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Multiple radially symmetric solutions for a quasilinear eigenvalue problem

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Abstract. In this paper we study an eigenvalue problem in $\mathbb{R}^N$, which involves the p-Laplacian $(1 < p < N)$, and the nonlinear term has a global $(p-1)$-sublinear growth. We guarantee an open interval of eigenvalues, for which the eigenvalue problem has three distinct radially symmetric solutions in a weighted Sobolev space. We use a compact embedding result of Su, Wang and Willem ([6]) and a Ricceri-type three critical points theorem of Bonanno ([1]).

1 Main result

Let $V, Q : (0,\infty) \to (0,\infty)$ be two continuous functions satisfying the following hypotheses

(V) there exist real numbers a and a_0 such that

$$\liminf_{r\to\infty} \frac{V(r)}{r^a} > 0, \liminf_{r\to 0} \frac{V(r)}{r^{a_0}} > 0.$$

(Q) there exist real numbers b and b_0 such that

$$\liminf_{r\to\infty} \frac{Q(r)}{r^b} < \infty, \liminf_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty.$$

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Let $C_0^\infty(\mathbb{R}^N)$ denote the collection of smooth functions with compact support and

$$C_{0,r}^\infty(\mathbb{R}^N) = \{u \in C_0^\infty(\mathbb{R}^N) | u \text{ is radially symmetric}\}.$$

We recall that a function $u \in C_0^\infty(\mathbb{R}^N)$ is radially symmetric, if $u(|x|) = u(x)$, for any $x \in \mathbb{R}^N$.
Let $D_r^{1,p}(\mathbb{R}^N)$ be the completion of $C_{0,r}^\infty(\mathbb{R}^N)$ under

$$||u||^p = \int_{\mathbb{R}^N} |\nabla u|^p dx.$$

Define the Lebesgue spaces for $p \geq 1$ and $q \geq 1$:

$$L^p(\mathbb{R}^N; V) = \{u : \mathbb{R}^N \to \mathbb{R} | u \text{ is measurable}, \int_{\mathbb{R}^N} V(|x|)|u|^p dx < \infty\}$$

$$L^q(\mathbb{R}^N; Q) = \{u : \mathbb{R}^N \to \mathbb{R} | u \text{ is measurable}, \int_{\mathbb{R}^N} Q(|x|)|u|^q dx < \infty\}$$

with the corresponding norms

$$||u||_{L^p(\mathbb{R}^N;V)} = \left(\int_{\mathbb{R}^N} V(|x|)|u|^p dx\right)^{1/p},$$

$$||u||_{L^q(\mathbb{R}^N;Q)} = \left(\int_{\mathbb{R}^N} Q(|x|)|u|^q dx\right)^{1/q}.$$

For these norms, we use the abbreviations: $||u||_{L^p(\mathbb{R}^N;V)} = ||u||_{p,V}$ and $||u||_{L^q(\mathbb{R}^N;Q)} = ||u||_{q,Q}$.
Then define $W_r^{1,p}(\mathbb{R}^N; V) = D_r^{1,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N; V)$, which is a Banach space under

$$||u||_W^p = \int_{\mathbb{R}^N} (|\nabla u|^p + V(|x|)|u|^p)\, dx.$$

In order to state the embedding theorem used in our proofs, we need to introduce the following notations:

$$q_* = \begin{cases} \dfrac{p^2(N-1+b) - ap}{p(N-1) + a(p-1)}, & b \geq a > -p, \\ \dfrac{p(N+b)}{N-p}, & b \geq -p \geq a, \\ p, & b \leq \max\{a, -p\} \end{cases}$$

$$q^* = \begin{cases} \dfrac{p(N+b_0)}{N-p}, & b_0 \geq -p, a_0 \geq -p, \\ \dfrac{p^2(N-1+b_0)-a_0 p}{p(N-1)+a_0(p-1)}, & -p \geq a_0 > -\frac{N-1}{p-1}p, b_0 \geq a_0, \\ \infty, & a_0 \leq -\frac{N-1}{p-1}p, b_0 \geq a_0. \end{cases}$$

We shall use the following embedding theorem.

