

New classes of local almost contractions

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Abstract. Contractions represents the foundation stone of nonlinear analysis. That is the reason why we propose to unify two different type of contractions: almost contractions, introduced by V. Berinde in [2] and local contractions (Martins da Rocha and Filipe Vailakis in [7]). These two types of contractions operate in different space settings: in metric spaces (almost contractions) and semimetric spaces (for local contractions). That new type of contraction was built up in a new space setting, which is the pseudometric space. The main results of this paper represent the extension for various type of operators on pseudometric spaces, such as: generalized ALC, Ćirić-type ALC, quasi ALC, Ćirić-Reich-Rus type ALC. We propose to study the existence and uniqueness of their fixed points, and also the continuity in their fixed points, with a large number of examples for ALC-s.

1 Introduction

First, we present the concept of almost contraction, following V. Berinde in [2].

Definition 1 (see [2]) *Let (X, d) be a metric space. $T : X \rightarrow X$ is called almost contraction or (δ, L) -contraction if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \cdot d(y, Tx), \forall x, y \in X. \quad (1)$$

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Remark 1 The term of almost contraction is equivalent to weak contraction, and it was first introduced by V. Berinde in [2].

Because of the symmetry of the distance, the almost contraction condition (1) includes the following dual one:

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \cdot d(x, Ty), \forall x, y \in X, \quad (2)$$

obtained from (1) by replacing $d(Tx, Ty)$ by $d(Ty, Tx)$ and $d(x, y)$ by $d(y, x)$.

Obviously, to prove the almost contractiveness of T , it is necessary to check both (1) and (2).

A strict contraction satisfies (1), with $\delta = \alpha$ and $L = 0$, therefore it is an almost contraction with a unique fixed point.

Many examples of almost contractions are given in [1]-[3]. Weak contractions represent a generous concept, due to various mappings satisfying the condition (1). Such examples of weak contraction was given by V. Berinde in [2].

Definition 2 [5] Let (X, d) be a metric space. Any mapping $T : X \rightarrow X$ is called Ćirić-Reich-Rus contraction if it is satisfied the condition:

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta \cdot [d(x, Tx) + d(y, Ty)], \forall x, y \in X, \quad (3)$$

where $\alpha, \beta \in \mathcal{R}_+$ and $\alpha + 2\beta < 1$.

Proposition 1 (see [8]) Let (X, d) be a metric space. Any Ćirić-Reich-Rus contraction, i.e., any mapping $T : X \rightarrow X$ satisfying the condition (3), represent an almost contraction.

Theorem 1 A mapping satisfying the contractive condition: there exists $0 \leq h < \frac{1}{2}$ such that

$$d(Tx, Ty) \leq h \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \quad (4)$$

for all $x, y \in X$, is a weak contraction.

An operator satisfying (4) with $0 < h < 1$ is called quasi-contraction.

Remark 2 Theorem 1 prove that quasi-contractions with $0 < h < \frac{1}{2}$ are always weak contractions. However, there exists quasi-contractions with $h \geq \frac{1}{2}$, presented in Example 1 by V. Berinde in [2], as it follows:

Example 1 Let $T : [0, 1] \rightarrow [0, 1]$ a mapping given by $Tx = \frac{2}{3}$ for $x \in [0, 1]$, and $T1 = 0$. Then T has the following properties:

- 1) T satisfies (4) with $h \in [\frac{2}{3}, 1)$, i.e., T is quasi-contraction;
- 2) T satisfies (1), with $\delta \geq \frac{2}{3}$ and $L \geq \delta$, i.e., T is also weak contraction;
- 3) T has a unique fixed point, $x^* = \frac{2}{3}$.

Since we were familiarized with the class of almost contractions, we introduce the concept of local contractions, another interesting type of operators with unexpected applications. The concept of local contraction was presented by Martins da Rocha and Filipe Vailakis in [7].

Definition 3 (see [7]) Let F be a set and let $\mathcal{D} = (d_j)_{j \in J}$ be a family of semidisstances defined on F . We let σ be the weak topology on F defined by the family $\mathcal{D}$. A sequence $(f_n)_{n \in \mathbb{N}^*}$ is said to be σ -Cauchy if it is d_j -Cauchy, $\forall j \in J$. A subset A of F is said to be sequentially σ -complete if every σ -Cauchy sequence in A converges in A for the σ -topology. A subset $A \subset F$ is said to be σ -bounded if $\text{diam}_j(A) \equiv \sup\{d_j(f, g) : f, g \in A\}$ is finite for every $j \in J$. Let r be a function from J to J . An operator $T : F \rightarrow F$ is called local contraction with respect $(\mathcal{D}, r)$ if, for every j , there exists $\beta_j \in [0, 1)$ such that

$$\forall f, g \in F, \quad d_j(Tf, Tg) \leq \beta_j d_{r(j)}(f, g).$$

Definition 4 The mapping $d(x, y) : X \times X \rightarrow \mathbb{R}_+$ is said to be a pseudometric if:

1. $d(x, y) = d(y, x)$;
2. $d(x, y) \leq d(x, z) + d(z, y)$;
3. $x = y$ implies $d(x, y) = 0$
(instead of $x = y \Leftrightarrow d(x, y) = 0$ in the metric case).

