proving that $\gamma_k(K_r \times K_s) \leq 3$. As we have observed earlier, we require at least two vertices to dominate a graph. And if we consider an edge, the two vertices in the edge say (u_{i_1}, v_{j_1}) and (u_{i_2}, v_{j_2}) can dominate all the vertices but (u_{i_1}, v_{j_2}) and (u_{i_2}, v_{j_1}), hence the graph cannot be dominated by an edge. Therefore K_3 is the smallest clique dominating the tensor product $K_r \times K_s$ □

The strong product $G \boxtimes H$ of two graphs G and H is the graph with $V(G \boxtimes H) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G \boxtimes H)$ if and only if either $uv \in E(G)$

and $u' = v'$ or $u = v$ and $u'v' \in E(H)$ or $uv \in E(G)$ and $u'v' \in E(H)$. Note that if $C \subseteq V(G \boxtimes H)$, then the G-projection and H-projection of C are, respectively, the sets $C_G = \{u \in V(G) \ : \ (u, b) \in C \text{ for some } b \in V(H)\}$ and $C_H = \{v \in V(H) \ : \ (a, v) \in C \text{ for some } a \in V(G)\}$. [17, 9]

Theorem 13 *The graph* $G \boxtimes H$ *has a dominating clique if and only if the graphs* G *and* H *have dominating cliques.*

Proof. Suppose $G \boxtimes H$ has a dominating clique. Let $C \subseteq V(G \boxtimes H)$ be the dominating clique of $G \boxtimes H$. Consider the projections C_G and C_H of C on G and H respectively. We claim that C_G is a dominating clique of G and C_H is a dominating clique of H. Strong product being commutative, it is sufficient to show that C_G is a dominating clique of G. Let u, $u' \in C_G$ be distinct vertices. By the definition of projection we can observe that there exist adjacent vertices (u, v) and (u', v') in C. We know that (u, v) and (u', v') are adjacent in $G \boxtimes H$ implies that either $uu' \in E(G)$ and $v = v'$ or $u = u'$ and $vv' \in E(H)$ or $uu' \in E(G)$ and $vv' \in E(H)$. Since u and u' are distinct we can easily conclude that $uu' \in E(G)$. Therefore C_G forms a clique in G. Now, to show that C_G is a dominating set. Let $u_1 \notin C_G$ be vertex of G. There exists a vertex (u_1, v_1) in $G \boxtimes H$. Since C is a dominating clique in $G \boxtimes H$, there exists a vertex $(u_0, v_0) \in C$ adjacent to (u_1, v_1). Since u_0 and u_1 are distinct, by definition of an edge in strong product u_1 and u_0 are adjacent. Therefore, C_G is a dominating clique of G.
Conversely, let S_G and S_H be the dominating cliques in the graphs G and H. We claim that $S_G \times S_H$ forms a dominating clique in $G \boxtimes H$. Firstly to show that $S_G \times S_H$ is a clique in $G \boxtimes H$. Let (u, v) and (u', v') be two distinct vertices in $S_G \times S_H$. Either $u = u'$ or $uu' \in E(G)$ and $v = v'$ or $vv' \in E(H)$. Either ways (u, v) and (u', v') are adjacent. Hence, $S_G \times S_H$ is a clique in $G \boxtimes H$. Now to show that $S_G \times S_H$ dominates $G \boxtimes H$. Consider a vertex (u_1, v_1) not in $S_G \times S_H$. If u_1 not in S_G then there exists a u_0 in S_G adjacent to u_1 in G and a v_0 in S_H where $v_0 = v_1$ or v_0 and v_1 are adjacent in H. By the definition of strong product of graphs (u_1, v_1) is adjacent to (u_0, v_0). And if u_1 is in S_G since (u_1, v_1) not in $S_G \times S_H$ there exist $v_0 \neq v_1$ in S_H dominating v_1 in H. Owing to the definition of strong product of graphs (u_1, v_1) is adjacent to (u_1, v_0). Therefore, $S_G \times S_H$ forms a dominating clique in $G \boxtimes H$. □

Theorem 14 *If* G *and* H *are connected graphs with dominating cliques, then* $\gamma_k(G \boxtimes H) = \gamma_k(G) \ \times \ \gamma_k(H)$