Theorem 1 [6, Theorem 1.] *Let* $1 < p < N$. *Assume* (V) *and* (Q). *Then we have the embedding*

$$W_r^{1,p}(\mathbb{R}^N; V) \hookrightarrow L^q(\mathbb{R}^N; Q) \tag{1}$$

for $q_* \leq q \leq q^*$, *when* $q^* < \infty$ *and for* $q_* \leq q < \infty$, *when* $q^* = \infty$.

Furthermore, the embedding is compact for $q_* < q < q^*$. *And if* $b < \max\{a, -p\}$ *and* $b_0 > \min\{-p, a_0\}$, *the embedding is also compact for* $q = p$.

Therefore, supposing besides (V) and (Q) the condition

(ab) $\qquad b < \max\{a, -p\}$ and $b_0 > \min\{-p, a_0\}$,

the embedding

$$W_r^{1,p}(\mathbb{R}^N; V) \hookrightarrow L^p(\mathbb{R}^N; Q) \tag{2}$$

is also compact.
The collection of those functions, which satisfy the conditions $(V), (Q)$ and (ab) is not empty. For example, taking

$$a = p, b = -p-1, a_0 = -p, b_0 = -p+1,$$

the functions V and Q defined by

$$V(r) = \max\left\{1, \frac{1}{r^p}\right\},$$

$$Q(x) = \min\left\{\frac{1}{r^{p+1}}, \frac{1}{r^{p-1}}\right\}$$

satisfy all three assumptions for every $1 < p < N$.

For $\lambda > 0$, we consider the following problem:

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u + V(|x|)|u|^{p-2}u = \lambda Q(|x|)f(u) \text{ in } \mathbb{R}^N \\ |u(x)| \to 0, \text{as } |x| \to \infty \end{cases},$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

We say that $u \in W_r^{1,p}(\mathbb{R}^N; V)$ is a *weak radial solution* of the problem (P_λ) if

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla v + V(|x|)|u|^{p-2}uv)dx - \lambda \int_{\mathbb{R}^N} Q(|x|)f(u(x))v(x)dx = 0,$$

for every $v \in W_r^{1,p}(\mathbb{R}^N; V)$.

We assume the following conditions on f :

(f_1) there exists $C > 0$ such that $|f(s)| \leq C(1 + |s|^{p-1})$, for every $s \in \mathbb{R}$;

(f_2) $\lim\limits_{s\to 0} \dfrac{f(s)}{|s|^{p-1}} = 0$;

(f_3) there exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$, where $F(s) = \int_0^s f(t)dt$.

Our main result is the following

Theorem 2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies $(f_1), (f_2)$, (f_3), and assume that (V), (Q) and (ab) are verified. Then, there exists an open interval $\Lambda \subset (0, \infty)$ and a constant $\mu > 0$ such that for every $\lambda \in \Lambda$ problem (P_λ) there are at least three distinct weak radial solutions in $W_r^{1,p}(\mathbb{R}^N; V)$, whose $W_r^{1,p}(\mathbb{R}^N; V)$–norms are less than μ.*

2 Auxiliary results

In this section we give a few preliminary results. These will be used in the proof of the main result in the next section.

We denote the best embedding constant of the embedding (1) by C_q, i.e. we have the inequality:

$$||u||_{q,Q} \leq C_q||u||_W.$$

We define the energy functional corresponding to (P_λ) as

$$\mathcal{E}_\lambda : W_r^{1,p}(\mathbb{R}^N; V) \to \mathbb{R}$$

$$\mathcal{E}_\lambda(u) = \frac{1}{p}||u||_W^p - \lambda J(u),$$

where $J : W_r^{1,p}(\mathbb{R}^N; V) \to \mathbb{R}$ is the functional defined by

$$J(u) = \int_{\mathbb{R}^N} Q(|x|)F(u(x))dx.$$

The functional $\mathcal{E}_\lambda$ is of class C^1 (see for instance [3, Lemma 4]), and its derivative is given by

$$\langle \mathcal{E}_\lambda'(u), v\rangle = \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla v + V(|x|)|u|^{p-2}uv)dx - \lambda \int_{\mathbb{R}^N} Q(|x|)f(u(x))v(x)dx,$$

for every $v \in W_r^{1,p}(\mathbb{R}^N; V)$. Therefore, the critical points of the energy functional are exactly the weak radial solutions of the problem (P_λ).