Definition 5 (see [11])

Let r be a function from J to J . An operator $T : F \rightarrow F$ is an almost local contraction (ALC) with respect $(\mathcal{D}, r)$ or (δ, L) -contraction, if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d_j(Tf, Tg) \leq \delta \cdot d_j(f, g) + L \cdot d_{r(j)}(g, Tf), \forall f, g \in F. \quad (5)$$

Theorem 2 [11] Assume that the space F is σ -Hausdorff, which means: for each pair $f, g \in F$, $f \neq g$, there exists $j \in J$ such that $d_j(f, g) > 0$.

If A is a nonempty subset of F , then for each h in F , we let

$$d_j(h, A) \equiv \{d_j(h, g) : g \in A\}.$$

Consider a function $r : J \rightarrow J$ and let $T : F \rightarrow F$ be an almost local contraction with respect to $(\mathcal{D}, r)$. Consider a nonempty, σ -bounded, sequentially σ -complete, and T -invariant subset $A \subset F$.

(E) If the condition

$$\forall j \in J, \quad \lim_{n \rightarrow \infty} \beta_j \beta_{r(j)} \cdots \beta_{r^n(j)} \text{diam}_{r^{n+1}(j)}(A) = 0 \quad (6)$$

is satisfied, then the operator T admits a fixed point f^* in A .

(S) Moreover, if $h \in F$ satisfies

$$\forall j \in J, \quad \lim_{n \rightarrow \infty} \beta_j \beta_{r(j)} \cdots \beta_{r^n(j)} d_{r^{n+1}(j)}(h, A) = 0, \quad (7)$$

then the sequence $(T^n h)_{n \in \mathbb{N}}$ is σ -convergent to f^* .

Example 2 Let $X = [0, n] \times [0, n] \subset \mathbb{R}^2$, $n \in \mathbb{N}^*$, $T : X \rightarrow X$,

$$T(x, y) = \begin{cases} (\frac{x}{2}, \frac{y}{2}) & \text{if } (x, y) \neq (1, 0) \\ (0, 0) & \text{if } (x, y) = (1, 0) \end{cases}$$

The diameter of the subset $X = [0, n] \times [0, n] \subset \mathbb{R}^2$ is given by the diagonal line of the square whose four sides have length n .

We shall use the pseudometric:

$$d_j((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| \cdot e^{-j}, \forall j \in J, \quad (8)$$

where J is a subset of $\mathbb{N}$. This is a pseudometric, but not a metric, take for example:

$$d_j((1, 4), (1, 3)) = |1 - 1| \cdot e^{-j} = 0, \text{ however } (1, 4) \neq (1, 3)$$

In this case, we shall use the function $r(j) = \frac{j}{2}$. By applying the inequality (5) to our mapping T , we get for all $x = (x_1, y_1), y = (x_2, y_2) \in X$

$$\left| \frac{x_1}{2} - \frac{x_2}{2} \right| \cdot e^{-j} \leq \theta \cdot |x_1 - x_2| \cdot e^{-\frac{j}{2}} + L \cdot \left| x_2 - \frac{x_1}{2} \right| \cdot e^{-\frac{j}{2}},$$

for all $j \in J$, which can be write as the equivalent form

$$|x_1 - x_2| \cdot e^{-\frac{j}{2}} \leq 2\theta \cdot |x_1 - x_2| + L \cdot |2x_2 - x_1|,$$

The last inequality became true if we take $\theta = \frac{1}{2} \in (0, 1)$, $L = 4 \geq 0$. Hence T is an almost local contraction, with the unique fixed point $(0, 0)$.

T is continuous in the fixed point, at $(0, 0) \in \text{Fix}(T)$, but is not continuous at $(1, 0) \notin \text{Fix}(T)$.

Example 3 With the assumptions from the previous example and the pseudometric defined by (8) where $j \in J$, and $r(j) = \frac{j}{2}$, we get another example for almost local contractions. Considering $T : X \rightarrow X$,

$$T(x, y) = \begin{cases} (x, -y) & \text{if } (x, y) \neq (1, 1) \\ (0, 0) & \text{if } (x, y) = (1, 1) \end{cases}$$

T is not a contraction because the contractive condition:

$$d_j(Tx, Ty) \leq \theta \cdot d_j(x, y), \quad (9)$$

is not valid $\forall x, y \in X$, and for any $\theta \in (0, 1)$. Indeed, (9) is equivalent with:

$$|x_1 - x_2| \cdot e^{-j} \leq \theta \cdot |x_1 - x_2| \cdot e^{-j}, \forall j \in J.$$

