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Multivalent $\beta-$uniformly starlike functions involving the Hurwitz-Lerch Zeta function

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Abstract. Making use of convolution product, we introduce a novel subclass of $p-$valent analytic functions with negative coefficients and obtain coefficient bounds, extreme points and radius of starlikeness for functions belonging to the generalized class $\mathrm{TP}^{k,p}_{b,\mu}(\lambda, \alpha, \beta)$. We also derive results for the modified Hadamard products of functions belonging to the class $\mathrm{TP}^{k,p}_{b,\mu}(\lambda, \alpha, \beta)$.

1 Introduction

Denote by $\mathcal{A}_p$ the class of functions f normalized by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \quad (p \in \mathbb{N} = 1, 2, 3, ...) \tag{1}$$

which are analytic and $p-$valent in the open disc $\mathbb{U} = \{z : z \in \mathbb{C}, \ |z| < 1\}$. Denote by $\mathcal{T}_p$ a subclass of $\mathcal{A}_p$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \quad (a_{p+k} \geq 0; p \in \mathbb{N} = 1, 2, 3, ..., z \in \mathbb{U}). \tag{2}$$

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For functions $f \in \mathcal{A}_p$ given by(1) and $g \in \mathcal{A}_p$ given by $g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} = (g * f)(z), \quad z \in U. \tag{3}$$

The following we recall a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (see [23])

$$\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s} \tag{4}$$

$$(a \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C}, \Re(s) > 1 \text{ and } |z| = 1)$$

where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \{\mathbb{N}\}$ $(\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, ...\}; \mathbb{N} := \{1, 2, 3, ...\})$. Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [4], Ferreira and Lopez [5], Garg et al. [7], Lin and Srivastava [10], Lin et al. [11], and others.

For the class of analytic functions denote by $\mathcal{A}$ consisting of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ Srivastava and Attiya [22] (see also Raducanu and Srivastava [17], and Prajapat and Goyal [14]) introduced and investigated the linear operator:

$$\mathcal{J}_{\mu,b} : \mathcal{A} \to \mathcal{A}$$

defined in terms of the Hadamard product (or convolution) by

$$\mathcal{J}_{\mu,b} f(z) = \mathcal{G}_{b,\mu} * f(z) \tag{5}$$

$(z \in U; b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}; f \in \mathcal{A})$, where, for convenience,

$$G_{\mu,b}(z) := (1+b)^{\mu}[\Phi(z, \mu, b) - b^{-\mu}] \quad (z \in U). \tag{6}$$

It is easy to observe from (given earlier by [14], [17]) (1), (5) and (6)that

$$\mathcal{J}_b^{\mu} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b} \right)^{\mu} a_k z^k. \tag{7}$$

Motivated essentially by the above-mentioned Srivastava-Attiya operator, we define the operator

$$\mathcal{J}_{b,\mu}^{n,p} : \mathcal{A}_p \to \mathcal{A}_p$$

which is defined as

$$\mathcal{J}_{b,\mu}^{k,p}f(z) = z^p + \sum_{k=1}^{\infty} C_b^{\mu}(k,p)a_{p+k}z^{p+k} \quad (z \in U; f(z) \in \mathcal{A}_p) \tag{8}$$

where

$$C_b^{\mu}(k,p) = \left| \left(\frac{p+b}{k+p+b} \right)^{\mu} \right| \tag{9}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as

$$b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C} \quad \text{and} \quad p, \in \mathbb{N}.$$

1. For $\mu = 1$ and $b = \nu(\nu > -1)$ generalized Libera Bernardi integral operators [16]

$$\mathcal{J}_{\nu,1}^{k,p}f(z) := \frac{p+\nu}{z^{\nu}} \int_0^z t^{\nu-1}f(t)dt := z + \sum_{k=1}^{\infty} \left(\frac{\nu+p}{k+p+\nu} \right) a_{p+k}z^{p+k} := \mathcal{L}_{\nu}^{p}f(z). \tag{10}$$