Proof. Let S_G and S_H be the γ_k sets of G and H respectively. We know that $S_G \times S_H$ forms a dominating clique in $G \boxtimes H$. This implies

$$\gamma_k(G \boxtimes H) \leq \gamma_k(G) \ \times \ \gamma_k(H)$$

To show that $\gamma_k(G \boxtimes H) \geq \gamma_k(G) \ \times \ \gamma_k(H)$ we need to show that $\forall \ (u_i, v_i) \in S_G \times S_H$, $(u_i, v_i) \in \gamma_k$-set of $(G \boxtimes H)$. $u_i \in S_G$ implies u_i has a private neighbor say u_1. Similarly v_i has a private neighbor v_1. We claim that (u_1, v_1) is a private neighbor of (u_i, v_i), i.e. there is no (u_2, v_2) adjacent to (u_1, v_1) in $S_G \times S_H$. If there exists a vertex, say, (u_2, v_2) adjacent to (u_1, v_1) then by definition of strong product $u_2 = u_1$ and $v_1 v_2 \in E(H)$ or $u_1 u_2 \in E(G)$ and $v_2 = v_1$ or $u_1 u_2 \in E(G)$ and $v_1 v_2 \in E(H)$ all contradicting the fact that u_1 is the private neighbor of u_i and v_1 is the private neighbor of v_i. Which implies that $\forall$ $(u_i, v_i) \in S_G \times S_H$, $(u_i, v_i) \in \gamma_k$-set of $(G \boxtimes H)$. Hence $S_G \times S_H$ is a minimal dominating clique of $G \boxtimes H$. To show that $S_G \times S_H$ is a γ_k set of $G \boxtimes H$, assume the contrary, if $S_G \times S_H$ is not a γ_k set of $G \boxtimes H$, then there exist a smaller dominating clique T whose projections T_G and T_H forms a smaller dominating clique for G and H respectively hence contradicting the minimality of S_G and S_H. □

5 Graphs with connected dominating structure of order four

A forbidden subgraph condition necessary for a graph to have a connected dominating set of size three was found by Cozzens and Kelleher [5]. We discuss a forbidden subgraph condition necessary to have a connected dominating set of size four. There are 6 connected graphs on four vertices :- K_4, C_4, P_4, Claw ($K_{1,3}$), Paw and Diamond ($K_4 - e$). Therefore a connected dominating set of size four can be any of the above mentioned graph.

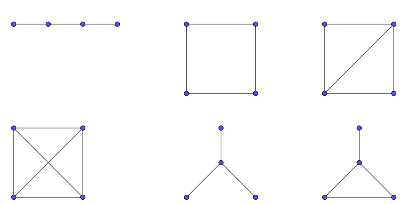

Figure 3: Connected graphs of order four

Theorem 15 *If G is a finite, connected graph with four or more vertices that has none of the graphs in $\mathcal{B}$ (Fig. 4) as an induced subgraph, then G has a connected dominating structure of order four.*

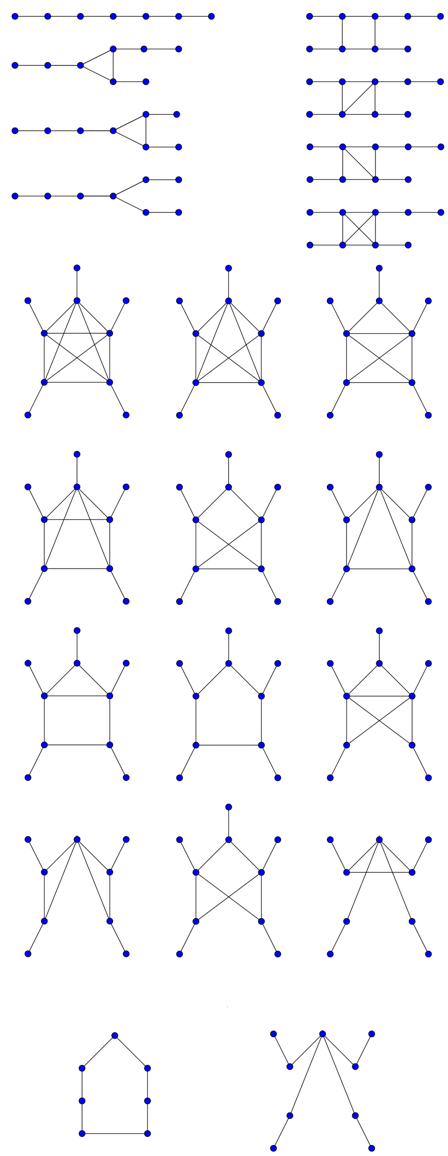

Figure 4: Class $\mathcal{B}$

Proof. By induction on n, the order of graph G,

(i) The theorem is true when $n = 4$.

(ii) Assume that any finite connected graph with n vertices, $n \geq 4$, that has none of the graphs in $\mathcal{B}$ as an induced subgraph has a connected dominating structure of size four.

(iii) Let G be a finite connected graph on $n + 1$ vertices, where $n \geq 4$, that has none of the graphs in class $\mathcal{B}$ as an induced subgraph. Let v be vertex of G which is not a cut vertex. Consider the graph G', subgraph of G induced by all vertices of G excluding v. Since G' is a finite connected graph with n vertices having no graphs from class $\mathcal{B}$ as an induced subgraph, by the induction hypothesis G' has a connected dominating structure of order four.