The last inequality leads us to $1 \leq \theta$, which is obviously false, considering $\theta \in [0, 1)$. However, T becomes an almost local contraction if:

$$|x_1 - x_2| \cdot e^{-j} \leq \theta \cdot |x_1 - x_2| \cdot e^{-\frac{j}{2}} + L \cdot |x_2 - x_1| \cdot e^{-\frac{j}{2}}$$

which is equivalent to : $e^{-\frac{j}{2}} \leq \theta + L$. For $\theta = \frac{1}{3} \in [0, 1)$, $L = 2 \geq 0$ and $j \in J$, the last inequality becomes true, i.e. T is an almost local contraction with many fixed points:

$$\text{Fix}T = \{(x, 0) : x \in \mathbb{R}\}.$$

In this case, we have:

$$\forall j \in J, \quad \lim_{n \rightarrow \infty} \theta^{n+1} \text{diam}_{r^{n+1}(j)}(A) = \lim_{n \rightarrow \infty} \left(\frac{1}{3}\right)^{n+1} \cdot (n-1)^2 = 0$$

This way, the existence of the fixed point is assured, according to condition (E) from Theorem 2. The continuity of T in $(0, 0) \in \text{Fix}(T)$ is valid, but we have discontinuity in $(1, 1)$, which is not a fixed point of T .

Example 4 Let X be the set of positive functions:

$$X = \{f : [0, \infty) \rightarrow [0, \infty)\},$$

which is a subset of the real functions $\mathcal{F} = \{f : \mathbb{R} \rightarrow \mathbb{R}\}$.

Let $d_j(f, g) = |f(0) - g(0)| \cdot e^{-j}$, $\forall f, g \in X$, $r(j) = \frac{j}{2}$, $\forall j \in J$. Indeed, d_j is a pseudometric, but not a metric, take for example $d_j(x, x^2) = 0$, but $x \neq x^2$.

Considering the mapping $Tf = |f|$, $\forall f \in X$, and using the inequality (1) from the definition of almost local contractions:

$$|f(0) - g(0)| \cdot e^{-j} \leq \theta \cdot |f(0) - g(0)| \cdot e^{-\frac{j}{2}} + L \cdot |g(0) - f(0)| \cdot e^{-\frac{j}{2}}$$

which is equivalent to: $e^{-j/2} \leq \theta + L$. This inequality becomes true if $j > 0$, $\theta = \frac{1}{4} \in (0, 1)$, $L = 3 > 0$. Hence, T is an ALC. However, T is not a contraction, because the contractive condition (9) leads us again to the false assumption: $1 \leq \theta$. The mapping T has infinite number of fixed points: $\text{Fix}T = \{f \in X\} = X$, by taking:

$$|f(x)| = f(x), \forall f \in X, x \in [0, \infty)$$

2 Main results

The main results of this paper represent the extension for various type of operators on pseudometric spaces, such as: generalized ALC, Ćirić-type ALC, quasi ALC, Ćirić-Reich-Rus type ALC.

a) Generalized ALC

Definition 6 Let r be a function from J to J . Let $A \subset F$ be a τ -bounded sequentially τ -complete and T -invariant subset of F . A mapping $T : A \rightarrow A$ is called generalized almost local contraction if there exist a constant $\theta \in (0, 1)$ and some $L \geq 0$ such that $\forall x, y \in X, \forall j \in J$ we have:

$$d_j(Tx, Ty) \leq \theta \cdot d_{r(j)}(x, y) + L \cdot \min\{d_{r(j)}(x, Tx), d_{r(j)}(y, Ty), d_{r(j)}(x, Ty), d_{r(j)}(y, Tx)\} \quad (10)$$

Remark 3 It is obvious that any generalized almost local contraction is an almost contraction, i.e., it does satisfy inequality (1).

Theorem 3 Let $T : A \rightarrow A$ be a generalized almost local contraction, i.e., a mapping satisfying (10), and also verifying the condition (7) for the unicity of fixed point. Let $\text{Fix}(T) = \{f\}$. Then T is continuous at f .

Proof. Since T is a generalized almost local contraction, there exist a constant $\theta \in (0, 1)$ and some $L \geq 0$ such that (10) is satisfied. We know by Theorem 7 that T has a unique fixed point, say f .