2. For $\mu = \sigma(\sigma > 0)$ and $b = 1$ Jung-Kim-Srivastava integral operator [12]

$$\mathcal{J}_{1,\sigma}^{k,p}f(z) := z + \sum_{k=1}^{\infty} \left(\frac{1+p}{k+p+1} \right)^{\sigma} a_{p+k}z^{p+k} = \mathcal{I}_{\sigma}^{p}f(z) \tag{11}$$

closely related to some multiplier transformatiom studied by Flett[6]. Making use of the operator $\mathcal{J}_{b,\mu}^{k,p}$, and motivated by earlier works of [1, 2, 3, 8, 9, 15, 13, 20, 21, 24, 25, 26], we introduced a new subclass of analytic functions with negative coefficients and discuss some some usual properties of the geometric function theory of this generalized function class.

For $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$ and $\beta \geq 0$, we let $P_{b,\mu}^{k,p}(\lambda, \alpha, \beta)$ be the subclass of $\mathcal{A}_p$ consisting of functions of the form (1) and satisfying the inequality

$$\begin{aligned} \operatorname{Re} &\left\{ \frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{J}_{b,\mu}^{k,p}f(z))' + \frac{\lambda}{p}z^2(\mathcal{J}_{b,\mu}^{k,p}f(z))''}{p(1-\lambda)\mathcal{J}_{b,\mu}^{k,p}f(z) + \lambda z(\mathcal{J}_{b,\mu}^{k,p}f(z))'} - \alpha \right\} \\ > \beta &\left| \frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{J}_{b,\mu}^{k,p}f(z))' + \frac{\lambda}{p}z^2(\mathcal{J}_{b,\mu}^{k,p}f(z))''}{p(1-\lambda)\mathcal{J}_{b,\mu}^{k,p}f(z) + \lambda z(\mathcal{J}_{b,\mu}^{k,p}f(z))'} - 1 \right| \end{aligned} \tag{12}$$

where $z \in U$, $\mathcal{J}^{k,p}_{b,\mu}f(z)$ is given by (8). We further let $TP^{k,p}_{b,\mu}(\lambda,\alpha,\beta) = P^{k,p}_{b,\mu}(\lambda,\alpha,\beta) \cap \mathcal{T}_p$.

In particular, for $0 \le \lambda \le 1$, the class $TP^{k,p}_{b,\mu}(\lambda,\alpha,\beta)$ provides a transition from k−uniformly starlike functions to k−uniformly convex functions.

Example 1 *If* $\lambda = 0$, *then*

$$\begin{aligned} TP^{k,p}_{b,\mu}(0,\alpha,\beta) \equiv TS^{k,p}_{b,\mu}(\alpha,\beta) &:= Re\left\{\frac{1}{p}\frac{z(\mathcal{J}^{k,p}_{b,\mu}f(z)')}{\mathcal{J}^{k,p}_{b,\mu}f(z)} - \alpha\right\} \\ &> \beta\left|\frac{1}{p}\frac{z(\mathcal{J}^{k,p}_{b,\mu}f(z))'}{\mathcal{J}^{k,p}_{b,\mu}f(z)} - 1\right|, \quad z \in U. \end{aligned} \tag{13}$$

Example 2 *If* $\lambda = 1$, *then*

$$\begin{aligned} TP^{k,p}_{b,\mu}(1,\alpha,\beta) \equiv UCT^{k,p}_{b,\mu}(\alpha,\beta) &:= Re\left\{\frac{1}{p}[1 + \frac{z(\mathcal{J}^{k,p}_{b,\mu}f(z))''}{(\mathcal{J}^{k,p}_{b,\mu}f(z))'}] - \alpha\right\} \\ &> \beta\left|\frac{1}{p}[1 + \frac{z(\mathcal{J}^{k,p}_{b,\mu}f(z))''}{(\mathcal{J}^{k,p}_{b,\mu}f(z))'}] - 1\right|, z \in U. \end{aligned} \tag{14}$$