Lemma 1 *For every* $\lambda > 0$*, the functional* $\mathcal{E}_\lambda : W_r^{1,p}(\mathbb{R}^N; V) \to \mathbb{R}$ *is sequentially weakly lower semicontinuous.*

Proof. Due to (f_2), for arbitrary small $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(s)| \leq \varepsilon p|s|^{p-1}, \text{ for every } s \leq |\delta|. \tag{3}$$

Combining this inequality with condition (f_1), we obtain

$$|f(s)| \leq \varepsilon p|s|^{p-1} + K(\delta)r|s|^{r-1}, \text{ for every } s \in \mathbb{R}, \tag{4}$$

where $r \in]q_*, q^*[$ is fixed and $K(\delta) > 0$ does not depend on s.
Let $\{u_n\}$ be a sequence from $W_r^{1,p}(\mathbb{R}^N; V)$, which is weakly convergent to some $u \in W_r^{1,p}(\mathbb{R}^N; V)$. Then there exists a positive constant $M > 0$ such that

$$||u_n||_W \leq M, ||u_n - u||_W \leq M, \forall n \in \mathbb{N}. \tag{5}$$

We claim that $|J(u_n) - J(u)| \to 0$ as $n \to \infty$. Using inequality (4), the standard mean value theorem for F and the Hölder's inequality, we obtain:

$$\begin{aligned}
|J(u_n) - J(u)| &\leq \int_{\mathbb{R}^N} Q(|x|)|F(u_n(x)) - F(u(x))|dx \leq \\
&\leq \int_{\mathbb{R}^N} Q(|x|)|f(\theta u_n(x) - (1-\theta)u(x))||u_n(x) - u(x)|dx \leq \\
&\leq \varepsilon p \int_{\mathbb{R}^N} Q(|x|)|\theta u_n(x) - (1-\theta)u(x)|^{p-1}|u_n(x) - u(x)|dx + \\
&+ K(\delta)r \int_{\mathbb{R}^N} Q(|x|)|\theta u_n(x) - (1-\theta)u(x)|^{r-1}|u_n(x) - u(x)|dx \leq \\
&\leq \varepsilon p \int_{\mathbb{R}^N} Q(|x|)(|u_n(x)|^{p-1} + |u(x)|^{p-1})|u_n(x) - u(x)|dx + \\
&+ K(\delta)r \int_{\mathbb{R}^N} Q(|x|)|(|u_n(x)|^{r-1} + |u(x)|^{r-1})|u_n(x) - u(x)|dx \leq \\
&\leq \varepsilon p(||u_n||_{p,Q}^{p-1} + ||u||_{p,Q}^{p-1})||u_n - u||_{p,Q} + \\
&+ K(\delta)r(||u_n||_{r,Q}^{r-1} + ||u||_{r,Q}^{r-1})||u_n - u||_{r,Q}.
\end{aligned}$$

Now, using the embeddings (1), (2) and the inequalities (5) we have

$$\begin{aligned} |J(u_n)-J(u)| &\leq \varepsilon p C_p^p(\|u_n\|_W^{p-1}+\|u\|_W^{p-1})\|u_n-u\|_W+ \\ &+ K(\delta) r C_r^{r-1}(\|u_n\|_W^{r-1}+\|u\|_W^{r-1})\|u_n-u\|_{r,Q} \leq \\ &\leq 2\varepsilon p C_p^p M^p + 2K(\delta) r C_r^{r-1} M^{r-1}\|u_n-u\|_{r,Q}. \end{aligned}$$

Since the embedding $W_r^{1,p}(\mathbb{R}^N;V) \hookrightarrow L^r(\mathbb{R}^N;Q)$ is compact for $r \in]q_*, q^*[$, we have that $\|u_n-u\|_{r,Q} \to 0$, whenever $n \to \infty$. Besides that, ε is chosen arbitrarily, so the claim follows from the last inequality. ■

Lemma 2 *For every* $\lambda > 0$*, the functional* $\mathcal{E}_\lambda : W_r^{1,p}(\mathbb{R}^N;V) \to \mathbb{R}$ *is coercive.*

Proof. Let η be a constant such that

$$0 < \eta < \frac{1}{pC_p^p}, \tag{6}$$

where C_p is the best embedding constant of the embedding $W_r^{1,p}(\mathbb{R}^N;V) \hookrightarrow L^p(\mathbb{R}^N;V)$. Due to conditions (f_1) and (f_2), there is a function $k \in L^1(\mathbb{R}^N;Q)$ such that

$$|F(s)| \leq \eta |s|^p + k(x), \forall s \in \mathbb{R}, \forall x \in \mathbb{R}^N. \tag{7}$$