Let $\{y_n\}_{n=0}^\infty$ be any sequence in X converging to f . Then by taking

$$y := y_n, \quad x := f$$

in the generalized almost local contraction condition (10), we get

$$d_j(Tf, Ty_n) \leq \theta \cdot d_{r(j)}(f, y_n), n = 0, 1, 2, \dots \quad (11)$$

since f is a fixed point for T , we have

$$\min\{d_{r(j)}(x, Tx), d_{r(j)}(y, Ty), d_{r(j)}(x, Ty), d_{r(j)}(y, Tx)\} = d_{r(j)}(f, Tf) = 0.$$

Now, by letting $n \rightarrow \infty$ in (11), we get $Ty_n \rightarrow Tf$, which shows that T is continuous at f . $\square$

b) Ćirić-type almost local contraction

Definition 7 (see Berinde, [4]) *Let (X, d) be a complete metric space.*

The mapping $T : X \rightarrow X$ is called Ćirić almost contraction if there exist a constant $\alpha \in [0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \alpha \cdot M(x, y) + L \cdot d(y, Tx), \text{ for all } x, y \in X, \quad (12)$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

From the above definition the following question arises: it is possible to expand it to the case of almost local contractions? The answer is affirmative and is given by the next definition. But first we need to remind the Lemma of Ćirić ([6]), which will be essential in proving our main results.

Lemma 1 *Let T be a quasi-contraction on X and let n be any positive integer. Then, for each $x \in X$, and all positive integers i, j , where $i, j \in \{1, 2, \dots, n\}$ implies*

$$d(T^i x, T^j x) \leq h \cdot \delta[O(x, n)],$$

where we denoted $\delta(A) = \sup\{d(a, b) : a, b \in A\}$ for a subset $A \subset X$.

Remark 4 *Observe that, by means of Lemma 1, for each n , there exist $k \leq n$ such that*

$$d(x, T^k x) = \delta[O(x, n)].$$

Lemma 2 (see [6]) *Let T be a quasi-contraction on X . Then the inequality*

$$\delta[O(x, n)] \leq \frac{1}{1-h} d(x, T^k x)$$

holds for all $x \in X$.

Definition 8 Under the assumptions of definition 5, the operator $T : A \rightarrow A$ is called Čirić-type almost local contraction with respect $(\mathcal{D}, r)$ if, for every $j \in J$, there exist the constants $\theta \in [0, 1)$ and $L \geq 0$ such that

$$d_j(Tf, Tg) \leq \theta \cdot M_{r(j)}(f, g) + L \cdot d_{r(j)}(g, Tf), \text{ for all } f, g \in A, \quad (13)$$

where

$$M_{r(j)}(f, g) = \max \{d_{r(j)}(f, g), d_{r(j)}(f, Tf), d_{r(j)}(g, Tg), d_{r(j)}(f, Tg), d_{r(j)}(g, Tf)\}.$$

Remark 5 Although this class is more wide than the one of almost local contractions, similar conclusions can be stated as in the case of almost local contractions, as it follows:

Theorem 4 Consider a function $r : J \rightarrow J$, let a nonempty, τ - bounded, sequentially τ - complete, and T - invariant subset $A \subset X$ and let $T : A \rightarrow A$ be Čirić- type almost local contraction with respect to $(\mathcal{D}, r)$. Then

1. T has a fixed point, i.e., $\text{Fix}(T) = \{x \in X : Tx = x\} \neq \emptyset$;
2. For any $x_0 = x \in A$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ converges to $x^* \in \text{Fix}(T)$;
3. The following a priori estimate is available:

$$d_j(x_n, x^*) \leq \frac{\theta^n}{(1 - \theta)^2} d_j(x, Tx), \quad n = 1, 2, \dots \quad (14)$$

Proof. For the conclusion of the Theorem, we have to prove that T has at least a fixed point in the subset $A \subset X$. To this end, let $x \in A$ be arbitrary, and let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$ with $x_0 = x$.

Take $x := x_{n-1}$, $y := x_n$ in (13) to obtain

$$d_j(x_n, x_{n+1}) = d_j(Tx_{n-1}, Tx_n) \leq \theta \cdot M_{r(j)}(x_{n-1}, x_n),$$

since $d_j(x_n, Tx_{n-1}) = d_j(Tx_{n-1}, Tx_{n-1}) = 0$. Continuing in this manner, for $n \geq 1$, by Lemma 1 we have

$$d_j(T^n x, T^{n+1} x) = d_j(TT^{n-1} x, T^2 T^{n-1} x) \leq \theta \cdot \delta[O(T^{n-1} x, 2)].$$

By using Remark 4, we can easily conclude: there exist a positive integer $k_1 \in \{1, 2\}$ such that

$$\delta[O(T^{n-1} x, 2)] = d_j(T^{n-1} x, T^{k_1} T^{n-1} x)$$

and therefore

$$d_j(x_n, x_{n+1}) \leq \theta \cdot d_j(T^{n-1}x, T^{k_1}T^{n-1}x).$$

By using once again Lemma 1, we obtain, for $n \geq 2$,

$$\begin{aligned} d_j(T^{n-1}x, T^{k_1}T^{n-1}x) &= d_j(TT^{n-2}x, T^{k_1+1}T^{n-2}x) \leq \\ &\leq \theta \cdot \delta[O(T^{n-2}x, k_1 + 1)] \leq \theta \cdot \delta[O(T^{n-2}x, 3)]. \end{aligned}$$