Example 3 *For* $\mu = 1, b = \nu(\nu > -1)$ *and* $f(z)$*is as defined in (10) is in* $\mathcal{L}^{k,p}_{\nu}(\lambda,\alpha,\beta)$ *if*

$$\begin{aligned} &Re\left(\frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{L}^p_\nu f(z))' + \frac{\lambda}{p}z^2(\mathcal{L}^p_\nu f(z))''}{p(1-\lambda)\mathcal{L}^p_\nu f(z) + \lambda z(\mathcal{L}^p_\nu f(z))'} - \alpha\right) \\ &\quad > \beta\left|\frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{L}^p_\nu f(z))' + \frac{\lambda}{p}z^2(\mathcal{L}^p_\nu f(z))''}{p(1-\lambda)\mathcal{L}^p_\nu f(z) + \lambda z(\mathcal{L}^p_\nu f(z))'} - 1\right|, z \in U. \end{aligned} \tag{15}$$

Also, let $\mathcal{L}^p_\nu(\lambda,\alpha,\beta) \cap \mathcal{T}_p = \mathcal{T}\mathcal{L}^p_\nu(\lambda,\alpha,\beta)$.

Example 4 *For* $\mu = \sigma(\sigma > 0)$, $b = 1$ *and* $f(z)$ *is defined in (11), is in* $\mathcal{I}^p_\sigma(\lambda,\alpha,\beta)$ *if*

$$\begin{aligned} &Re\left(\frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{I}^p_\sigma f(z))' + \frac{\lambda}{p}z^2(\mathcal{I}^p_\sigma f(z))''}{p(1-\lambda)\mathcal{I}^p_\sigma f(z) + \lambda z(\mathcal{I}^p_\sigma f(z))'} - \alpha\right) \\ &\quad > \beta\left|\frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{I}^p_\sigma f(z))' + \frac{\lambda}{p}z^2(\mathcal{I}^p_\sigma f(z))''}{p(1-\lambda)\mathcal{I}^p_\sigma f(z) + \lambda z(\mathcal{I}^p_\sigma f(z))'} - 1\right|, z \in U. \end{aligned} \tag{16}$$

Also, let $\mathcal{I}^p_\sigma(\lambda,\alpha,\beta) \cap \mathcal{T}_p = \mathcal{T}\mathcal{I}^p_\sigma(\lambda,\alpha,\beta)$.

The main object of this paper is to study the coefficient bounds, extreme points and radius of starlikeness for functions belong to the generalized class $TP^{k,p}_{b,\mu}(\lambda,\alpha,\beta)$ empolying the technique of Silverman[18] and also derive results for the modified Hadamard products of functions belonging to the class $TP^{k,p}_{b,\mu}(\lambda,\alpha,\beta)$using the techniques of Schild and Silverman [19]

2 Coefficient Bounds

In this section we obtain a necessary and sufficient condition for functions $f(z)$ in the classes $P^{k,p}_{b,\mu}(\lambda,\alpha,\beta)$ and $TP^{k,p}_{b,\mu}(\lambda,\alpha,\beta)$.

Theorem 1 *A function* $f(z)$ *of the form (1) is in* $P^{k,p}_{b,\mu}(\lambda,\alpha,\beta)$ *if*

$$\sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C^{\mu}_{b}(k,p)||a_{p+k}|\leq p^2(1-\alpha), \tag{1}$$

$0\leq\lambda\leq 1,\ -1\leq\alpha<1,\ \beta\geq 0.$

Proof. It suffices to show that

$$\beta\left|\frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{J}^{k,p}_{b,\mu}f(z))'+\frac{\lambda}{p}z^2(\mathcal{J}^{k,p}_{b,\mu}f(z))''}{p(1-\lambda)\mathcal{J}^{k,p}_{b,\mu}f(z)+\lambda z(\mathcal{J}^{k,p}_{b,\mu}f(z))'}-1\right|$$
$$-\operatorname{Re}\left\{\frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{J}^{k,p}_{b,\mu}f(z))'+\frac{\lambda}{p}z^2(\mathcal{J}^{k,p}_{b,\mu}f(z))''}{p(1-\lambda)\mathcal{J}^{k,p}_{b,\mu}f(z)+\lambda z(\mathcal{J}^{k,p}_{b,\mu}f(z))'}-1\right\}\leq 1-\alpha$$