Let $S = \{a, b, c, d\}$ induce the connected dominating structure of order four of G'. If v is adjacent to any vertex in S, then S dominates G also.

Suppose that in G, v is not adjacent to any vertex in S. Since G is connected, v must be adjacent to some vertex of G, say x. And S being the connected dominating set of G', x must be adjacent to some vertex in S. The set $\{a, b, c, d, x\}$ induces a connected graph of 5 vertices. Therefore, the graph induced by $S \cup N(S) \cup \{v\}$ has one of the graphs from $\mathcal{B}$ as a subgraph, not necessarily induced, i.e, there might be edges between the pendant vertices and other vertices. If there are no edges between the pendant vertices and the other vertices, this implies that the subgraphs are induced, which contradicts the assumption that G has none of the graphs in class $\mathcal{B}$ as an induced subgraph.

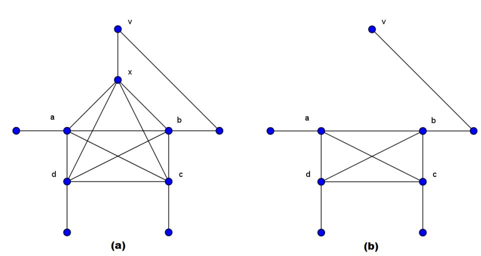

Figure 5: Graphs used in the proof of Theorem 15

Suppose G has at least one edge between the pendant vertices. If G has exactly one edge between vertices as shown in Figure 5(a) , then G has an induced subgraph shown in Figure 5(b), which is a forbidden subgraph from class $\mathcal{B}$.

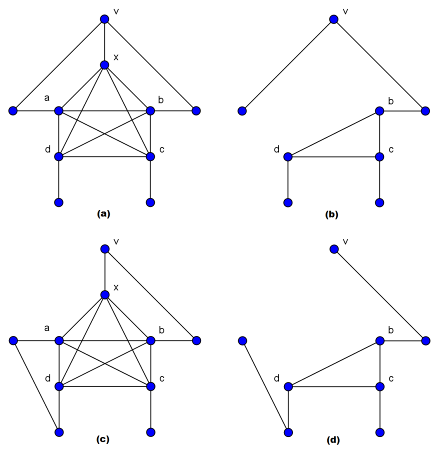

Figure 6: Graphs used in the proof of Theorem 15

If G has exactly two edges between the pendant vertices as shown in Figure 6(a) or 6(c), then G has an induced subgraph shown in Figure 6(b) or 6(d) , which is a forbidden subgraph from class $\mathcal{B}$.

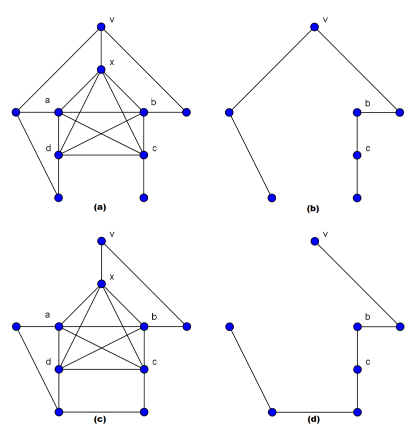

Figure 7: Graphs used in the proof of Theorem 15

If G has exactly three edges between the pendant vertices as shown in Figure 7(a) or 7(c), then G has an induced P_7, which is a forbidden subgraph from class $\mathcal{B}$.

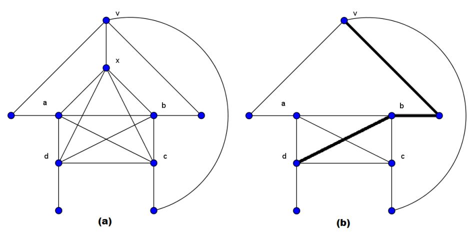

Figure 8: Graphs used in the proof of Theorem 15

If G has exactly three edges between the pendant vertices as shown in Figure 8(a), then G has a P_4 as shown in Figure 8(b) which is a connected dominating structure of order four.

We can now observe that an edge between the pendant vertices in the graphs in Class $\mathcal{B}$ will lead to obtaining a connected dominating structure of order four or a contradiction to the absence of an induced forbidden structure from class $\mathcal{B}$. Therefore, G has a connected dominating set of size four. □

As we have seen before, the converse of this theorem need not be true. A finite connected graph having graph from $\mathcal{B}$ as an induced subgraph can have a dominating clique of size four. An example is given in Fig. 9.

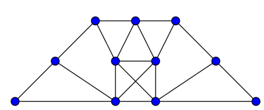

Figure 9: Graph with induced P_7 dominated by K_4

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