Continuing in this manner, we get

$$d_j(T^n x, T^{n+1}x) \leq \theta \cdot \delta[O(T^{n-1}x, 2)] \leq \theta^2 \cdot \delta[O(T^{n-2}x, 3)].$$

By applying repeatedly the last inequality, we get

$$d_j(T^n x, T^{n+1}x) \leq \theta \cdot \delta[O(T^{n-1}x, 2)] \leq \dots \leq \theta^n \cdot \delta[O(x, n+1)]. \quad (15)$$

At this point, by Lemma 2, we obtain

$$\delta[O(x, n+1)] \leq \delta[O(x, \infty)] \leq \frac{1}{1-\theta} d_j(x, Tx),$$

which by (15) yields

$$d_j(T^n x, T^{n+1}x) \leq \frac{\theta^n}{1-\theta} d_j(x, Tx). \quad (16)$$

The last inequality and the triangle inequality can be merged to obtain the following estimate:

$$d_j(T^n x, T^{n+p}x) \leq \frac{\theta^n}{1-\theta} \cdot \frac{1-\theta^p}{1-\theta} d_j(x, Tx). \quad (17)$$

Let us remind the fact that $0 \leq \theta \leq 1$, then, by using (17), we can conclude that $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence. The subset A is assumed to be sequentially τ -complete, there exists x^* in A such that $\{x_n\}$ is τ -convergent to x^* . After simple computations involving the triangular inequality and the Definition (13), we get

$$\begin{aligned} d_j(x^*, Tx^*) &\leq d_j(x^*, x_{n+1}) + d_j(x_{n+1}, Tx^*) \\ &= d_j(T^{n+1}x, x^*) + d_j(T^n x, Tx^*) \\ &\leq d_j(T^{n+1}x, x^*) + \theta \max\{d_j(T^n x, u), d_j(T^n x, T^{n+1}x), d_j(x^*, Tx^*), \\ &\quad d_j(T^n x, Tx^*), d_j(T^{n+1}x, x^*)\} + L \cdot d_j(x^*, Tx_n) \end{aligned}$$

Continuing in this manner, we obtain

$$\begin{aligned} d_j(x^*, Tx^*) &\leq d_j(T^{n+1}x, x^*) + \theta \cdot [d_j(T^n x, u) + d_j(T^n x, T^{n+1}x) \\ &\quad + d_j(x^*, Tx^*) + d_j(T^{n+1}x, x^*)] + L \cdot d_j(x^*, Tx_n). \end{aligned}$$

These relations leads us to the following inequalities:

$$\begin{aligned} d_j(x^*, Tx^*) &\leq \frac{1}{1-\theta} [(1+\theta)d_j(T^{n+1}x, x^*) \\ &\quad + (\theta+L)d_j(x^*, Tx_n) + \theta d_j(T^n x, T^{n+1}x)]. \end{aligned} \quad (18)$$

Letting $n \rightarrow \infty$ in (18) we obtain

$$d_j(x^*, Tx^*) = 0,$$

which means that x^* is a fixed point of T . The estimate (14) can be obtained from (16) by letting $p \rightarrow \infty$.

This completes the proof. $\square$

Remark 6 1) Theorem 4 represent a very important extension of Banach's fixed point theorem, Kannan's fixed point theorem, Chatterjea's fixed point theorem, Zamfirescu's fixed point theorem, as well as of many other related results obtained on the base of similar contractive conditions. These fixed point theorems mentioned before ensures the uniqueness of the fixed point, but the Ćirić type almost local contraction need not have a unique fixed point.

2) Let us remind (see Rus [9], [10]) that an operator $T : X \rightarrow X$ is said to be a weakly Picard operator (WPO) if the sequence $\{T^n x_0\}_{n=0}^{\infty}$ converges for all $x_0 \in X$ and the limits are fixed point of T . The main merit of Theorem 4 is the very large class of Weakly Picard operators assured by using it.

The uniqueness of the fixed point of a Ćirić type almost local contraction can be assured by imposing an additional contractive condition, quite similar to (13), according to the next theorem.

Theorem 5 With the assumptions of Theorem 4, let $T : A \rightarrow A$ be a Ćirić type almost local contraction with the additional inequality, which actually means the monotonicity of the pseudometric:

$$d_{r(j)}(f, g) \leq d_j(f, g), \forall f, g \in A, \forall j \in J. \quad (19)$$

If the mapping T satisfies the supplementary condition: there exist the constants $\theta \in [0, 1]$ and some $L_1 \geq 0$ such that

$$d_j(Tf, Tg) \leq \theta \cdot d_{r(j)}(f, g) + L_1 \cdot d_{r(j)}(f, Tf), \text{ for all } f, g \in A, \forall j \in J, \quad (20)$$

then

- 1) T has a unique fixed point, i.e., $\text{Fix}(T) = \{f^*\}$;
- 2) The Picard iteration $\{x_n\}_{n=0}^\infty$ given by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$ converges to f^* , for any $x_0 \in A$;
- 3) The a priori error estimate (14) holds;
- 4) The rate of the convergence of the Picard iteration is given by

$$d_j(x_n, f^*) \leq \theta \cdot d_{r(j)}(x_{n-1}, f^*), \quad n = 1, 2, \dots, \forall j \in J \quad (21)$$