We have

$$\beta\left|\frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{J}^{k,p}_{b,\mu}f(z))'+\frac{\lambda}{p}z^2(\mathcal{J}^{k,p}_{b,\mu}f(z))''}{p(1-\lambda)\mathcal{J}^{k,p}_{b,\mu}f(z)+\lambda z(\mathcal{J}^{k,p}_{b,\mu}f(z))'}-1\right|$$
$$-\operatorname{Re}\left\{\frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{J}^{k,p}_{b,\mu}f(z))'+\frac{\lambda}{p}z^2(\mathcal{J}^{k,p}_{b,\mu}f(z))''}{p(1-\lambda)\mathcal{J}^{k,p}_{b,\mu}f(z)+\lambda z(\mathcal{J}^{k,p}_{b,\mu}f(z))'}-1\right\}$$
$$\leq(1+\beta)\left|\frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{J}^{k,p}_{b,\mu}f(z))'+\frac{\lambda}{p}z^2(\mathcal{J}^{k,p}_{b,\mu}f(z))''}{p(1-\lambda)\mathcal{J}^{k,p}_{b,\mu}f(z)+\lambda z(\mathcal{J}^{k,p}_{b,\mu}f(z))'}-1\right|$$
$$\leq\frac{(1+\beta)\sum\limits_{k=1}^{\infty}k[\frac{p+k\lambda}{p}]|C^{\mu}_{b}(k,p)||a_{p+k}|}{p-\sum\limits_{k=1}^{\infty}[p+k\lambda]|C^{\mu}_{b}(k,p)||a_{p+k}|}.$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]\ |C_b^{\mu}(k,p)|\ |a_{p+k}|\leq p^2(1-\alpha)$$

and hence the proof is complete. □

Theorem 2 *A necessary and sufficient condition for* $f(z)$ *of the form (2) to be in the class* $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$, $-1\leq\alpha<1, 0\leq\lambda\leq 1, \beta\geq 0$ *is that*

$$\sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]\ |C_b^{\mu}(k,p)|a_{p+k}\leq p^2(1-\alpha), \tag{2}$$

Proof. In view of Theorem 1, we need only to prove the necessity. If $f\in P_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$ and z is real then

$$\frac{1-\sum\limits_{k=1}^{\infty}(\frac{p+k}{p})[\frac{p+k\lambda}{p}]|C_b^{\mu}(k,p)|a_{p+k}|z|^k}{1-\sum\limits_{k=1}^{\infty}[\frac{p+k\lambda}{p}]|C_b^{\mu}(k,p)|a_{p+k}|z|^k}-\alpha\geq\beta\frac{\sum\limits_{k=1}^{\infty}k[\frac{p+k\lambda}{p}]|C_b^{\mu}(k,p)|a_{p+k}|z|^k}{1-\sum\limits_{k=1}^{\infty}[\frac{p+k\lambda}{p}]|C_b^{\mu}(k,p)a_{p+k}|z|^k}$$

Letting $z\to 1$ along the real axis, we obtain the desired inequality

$$\sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|\ a_{p+k}\leq p^2(1-\alpha).$$

□

In view of the Examples 1 to 4 in Section 1 and Theorem 2, we have following corollaries for the classes defined in these examples.