Then, we obtain

$$\begin{aligned} \mathcal{E}_\lambda(u) &\geq \frac{1}{p}\|u\|_W^p - \eta\int_{\mathbb{R}^N} Q(|x|)|u(x)|^p dx - \int_{\mathbb{R}^N} Q(|x|)k(x)dx \geq \\ &\geq \frac{1}{p}\|u\|_W^p - \eta C_p^p\|u\|_W^p - \|k\|_{1,Q} = \\ &= \left(\frac{1}{p} - \eta C_p^p\right)\|u\|_W^p - \|k\|_{1,Q} \end{aligned}$$

By the choice of the function k, we have that $\|k\|_{1,Q}$ is bounded. Therefore, using the inequality (6), we obtain that $\mathcal{E}_\lambda(u) \to \infty$, as $\|u\|_W \to \infty$, concluding the proof. ■

Lemma 3 *For every* $\lambda > 0$*, the functional* $\mathcal{E}_\lambda : W_r^{1,p}(\mathbb{R}^N;V) \to \mathbb{R}$ *satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^N;V)$ be a (PS)-sequence for the function $\mathcal{E}_\lambda$, i.e.

(1) $\{\mathcal{E}_\lambda(u_n)\}$ is bounded;

(2) $\mathcal{E}'_\lambda(u_n) \to 0$.
Since $\mathcal{E}_\lambda$ is coercive, we have that $\{u_n\}$ is bounded. The reflexivity of the Banach space $W_r^{1,p}(\mathbb{R}^N;V)$ implies the existence of a subsequence (notated also by $\{u_n\}$), such that $\{u_n\}$ is weakly convergent to an element $u \in W_r^{1,p}(\mathbb{R}^N;V)$. Therefore, we have

$$\langle \mathcal{E}'_\lambda(u), u_n - u\rangle \to 0 \text{ as } n \to \infty. \tag{8}$$

Because the inclusion $W_r^{1,p}(\mathbb{R}^N;V) \hookrightarrow L^p(\mathbb{R}^N;V)$ is compact, we have that $u_n \to u$ strongly in $L^p(\mathbb{R}^N;V)$. We would like to prove that u_n converges strongly to u in $W_r^{1,p}(\mathbb{R}^N;V)$. For this, we will use the following estimates from [2, Lemma 4.10]

$$|\xi - \zeta|^p \leq M_1(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta), \quad \text{for } p \geq 2 \tag{9}$$

$$|\xi - \zeta|^2 \leq M_2(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta)(|\xi| + |\zeta|)^{2-p}, \quad \text{for } p \in]1,2[, \tag{10}$$

where M_1 and M_2 are some positive constants. We separate two cases. In the first case let $p \geq 2$. Then we have:

$$\begin{aligned}
||u_n - u||_W^p &= \int_{\mathbb{R}^N} |\nabla u_n(x) - \nabla u(x)|^p dx + \int_{\mathbb{R}^N} V(|x|)|u_n(x) - u(x)|^p dx \\
&\leq M_1 \int_{\mathbb{R}^N} \left[|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x)\right](\nabla u_n(x) - \nabla u(x))dx \\
&+ M_1 \int_{\mathbb{R}^N} V(|x|)\left[|u_n(x)|^{p-2}u_n(x) - |u(x)|^{p-2}u(x)\right](u_n(x) - u(x))dx \\
&= M_1(\langle \mathcal{E}'_\lambda(u_n), u_n - u\rangle - \langle \mathcal{E}'_\lambda(u), u_n - u\rangle + \lambda\langle J'(u_n) - J'(u), u_n - u\rangle) \\
&\leq M_1\left(||\mathcal{E}'_\lambda(u_n)||_{W_r^{1,p}(\mathbb{R}^N;V)^*} + \lambda||J'(u_n) - J'(u)||_{W_r^{1,p}(\mathbb{R}^N;V)^*}\right)||u_n - u||_W \\
&- M_1\langle \mathcal{E}'_\lambda(u), u_n - u\rangle.
\end{aligned}$$