Proof. 1) Suppose, by contradiction, there are two distinct fixed points f^* and g^* of T . Then, by using (20), and condition (19) for every fixed $j \in J$ with $f := f^*, g := g^*$ we get:

$$d_j(f^*, g^*) \leq \theta \cdot d_{r(j)}(f^*, g^*) \leq \theta \cdot d_j(f^*, g^*) \Leftrightarrow (1 - \theta) \cdot d_j(f^*, g^*) \leq 0,$$

which is obviously a contradiction with $d_j(f^*, g^*) > 0$. So, we prove the uniqueness of the fixed point.

The proof for 2) and 3) is quite similar to the proof from the Theorem 4.

4) At this point, letting $g := x_n, f := f^*$ in (20), it results the rate of convergence given by (21). The proof is complete. $\square$

The contractive conditions (13) and (20) can be merged to maintain the unicity of the fixed point, stated by the next theorem.

Theorem 6 Under the assumptions of definition 8, let $T : A \rightarrow A$ be a mapping for which there exist the constants $\theta \in [0, 1)$ and some $L \geq 0$ such that for all $f, g \in A$ and $\forall j \in J$

$$d_j(Tf, Tg) \leq \theta \cdot M_{r(j)}(f, g) + L \cdot \min\{d_{r(j)}(f, Tf), d_{r(j)}(g, Tg), d_{r(j)}(f, Tg), d_{r(j)}(g, Tf)\}, \quad (22)$$

where

$$M_{r(j)}(f, g) = \max\{d_{r(j)}(f, g), d_{r(j)}(f, Tf), d_{r(j)}(g, Tg), d_{r(j)}(f, Tg), d_{r(j)}(g, Tf)\}.$$

Then

1. T has a unique fixed point, i.e., $\text{Fix}(T) = \{f^*\}$;
2. The Picard iteration $\{x_n\}_{n=0}^\infty$ given by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$ converges to f^* , for any $x_0 \in A$;

3. The *a priori* error estimate (14) holds.

Particular case

1. The famous Ćirić's fixed point theorem for single valued mappings given in [6] can be obtain from Theorems 4, 6, 5 by taking $L = L_1 = 0$ and considering r the identity mapping: $r(j) = j$. The Ćirić's contractive condition represent one of the most general metrical condition that provide a unique fixed point by means of Picard iteration. Despite this observation, the contractive condition given for Ćirić-type almost local contraction (in (13)) possess a very high level of generalisation. Note that the fixed point could be approximated by means of Picard iteration, just like in the case of Ćirić's fixed point theorem, although the uniqueness of the fixed point is not ensured by using (13).

2. If the maximum from Theorem 6 becomes:

$$\max \{d_{r(j)}(f, g), d_{r(j)}(f, Tf), d_{r(j)}(g, Tg), d_{r(j)}(f, Tg), d_{r(j)}(g, Tf)\} = d_j(f, g),$$

for all $f, g \in A$, then we can easily obtain Theorem 2 (E) from Theorem 4. Also, by Theorem 5 we obtain Theorem 2 (U) (see Zakany,[11]).

In the light of the above informations about the Ćirić-type ALC-s, it is natural to extend it to the Ćirić-type strict almost local contractions.

Definition 9 Let X be a set and let $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics defined on X . In order to underline the local character of these type of contractions, we let $A \subset X$ a subset of X . We let τ be the weak topology on X defined by the family $\mathcal{D}$. Let r be a function from J to J . The operator $T : A \rightarrow A$ is called Ćirić-type strict almost local contraction with respect $(\mathcal{D}, r)$ if it simultaneously satisfies conditions (Ci – ALC) and (ALC – U), with some real constants $\theta_c \in [0, 1)$, $L_c \geq 0$ and $\theta_u \in [0, 1)$, $L_u \geq 0$, respectively.

$$(Ci - ALC) \quad d_j(Tf, Tg) \leq \theta_c \cdot M_{r(j)}(f, g) + L_c \cdot d_{r(j)}(g, Tf), \text{ for all } f, g \in A,$$

for every $j \in J$, where

$$M_{r(j)}(f, g) = \max \{d_{r(j)}(f, g), d_{r(j)}(f, Tf), d_{r(j)}(g, Tg), d_{r(j)}(f, Tg), d_{r(j)}(g, Tf)\}.$$

$$(ALC - U) \quad d_j(Tf, Tg) \leq \theta_u \cdot d_{r(j)}(f, g) + L_u \cdot d_{r(j)}(f, Tf), \text{ for all } f, g \in A, \forall j \in J,$$

We end with a few examples that have an illustrative role. They presents Ćirić' type almost local contractions, without having unique fixed point.