Corollary 1 *A necessary and sufficient condition for* $f(z)$ *of the form (2) to be in the class* $TS_{b,\mu}^{k,p}(\alpha,\beta)$, $0\leq\alpha<1, \beta\geq 0$ *is that*

$$\sum_{k=1}^{\infty}[k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|\ a_{p+k}\leq p(1-\alpha),$$

Corollary 2 *A necessary and sufficient condition for* $f(z)$ *of the form (2) to be in the class* $UCT_{b,\mu}^{k,p}(\alpha,\beta)$, $0\leq\alpha<1, \beta\geq 0$ *is that*

$$\sum_{k=1}^{\infty}(p+k)[k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|\ a_{p+k}\leq p^2(1-\alpha),$$

Corollary 3 *A necessary and sufficient condition for* $f(z)$ *of the form (2) to be in the class* $T\mathcal{L}_{\nu}^{k,p}(\lambda,\alpha,\beta)$, $0 \le \alpha < 1, \beta \ge 0$ *is that*

$$\sum_{k=1}^{\infty}(p+k\lambda)[k(1+\beta)+p(1-\alpha)]\left(\frac{p+\nu}{k+p+\nu}\right)a_{p+k} \le p^2(1-\alpha).$$

Corollary 4 *A necessary and sufficient condition for* $f(z)$ *of the form (2) to be in the class* $\mathcal{TI}_{\sigma}^{p}(\lambda,\alpha,\beta)$, $0 \le \alpha < 1, \beta \ge 0$ *is that*

$$\sum_{k=1}^{\infty}(p+k\lambda)[k(1+\beta)+p(1-\alpha)]\left(\frac{1+p}{k+p+1}\right)^{\sigma} a_{p+k} \le p^2(1-\alpha).$$

Corollary 5 *If* $f \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$, *then*

$$a_{p+k} \le \frac{p^2(1-\alpha)}{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}, \quad k \ge 1, \tag{3}$$

where $0 \le \lambda \le 1$, $-1 \le \alpha < 1$ *and* $\beta \ge 0$. *Equality in* (3) *holds for the function*

$$f(z) = z - \frac{p^2(1-\alpha)}{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}z^{p+k} \qquad (p \in N). \tag{4}$$

It is of interest to note that ,when $p = 1$ and $k = n-1$, the above results reduces to the results studied in [2, 8, 9, 20, 21] Similarly many known results can be obtained as particular cases of the following theorems, so we omit stating the particular cases for the following theorems.

3 Closure Properties

Theorem 1 *Let*

$$\begin{aligned} f_p(z) &= z^p \qquad (p \in N) \quad \text{and} \\ f_{p+k}(z) &= z^p - \frac{p^2(1-\alpha)}{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}z^{p+k}. \end{aligned} \tag{1}$$

Then $f \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$, *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=0}^{\infty}\omega_{p+k}f_{p+k}(z), \qquad \omega_{p+k} \ge 0, \quad \sum_{k=0}^{\infty}\omega_{p+k} = 1. \tag{2}$$

Proof. Let us suppose that $f(z)$ is given by (2),that is by

$$f(z) = z^p - \sum_{k=1}^{\infty} \frac{p^2(1-\alpha)}{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}\omega_{p+k}z^{p+k}.$$

Then, since

$$\sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|\; p^2(1-\alpha)}{p^2(1-\alpha)[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}\omega_{p+k}$$
$$= \sum_{k=1}^{\infty} \omega_{p+k} = 1-\omega_p \leq 1.$$

Thus $f \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. Conversely, let us have $f \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. Then by using (3), we set

$$\omega_{p+k} = \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(1-\alpha)}a_{p+k}, \quad (k \in N)$$

and $\omega_p = 1 - \sum_{k=1}^{\infty} \omega_{p+k}$,we can readily see that $f(z)$ can be expressed precisely as in (1).This evidently completes the proof of Theorem 1. □

Theorem 2 *The class* $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$ *is a convex set.*

Proof. Let the function

$$f_j(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,j}z^{p+k}, \quad (a_{p+k,j} \geq 0, p \in N; \quad j = 1,2...) \tag{3}$$

be in the class $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. It sufficient to show that the function $h(z)$ defined by

$$h(z) = \eta f_1(z) + (1-\eta)f_2(z), \quad 0 \leq \eta \leq 1,$$

is in the class $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. Since

$$h(z) = z^p - \sum_{k=1}^{\infty} [\eta a_{p+k,1} + (1-\eta)a_{p+k,2}]z^{p+k},$$

an easy computation with the aid of Theorem 2 gives,

$$\sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]\eta|C_b^{\mu}(k,p)|a_{p+k,1}$$
$$+\sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)](1-\eta)|C_b^{\mu}(k,p)|a_{p+k,2}$$
$$\leq p^2\eta(1-\alpha)+p^2(1-\eta)(1-\alpha)$$
$$\leq p^2(1-\alpha),$$

which implies that $h \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. Hence $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$ is convex. □

Now we provide the radii of $p-valently$ close-to-convexity, starlikeness and convexity for the class $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$.