Since $u_n \to u$ weakly in $W_r^{1,p}(\mathbb{R}^N;V)$ and J' are compact (see [3, Lemma 4]), we have that $||J'(u_n) - J'(u)||_{W_r^{1,p}(\mathbb{R}^N;V)^*} \to 0$. Moreover $||\mathcal{E}'_\lambda(u_n)|| \to 0$, hence using (8), we have that $||u_n - u||_W \to 0$, as $n \to \infty$.
In the second case, when $1 < p < 2$, we recall the following result: for all $s \in (0,\infty)$ there is a constant $c_s > 0$ such that

$$(x + y)^s \leq c_s(x^s + y^s), \quad \text{for any } x, y \in (0,\infty). \tag{11}$$

Then we obtain

$$||u_n - u||_W^2 = \left(\int_{\mathbb{R}^N} |\nabla u_n(x) - \nabla u(x)|^p dx + \int_{\mathbb{R}^N} V(|x|)|u_n(x) - u(x)|^p dx\right)^{\frac{2}{p}} \tag{12}$$

$$\leq c_p\left[\left(\int_{\mathbb{R}^N}|\nabla u_n(x)-\nabla u(x)|^p dx\right)^{\frac{2}{p}}+\left(\int_{\mathbb{R}^N}V(|x|)|u_n(x)-u(x)|^p dx\right)^{\frac{2}{p}}\right].$$

Now, using (10) and the Hölder inequalities, we get:

$$\begin{aligned}
&\int_{\mathbb{R}^N}|\nabla u_n(x)-\nabla u(x)|^p dx=\int_{\mathbb{R}^N}(|\nabla u_n(x)-\nabla u(x)|^2)^{\frac{p}{2}}dx\leq\\
\leq\ & M_2\cdot\int_{\mathbb{R}^N}\Big((|\nabla u_n(x)|^{p-2}\nabla u_n(x)-|\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x)-\nabla u(x))\Big)^{\frac{p}{2}}\cdot\\
&\cdot(|\nabla u_n(x)|+|\nabla u(x)|)^{\frac{p(2-p)}{2}}dx=\\
=\ & M_2\cdot\int_{\Omega}\Big[(|\nabla u_n(x)|^{p-2}\nabla u_n(x)-|\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x)-\nabla u(x))\Big]^{\frac{p}{2}}\cdot\\
&[(|\nabla u_n(x)|+|\nabla u(x)|)^p]^{\frac{2-p}{2}}dx=\\
\leq\ & \widetilde{M_2}\left(\int_{\mathbb{R}^N}|\nabla u_n(x)|^p dx+\int_{\mathbb{R}^N}|\nabla u(x)|^p dx\right)^{\frac{2-p}{2}}\cdot\\
&\left(\int_{\mathbb{R}^N}(|\nabla u_n(x)|^{p-2}\nabla u_n(x)-|\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x)-\nabla u(x))dx\right)^{\frac{p}{2}}\\
\leq\ & \overline{M}_2\left[\left(\int_{\mathbb{R}^N}|\nabla u_n(x)|^p dx\right)^{\frac{2-p}{2}}+\left(\int_{\mathbb{R}^N}|\nabla u(x)|^p dx\right)^{\frac{2-p}{2}}\right]\cdot\\
&\left(\int_{\mathbb{R}^N}(|\nabla u_n(x)|^{p-2}\nabla u_n(x)-|\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x)-\nabla u(x))dx\right)^{\frac{p}{2}}\\
\leq\ & \widehat{M_2}\cdot\left(\int_{\mathbb{R}^N}(|\nabla u_n(x)|^{p-2}\nabla u_n(x)-|\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x)-\nabla u(x))dx\right)^{\frac{p}{2}}\\
&\left(||u_n||_W^{\frac{(2-p)p}{2}}+||u||_W^{\frac{(2-p)p}{2}}\right).
\end{aligned}$$

Then, using again relation (11) and the above inequality, we have the estimate:

$$\left(\int_{\mathbb{R}^N}|\nabla u_n(x)-\nabla u(x)|^p dx\right)^{\frac{2}{p}}\leq \tag{13}$$

$$\begin{aligned}
\leq\ & M_2'\cdot\left(\int_{\mathbb{R}^N}(|\nabla u_n(x)|^{p-2}\nabla u_n(x)-|\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x)-\nabla u(x))dx\right)\cdot\\
&\left(||u_n||_W^{2-p}+||u||_W^{2-p}\right).
\end{aligned}$$