Example 5 Let A be the set of positive functions $A = \{f|f : [0, \infty) \rightarrow [0, \infty)\}$, which is the subset of all real functions $X = \{f : \mathbb{R} \rightarrow \mathbb{R}\}$, $A \subset X$.

We shall use the pseudometric:

$$d_j(f, g) = |f(0) - g(0)| \cdot j, \quad \forall j \in J; J \subset \mathbb{N}, \quad \forall f, g \in A.$$

Indeed, d_j is a pseudometric, but not a metric, take for example $d_j(x^3, x^2) = 0$, but $x^3 \neq x^2$. Considering the mapping $Tf = |f|$, $\forall f \in A$, $r(j) = j + 1$. Note that the restrictive condition (19) is also verified. By using condition (5) for almost local contractions:

$$|f(0) - g(0)| \cdot j \leq \theta \cdot |f(0) - g(0)| \cdot (j + 1) + L \cdot |g(0) - f(0)| \cdot (j + 1)$$

which is equivalent to: $j \leq (\theta + L)(j + 1)$. This inequality becomes true if $j > 1$, $\theta = \frac{1}{5} \in (0, 1)$, $L = 3 > 0$, and $\frac{j}{j+1} \in (1, 2)$. Hence, T is an almost local contraction. However, T is not a contraction, because the contractive condition

$$d(Tx, Ty) \leq \theta \cdot d(x, y)$$

leads us to the false assumption: $1 \leq \theta$.

The map T is Ćirić-type almost local contraction, because

$$M_{r(j)}(f, g) = |f(0) - g(0)| \cdot (j - 1),$$

and from (13) we have the equivalent form

$$|f(0) - g(0)| \cdot j \leq \theta \cdot |f(0) - g(0)| \cdot (j - 1) + L \cdot |f(0) - f(0)| \cdot (j - 1).$$

Again, we get the inequality $j \leq (\theta + L)(j - 1)$. The mapping T has infinite number of fixed points: $\text{Fix}T = \{f \in A\} = A$, by taking:

$$|f(x)| = f(x), \quad \forall f \in A, \quad x \in [0, \infty).$$

In fact, the uniqueness condition (20) is not valid, having in view the equivalent form:

$$|f(0) - g(0)| \cdot j \leq \theta \cdot |f(0) - g(0)| \cdot (j - 1) + L_1 \cdot |f(0) - f(0)| \cdot (j - 1),$$

which leads us to the contradiction $j \leq \theta(j - 1)$, i.e. the mapping T not satisfy the uniqueness condition (20).

In fact, not even (22) is satisfied, by computing $M_{r(j)}(f, g) = |f(0) - g(0)| \cdot (j - 1)$ and $\min\{d_{r(j)}(f, Tf), d_{r(j)}(g, Tg), d_{r(j)}(f, Tg), d_{r(j)}(g, Tf)\} = |f(0) - g(0)| \cdot (j - 1)$ (since $j > 1$). By replacing these values in (22), we get

$$|f(0) - g(0)| \cdot j \leq \theta \cdot |f(0) - g(0)| \cdot (j - 1) + L \cdot |f(0) - f(0)| \cdot (j - 1),$$

which also lead to the previous contradiction.

Example 6 By taking the mapping from Example 4, with a small modification, which is: let X be the set of positive functions

$$X = \{f \mid f : [0, \infty) \rightarrow [0, \infty)\},$$

which is a subset of the real functions $\mathcal{F} = \{f : \mathbb{R} \rightarrow \mathbb{R}\}$.

Let $d_j(f, g) = |f(x_0) - g(x_0)| \cdot e^j$, $\forall f, g \in X$, $r(j) = \frac{j}{2}$, $\forall j \in \mathbb{Z}$.

We can conclude in the same manner that T is also a Ćirić type almost local contraction, i.e., it satisfy the contractive condition (13).

Indeed, we have $M_{r(j)}(f, g) = |f(x_0) - g(x_0)| \cdot e^{\frac{j}{2}}$. This way, the condition (13) became the contractive condition for almost local contractions (5).

By considering $L = 0$ in the definition 8 of Ćirić-type almost local contraction, we get a new type of ALC, that is the quasi-almost local contraction.

c) Quasi-almost local contractions

Definition 10 Under the assumptions of definition 5, the operator

$T : A \rightarrow A$ is called quasi-almost local contraction with respect $(\mathcal{D}, r)$ if, for every $j \in J$, there exist the constant $\theta \in [0, 1)$ such that

$$d_j(Tf, Tg) \leq \theta \cdot M_{r(j)}(f, g), \text{ for all } f, g \in A, \quad (23)$$

where

$$M_{r(j)}(f, g) = \max\{d_{r(j)}(f, g), d_{r(j)}(f, Tf), d_{r(j)}(g, Tg), d_{r(j)}(f, Tg), d_{r(j)}(g, Tf)\}.$$

Theorem 7 Consider a function $r : J \rightarrow J$, let a nonempty, τ - bounded, sequentially τ - complete, and T - invariant subset $A \subset X$ and let $T : A \rightarrow A$ be quasi-almost local contraction with respect to $(\mathcal{D}, r)$.