Theorem 3 *Let the function* $f(z)$ *defined by (2) be in the class* $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. *Then* $f(z)$ *is p-valently close-to-convex of order* δ $(0 \leq \delta < p)$ *in the disc* $|z| < r_1$, *where*

$$r_1 := \inf_{k\in\mathbb{N}}\left[\frac{(1-\delta)[k(1+\beta)+p(1-\alpha)][p+k\lambda]|C_b^{\mu}(k,p)|}{p^2(p+k)(1-\alpha)}\right]^{\frac{1}{k}}. \tag{4}$$

The result is sharp, with extremal function $f(z)$ *given by (1).*

Proof. Given $f \in \mathcal{T}_p$, and f is close-to-convex of order δ, we have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| < p-\delta. \tag{5}$$

For the left hand side of (5) we have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \leq \sum_{k=1}^{\infty}(p+k)a_{p+k}|z|^k.$$

The last expression is less than $p-\delta$ if

$$\sum_{k=1}^{\infty}\frac{p+k}{p-\delta}a_{p+k}|z|^k < 1.$$

Using the fact, that $f \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$ if and only if

$$\sum_{k=1}^{\infty}\frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(1-\alpha)}a_n \leq 1,$$

We can say (5) is true if

$$\frac{p+k}{p-\delta}|z|^k \leq \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(1-\alpha)}a_n$$

Or, equivalently,

$$|z|^k = \left[\frac{(p-\delta)[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(p+k)(1-\alpha)}\right],$$

the last inequality leads us immediately to the disc $|z| < r_1$, where r_1 given by (4)and the proof of Theorem 3 is completed. □

Theorem 4 *If* $f \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$, *then*

(i) f *is p-valently starlike of order* $\delta(0 \leq \delta < p)$ *in the disc* $|z| < r_2$; *that is, Re* $\left\{\frac{zf'(z)}{f(z)}\right\} > \delta$, *where*

$$r_2 = \inf_{k\in N}\left[\left(\frac{p-\delta}{p+k-\delta}\right)\frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(1-\alpha)}\right]^{\frac{1}{k}}. \quad (6)$$

(ii) f *is convex of order* δ $(0 \leq \delta < p)$ *in the unit disc* $|z| < r_3$, *that is Re* $\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta$, *where*

$$r_3 = \inf_{k\in N}\left[\left(\frac{p-\delta}{(k+p)(p+k-\delta)}\right)\frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(1-\alpha)}\right]^{\frac{1}{k}}. \quad (7)$$

Each of these results are sharp for the extremal function $f(z)$ *given by (1).*

Proof.(i) Given $f \in \mathcal{T}_p$, and f is starlike of order δ, we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p-\delta. \quad (8)$$

For the left hand side of (8) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{k=1}^{\infty} ka_{p+k}\,|z|^k}{1-\sum\limits_{k=1}^{\infty} a_{p+k}\,|z|^k}.$$

The last expression is less than $p-\delta$ if

$$\sum_{k=1}^{\infty} \frac{k+p-\delta}{p-\delta} a_{p+k}\, |z|^k < 1.$$

Using the fact, that $f \in TP^{k,p}_{b,\mu}(\lambda,\alpha,\beta)$ if and only if

$$\sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]}{p^2(1-\alpha)} a_{p+k}|C^{\mu}_{b}(k,p)| \leq 1.$$

We can say (8) is true if

$$\frac{p+k-\delta}{p-\delta}|z|^k < \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C^{\mu}_{b}(k,p)|}{p^2(1-\alpha)}$$

Or, equivalently,

$$|z|^k = \left[\left(\frac{p-\delta}{p+k-\delta}\right)\frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C^{\mu}_{b}(k,p)|}{p^2(1-\alpha)}\right]$$

which yields the starlikeness of the family.