We introduce the following notation: $I(u) = \frac{1}{p}||u||_W^p$. As we used before, the directional derivative of I, in the direction $v \in E$ is

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^N} |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx - \int_{\mathbb{R}^N} V(|x|)|u(x)|^{p-2}u(x)v(x)dx.$$

Using the inequalities (12), (13) we have

$$||u_n - u||_W^2 < M_2' \cdot \langle I'(u_n) - I'(u), u_n - u\rangle \cdot (||u_n||_W^{p-2} + ||u||_W^{2-p}).$$

Since u_n is bounded, the same argument as in the first case (when $p \geq 2$) shows that u_n converges to u strongly in $W_r^{1,p}(\mathbb{R}^N; V)$.

Thus $\mathcal{E}_\lambda$ satisfies the (PS) condition for all $\lambda > 0$. ■

Lemma 4

$$\lim_{t\to 0^+} \frac{\sup\{J(u) : ||u||_W^p < pt\}}{t} = 0.$$

Proof. From inequality (4) we obtain:

$$|F(s)| \leq \varepsilon|s|^p + K(\delta)|s|^r, \text{ for every } s \in \mathbb{R}, \tag{14}$$

where $r \in]q_*, q^*[$ is fixed and $K(\delta)$ does not depend on s. Then

$$J(u) \leq \varepsilon||u||_{p,Q}^p + K(\delta)||u||_{r,Q}^r.$$

Now, using emeddings (1), (2), we get:

$$J(u) \leq \varepsilon C_p^p||u||_W^p + K(\delta)C_r^r||u||_W^r.$$

Therefore,

$$\sup\{J(u) : ||u||_W^p < pt\} \leq \varepsilon C_p^p pt + K(\delta)C_r^r(pt)^{\frac{r}{p}}.$$

Since ε is chosen arbitrarily and $r > p$, by dividing this last inequality with t and taking the limit, whenever $t \to 0^+$, we conclude the proof. ■

3 Proof of theorem 2

The main tool in the proof of Theorem 2 is a Ricceri-type critical points theorem (see [4], [5]) refined by Bonanno in [1].

Theorem 3 (G. Bonanno [1]) *Let* X *be a separable and reflexive real Banach space, and let* $\Phi, J : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functionals. Assume that*

(i) *there exists* $x_0 \in X$*, such that* $\Phi(x_0) = J(x_0) = 0$*;*

(ii) $\Phi(x) \geq 0$ *for every* $x \in X$*;*

(iii) *there exists* $x_1 \in X$, $\rho > 0$, *such that* $\rho < \Phi(x_1)$ *and* $\sup\{J(x) : \Phi(x) < \rho\} < \rho \frac{J(x_1)}{\Phi(x_1)}$.

(iv) *the functional* $\Phi - \lambda J$ *is sequentially weakly lower semicontinuous, satisfies the Palais–Smale condition for every* $\lambda > 0$ *and it is coercive, for every* $\lambda \in [0, \bar{a}]$*, where* $\bar{a} = \dfrac{\zeta\rho}{\rho\dfrac{J(x_1)}{\Phi(x_1)} - \sup\limits_{\Phi(x)<\rho} J(x)}$*, with* $\zeta > 1$.

Then there is an open interval $\Lambda \subset [0, \bar{a}]$ *and a number* $\sigma > 0$*, such that for each* $\lambda \in \Lambda$*, the equation* $\Phi'(x) - \lambda J'(x) = 0$ *admits at least three distinct solutions in X, having norm less than* σ*.*

We also need the following result of Su, Wang, Willem.

Lemma 5 [6, Lemma 4] *Assuming* (V) *with* $a > -\frac{N-1}{p-1}p$*, there exists* $C > 0$*, such that for all* $u \in W_r^{1,p}(\mathbb{R}^N; V)$

$$|u(x)| \leq C|x|^{-\frac{p(N-1)+a(p-1)}{p^2}} \|u\|_W, \quad |x| \gg 1. \tag{15}$$