Then

1. T has a fixed point, i.e., $\text{Fix}(T) = \{x \in X : Tx = x\} \neq \emptyset$;
2. For any $x_0 = x \in A$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ converges to $x^* \in \text{Fix}(T)$;
3. The following a priori estimate is available:

$$d_j(x_n, x^*) \leq \frac{\theta^n}{(1 - \theta)^2} d_j(x, Tx), \quad n = 1, 2, \dots \quad (24)$$

Proof. Obviously, we have to follow the steps from the proof of Theorem 4, with the only difference that the constant $L = 0$, as in the case of quasi ALC-s. $\square$

The uniqueness of the fixed point is also assured by imposing an additional condition, just like in the class of Ćirić-type almost local contraction, as it follows.

Theorem 8 *With the assumptions of Theorem 4, let $T : A \rightarrow A$ be a quasi-almost local contraction with the additional inequality:*

$$d_{r(j)}(f, g) \leq d_j(f, g), \forall f, g \in A, \forall j \in J. \quad (25)$$

If the mapping T satisfies the supplementary condition: there exist the constants $\theta \in [0, 1)$ such that

$$d_j(Tf, Tg) \leq \theta \cdot d_{r(j)}(f, g) + L_1 \cdot d_{r(j)}(f, Tf), \text{ for all } f, g \in A, \forall j \in J, \quad (26)$$

then

1. T has a unique fixed point, i.e., $\text{Fix}(T) = \{f^*\}$;
2. The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$ converges to f^* , for any $x_0 \in A$;
3. The a priori error estimate (14) holds;
4. The rate of the convergence of the Picard iteration is given by

$$d_j(x_n, f^*) \leq \theta \cdot d_{r(j)}(x_{n-1}, f^*), \quad n = 1, 2, \dots, \forall j \in J \quad (27)$$

d) Ćirić-Reich-Rus type almost local contraction

Definition 11 *Under the assumptions of definition 5, the operator $T : A \rightarrow A$ is called Ćirić-Reich-Rus type almost local contraction with respect (D, r) if the mapping $T : A \rightarrow A$ satisfying the condition*

$$d_j(Tf, Tg) \leq \delta \cdot d_{r(j)}(f, g) + L \cdot [d_{r(j)}(f, Tf) + d_{r(j)}(g, Tg)], \quad (28)$$

for all f, g in A , where $\delta, L \in \mathcal{R}_+$ and $\delta + 2L < 1$

Theorem 9 *If the pseudometric d satisfy the condition:*

$d_{r(j)}(f, g) < d_j(f, g), \forall j \in J, \quad \forall f, g \in A$, *then any Ćirić- Reich- Rus type almost local contraction, i.e. any mapping $T : A \rightarrow A$ satisfying the condition (28) with $L \neq 1$ is an almost local contraction.*

Proof. Using condition (28) and the triangle rule, we get

$$\begin{aligned} d_j(Tf, Tg) &\leq \delta \cdot d_{r(j)}(f, g) + L \cdot [d_{r(j)}(f, Tf) + d_{r(j)}(g, Tg)] \\ &\leq \delta \cdot d_{r(j)}(f, g) + L \cdot [d_{r(j)}(g, Tf) \\ &\quad + d_{r(j)}(Tf, Tg) + d_{r(j)}(f, g) + d_{r(j)}(g, Tf)] \end{aligned}$$

The condition for the pseudometric leads us to:

$$\begin{aligned} d_j(f, g) &> d_{r(j)}(f, g), \\ d_j(Tf, Tg) &> d_{r(j)}(Tf, Tg), \\ d_j(g, Tf) &> d_{r(j)}(g, Tf) \end{aligned}$$

From this point, we get after simple computations:

$$(1 - L) \cdot d_j(Tf, Tg) \leq (\delta + L) \cdot d_j(f, g) + 2L \cdot d_{r(j)}(g, Tf) \quad (29)$$

and which implies

$$d_j(Tf, Tg) \leq \frac{\delta + L}{1 - L} \cdot d_j(f, g) + \frac{2L}{1 - L} \cdot d_{r(j)}(g, Tf), \forall f, g \in A \quad (30)$$

Considering $\delta, L \in \mathcal{R}_+$ and $\delta + 2L < 1$, the inequality (28) holds, with $\frac{\delta+L}{1-L} \in (0, 1)$ and $\frac{2L}{1-L} \geq 0$. Therefore, any Ćirić-Reich-Rus type almost local contraction with the condition for the pseudometric, is an almost local contraction. $\square$

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