(ii) Using the fact that f is convex if and only if zf' is starlike, we can prove (ii), on lines similar to the proof of (i). □

4 Convolution Results

Let the functions

$$f_j(z) = z^p + \sum_{k=1}^{\infty} a_{j,p+k} z^{p+k}, \quad (p \in N = 1,2,3,...)(j=1,2) \tag{9}$$

then the modified Hadamard product of $f_1(z)$ and $f_2(z)$ is given by

$$(f_1 * f_2)(z) \ = \ z^p - \sum_{n=2}^{\infty} a_{1,p+k} a_{2,p+k}\, z^{p+k} = (f_2 * f_1)(z), (a_{1,p+k} \geq 0; a_{2,p+k} \geq 0).$$

Using the techniques of we prove the following results.

Theorem 5 *For functions* $f_j(z)(j = 1, 2)$ *defined by (9), be in the class* $TP^{k,p}_{b,\mu}(\lambda, \alpha, \beta)$*. Then* $(f_1 * f_2) \in TP^{k,p}_{b,\mu}(\lambda, \xi, \beta)$ *where*

$$\xi == 1 - \frac{p^2(1-\alpha)^2(1+\beta)}{[p+\lambda][(1+\beta)+p(1-\alpha)]^2|C^{\mu}_{b}(1,p)| - p^3(1-\alpha)^2} \tag{10}$$

where $C^{\mu}_{b}(1,p)$ *is given by (9).*

Proof. Employing the technique used earlier by Schild and Silverman[19], we need to find the largest ξ such that

$$\sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\xi)]|C^{\mu}_{b}(k,p)|}{p^2(1-\xi)} a_{1,p+k} a_{2,p+k} \le 1, \quad (0 \le \xi < 1)$$

for $f_j \in TP^{k,p}_{b,\mu}(\lambda, \alpha, \beta)(j = 1, 2)$ where ξ is defined by (10). On the other hand, under the hypothesis, it follows from (1) and the Cauchy's-Schwarz inequality that

$$\sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C^{\mu}_{b}(k,p)|}{p^2(1-\alpha)} \sqrt{a_{1,p+k} a_{2,p+k}} \le 1. \tag{11}$$

Thus we need to find the largest ξ such that

$$\sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\xi)]|C^{\mu}_{b}(k,p)|}{p^2(1-\xi)} a_{1,p+k}\, a_{2,p+k}$$
$$\le \sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C^{\mu}_{b}(k,p)|}{p^2(1-\alpha)} \sqrt{a_{1,p+k} a_{2,p+k}}$$

or, equivalently that

$$\sqrt{a_{1,p+k} a_{2,p+k}} \le \frac{(1-\xi)}{(1-\alpha)} \frac{[k(1+\beta)+p(1-\alpha)]}{[k(1+\beta)+p(1-\xi)]}, \quad (k \ge 1).$$

Hence by making use of the inequality (11), it is sufficient to prove that

$$\frac{p^2(1-\alpha)}{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C^{\mu}_{b}(k,p)|} \le \frac{(1-\xi)}{(1-\alpha)} \frac{[k(1+\beta)+p(1-\alpha)]}{[k(1+\beta)+p(1-\xi)]}$$

which yields

$$\xi = \Psi(k) = 1 - \frac{kp^2(1-\alpha)^2(1+\beta)}{[p+k\lambda][k(1+\beta)+p(1-\alpha)]^2|C^{\mu}_{b}(k,p)| - p^3(1-\alpha)^2} \tag{12}$$

for $k \geq 1$ is an increasing function of k and letting $k = 1$ in (12), we have

$$\xi = \Psi(1) = 1 - \frac{p^2(1-\alpha)^2(1+\beta)}{[p+\lambda][(1+\beta)+p(1-\alpha)]^2|C_b^\mu(1,p)| - p^3(1-\alpha)^2}$$

where $C_b^\mu(1,p)$ is given by (9). □

Theorem 6 *Let the function* $f(z)$ *defined by (2) be in the class* $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. *Also let* $g(z) = z^p - \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$ *for* $|b_{p+k}| \leq 1$. *Then* $(f * g) \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$.