Proof of Theorem 2. Let $s_0 \in \mathbb{R}$ be from (f_3), i.e. $F(s_0) > 0$. We denote by B_r the N-dimensional closed ball with center 0 and radius $r > 0$.
Since Q and V are positive continuous functions, for an $R > 0$ there exist the positive constants m_Q, M_Q, M_V such that:

$$m_Q = \min_{|x|\leq R} Q(|x|), M_Q = \max_{x\leq R} Q(|x|);$$

$$M_V = \max_{|x|\leq R} V(|x|).$$

For a $\sigma \in]0, 1[$ we define the function $u_\sigma : \mathbb{R}^N \to \mathbb{R}$ by

$$u_\sigma(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R}^N \setminus B_R \\ s_0, & \text{if } x \in B_{\sigma R} \\ \dfrac{s_0}{R(1-\sigma)}(R - |x|), & \text{if } x \in B_R \setminus B_{\sigma R}. \end{cases}$$

It is clear that u_σ belongs to $W_r^{1,p}(\mathbb{R}^N;V)$. Denoting the volume of the ball B_1 by ω_N, we obtain:

$$\begin{aligned}
||u_\sigma||_W^p &= \int_{B_{\sigma R}} V(|x|)|s_0|^p dx + \int_{B_R \setminus B_{\sigma R}} \left|\frac{s_0}{R(1-\sigma)}\right|^p dx + \\
&+ \int_{B_R \setminus B_{\sigma R}} V(|x|)\left|\frac{s_0}{R(1-\sigma)}\right|^p (R-|x|)^p dx \leq \\
&\leq |s_0|^p \omega_N R^N(\sigma^N M_V + R^{-p}(1-\sigma)^{-p}(1-\sigma^N)) + \\
&+ |s_0|^p R^{-p}(1-\sigma)^{-p} M_V \int_{B_R \setminus B_{\sigma R}} (R-|x|)^p dx \leq \\
&\leq |s_0|^p \omega_N R^N (M_V + R^{-p}(1-\sigma)^{-p}(1-\sigma^N))
\end{aligned}$$

and

$$J(u_\sigma) \geq \omega_N R^N (m_Q F(s_0)\sigma^N - M_Q \max_{|t|\leq|s_0|} F(t)(1-\sigma^N)). \tag{16}$$

By the choice of m_Q and M_Q, we have that $0 < \frac{m_Q F(s_0)}{M_Q \max\limits_{|t|\leq|s_0|} F(t)} < 1$. Therefore, we can choose a $\sigma_0 \in \left]\left(1 + \frac{m_Q F(s_0)}{M_Q \max\limits_{|t|\leq|s_0|} F(t)}\right)^{-\frac{1}{N}}, 1\right[\subseteq]0,1[$, such that

$$J(u_{\sigma_0}) > 0. \tag{17}$$

By Lemma 4 and inequality (16) it follows the existence of a positive constant $\rho_0 > 0$ so small that

$$\rho_0 < \frac{||u_{\sigma_0}||_W^p}{p} \tag{18}$$

$$\frac{\sup\{J(u) : ||u||_W^p < \rho_0\}}{\rho_0} < \frac{pJ(u_{\sigma_0})}{||u_{\sigma_0}||_W^p}. \tag{19}$$

Using the Lemmas from the previous section and inequalities (18), (19), all the assumptions of Theorem 3 are satisfied with the choices: $E = W_r^{1,p}(\mathbb{R}^N;V)$, $\Phi = \frac{1}{p}||u||_W^p$, $x_1 = u_{\sigma_0}$, $x_0 = 0$ and $\zeta = 1 + \rho_0$ and

$$a = \frac{1+\rho_0}{pJ(u_{\sigma_0})||u_{\sigma_0}||_W^{-p} - \sup\{J(u) : ||u||_W^p < r\}\rho_0^{-1}}.$$

Then, there exists an open interval $\Lambda \subset (0,\infty)$ and a constant $\mu > 0$ such that for every $\lambda \in \Lambda$ the equation $\mathcal{E}_\lambda = \Phi - \lambda J$ admits at least three distinct critical points: $u_\lambda^1, u_\lambda^2, u_\lambda^3 \in W_r^{1,p}(\mathbb{R}^N; V)$ such that

$$\max\{\|u_\lambda^1\|_W, \|u_\lambda^2\|_W, \|u_\lambda^3\|_W\} < \mu. \tag{20}$$

It remains to show that $|u_\lambda^i(x)| \to 0$ as $|x| \to \infty$, for $i \in \{1, 2, 3\}$. Using Lemma 5 and taking into account the estimate (20), the claim follows immediately. ■

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