Proof. Since

$$\begin{aligned}
&\sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^\mu(k,p)|\ |a_{p+k}b_{p+k}| \\
&\leq \sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^\mu(k,p)|a_{p+k}|b_{p+k}| \\
&\leq \sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^\mu(k,p)|a_{p+k} \\
&\leq p^2(1-\alpha)
\end{aligned}$$

it follows that $(f * g) \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$, by the view of Theorem 2. □

Theorem 7 *Let the functions* $f_j(z)(j = 1,2)$ *defined by (9) be in the class* $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. *Then the function* $h(z)$ *defined by*

$$h(z) = z^p - \sum_{n=2}^{\infty}(a_{1,p+k}^2 + a_{2,p+k}^2)z^{p+k}$$

is in the class $TP_{b,\mu}^{k,p}(\lambda,\xi,\beta)$, *where*

$$\xi = 1 - \frac{2p^2(1-\alpha)^2\ (1+\beta)}{[p+\lambda][(1+\beta)+p(1-\alpha)]^2|C_b^\mu(1,p)| - 2p^3(1-\alpha)^2}$$

where $C_b^\mu(1,p)$ *is given by (9).*

Proof. By virtue of Theorem 2, it is sufficient to prove that

$$\sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\xi)]|C_b^{\mu}(k,p)|}{p^2(1-\xi)}[a_{1,p+k}^2 + a_{2,p+k}^2] \leq 1 \tag{13}$$

where $f_j \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$ we find from (9) and Theorem 2, that

$$\begin{aligned} &\sum_{k=1}^{\infty} \left[\frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(1-\alpha)}\right]^2 a_{j,p+k}^2 \\ &\leq \left[\sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(1-\alpha)} a_{j,p+k}\right]^2 \qquad (14) \\ &\leq 1, (j=1,2) \qquad (15) \end{aligned}$$

which would readily yield

$$\sum_{k=1}^{\infty} \frac{1}{2}\left[\frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(1-\alpha)}\right]^2 (a_{1,p+k}^2 + a_{2,p+k}^2) \leq 1. \tag{16}$$

By comparing (14) and (16), it is easily seen that the inequality (13) will be satisfied if

$$\begin{aligned} &\frac{[p+k\lambda][k(1+\beta)+p(1-\xi)]|C_b^{\mu}(k,p)|}{p^2(1-\xi)} \\ \leq\ &\frac{1}{2}\left[\frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(1-\alpha)}\right]^2, \quad \text{for } k \geq 1. \end{aligned}$$

That is if

$$\xi = \Psi(k) = 1 - \frac{2p^2(1-\alpha)^2\ k(1+\beta)}{[p+k\lambda][k(1+\beta)+p(1-\alpha)]^2|C_b^{\mu}(k,p)| - 2p^3(1-\alpha)^2} \tag{17}$$

Since$\Psi(k)$ is an increasing function of k $(k \geq 1)$. Taking $k = 1$ in (17), we have,

$$\xi = \Psi(1) = 1 - \frac{2p^2(1-\alpha)^2\ (1+\beta)}{[p+\lambda][(1+\beta)+p(1-\alpha)]^2|C_b^{\mu}(1,p)| - 2p^3(1-\alpha)^2}$$

which completes the proof. □

Concluding Remarks: In fact, by appropriately selecting the arbitrary sequences given in (10) and (11), suitably specializing the values of μ, α, β and p the results presented in this paper would find further applications for the class of p-valent functions stated in Examples 1 to 4 in Section 1.

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