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Minimum covering reciprocal distance signless Laplacian energy of graphs

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Abstract. Let G be a simple connected graph. The reciprocal transmission $Tr'_G(v)$ of a vertex v is defined as

$$Tr'_G(v) = \sum_{u \in V(G)} \frac{1}{d_G(u,v)}, \quad u \neq v.$$

The reciprocal distance signless Laplacian (briefly RDSL) matrix of a connected graph G is defined as $RQ(G) = \mathrm{diag}(Tr'(G)) + RD(G)$, where $RD(G)$ is the Harary matrix (reciprocal distance matrix) of G and $\mathrm{diag}(Tr'(G))$ is the diagonal matrix of the vertex reciprocal transmissions in G. In this paper, we investigate the RDSL spectrum of some classes of graphs that are arisen from graph operations such as cartesian product, extended double cover product and InduBala product. We

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introduce minimum covering reciprocal distance signless Laplacian matrix (or briefly MCRDSL matrix) of G as the square matrix of order n, $RQ_C(G) := (q_{i,j})$,

$$q_{i,j} = \begin{cases} 1 + Tr'(v_i) & \text{if} \quad i = j \quad and \quad v_i \in C \\ Tr'(v_i) & \text{if} \quad i = j \quad and \quad v_i \notin C \\ \dfrac{1}{d(v_i, v_j)} & \text{otherwise,} \end{cases}$$

where C is a minimum vertex cover set of G. MCRDSL energy of a graph G is defined as sum of eigenvalues of RQ_C. Extremal graphs with respect to MCRDSL energy of graph are characterized. We also obtain some bounds on MCRDSL energy of a graph and MCRDSL spectral radius of G, which is the largest eigenvalue of the matrix $RQ_C(G)$ of graphs.

1 Introduction

Throughout the paper, we consider G as a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. A graph G of order n and size m is called an (n, m) graph. Distance between two vertices u and v is denoted by $d(u, v)$. The diameter of G is the maximum distance between any pair of vertices and is denoted by diam(G).

For a vertex v, deg(v) denotes the degree of v. Energy of a graph introduced by Ivan Gutman [12] as the sum of the absolute values of the eigenvalues of adjacency matrix of G. The concept of energy of graph have been extensively studied; for more information we refer to surveys [13, 23, 24]. Various kinds of graph energy such as Laplacian energy [14], minimum covering energy [1], minimum covering distance energy [21], and minimum covering Harary energy [22] of a graph were proposed and some mathematical aspects of them were investigated. A subset C of $V(G)$ is called a vertex covering set of G if every edge of G is incident to at least one vertex of C. A vertex covering set with minimum cardinality is called minimum vertex covering set. The cardinality of a minimum vertex covering set in a graph G is known as the vertex covering number of G, denoted by $\tau(G)$. A set of vertices that no pair of which are adjacent is called vertex independent set.

A vertex independent set with maximum cardinality is called maximum vertex independent set.

The cardinality of maximum independent set in G is called independence number of G, denoted by $\alpha(G)$. Clearly if C is a vertex covering set of G, the $V(G) - C$ make an independent set for G. This follows the well known relation $\tau(G) + \alpha(G) = n$, where n is order of G. Two distinct edges in a graph G

are independent if they do not share a common vertex in G. A matching of a graph G is a set of pairwise independent edges in G. The matching number of G, $\beta(G)$ is the number of edges in the largest matching of G. The investigation of matrices related to various graph structures is a very large and growing area of research. In what follows, some of such matrices are introduced.

Let C be a minimum covering set of a graph G. The minimum covering matrix [1] of G is defined as $A_C(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G) \\ 1 & \text{if } i = j \text{ and } v_i \in C \\ 0 & \text{otherwise.} \end{cases}$$

The Harary matrix of a graph G, $RD(G)$ was introduced by Ivanciuc et al [17] and successfully used in computer generation of acyclic graphs based on local vertex invariants and topological indices. The Harary matrix $RD(G) = (RD_{ij})$ is a square matrix of order n, where

$$RD_{ij} = \begin{cases} 0 & \text{if } i = j \\ \dfrac{1}{d(v_i, v_j)} & \text{otherwise.} \end{cases}$$

The Harary matrix can be used to derive a variant of the Balaban index, Harary index and topological indices based on reciprocal distance in graphs. The minimum covering energy of G is defined to be absolute values of the eigenvalues of $A_C(G)$. The minimum covering Harary matrix [22] of G, is a square matrix $n \times n$ defined as $RDC(G) = (RDC_{ij})$, where

$$RDC_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } v_i \in C \\ 0 & \text{if } i = j \text{ and } v_i \notin C \\ \dfrac{1}{d(v_i, v_j)} & \text{otherwise.} \end{cases}$$

Analogously, minimum covering Harary energy of G is defined as $HE_C(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $H_C(G)$. The mathematical aspects of the minimum covering Harary energy was reported in [22].

The reciprocal transmission $Tr'_G(v)$ of a vertex v is defined as

$$Tr'_G(v) = \sum_{u \in V(G)} \frac{1}{d_G(u, v)}, \quad u \neq v$$

and $Tr'(G)$ is the diagonal matrix whose main entries are the vertex reciprocal transmissions in G. For $1 \leq i \leq n$, one can easily see that $Tr'_G(v_i)$ is just the i-th row sum of $RD(G)$. The Harary index of a graph G, denoted by $H(G)$, has been introduced independently by Plavšić et al. [19] and by Ivanciuc et al. [17] in 1993. It has been named in honor of Professor Frank Harary on the occasion of his 70th birthday. The Harary index is defined as: $H(G) = \sum_{\{u,v\}\subseteq V(G)} \dfrac{1}{d(u,v)}$.

Let α be a real number, we use notations $H_\alpha(G)$ $\sigma_\alpha(G)$ for $\sum_{\{u,v\}\subseteq V(G)} \dfrac{1}{d(u,v)^\alpha}$ and $\sum_{v\in V(G)} Tr'(v)^\alpha$, respectively. Note that if $\alpha \neq 1$ then $H_\alpha(G) = H(G)$ if and only if G is a complete graph. The first and the second Zagreb indices of a graph G, denoted by $M_1(G)$ and $M_2(G)$ are defined as:

$$M_1(G) = \sum_{uv\in E(G)} deg(u) + deg(v),$$

$$M_2(G) = \sum_{uv\in E(G)} deg(u)deg(v).$$

The *reciprocal distance signless Laplacian matrix* (or briefly RDSL matrix) is defined as $RQ(G) = Tr'(G) + RD(G)$. Since the matrix $RQ(G)$ is irreducible, non-negative, symmetric and positive semi-definite, all its eigenvalues are non-negative [2]. The set of eigenvalue of $RQ(G)$ is called RDSL spectrum of G.

Motivated by the concept of minimum covering distance matrix, we define the *minimum covering reciprocal distance signless Laplacian matrix* (or briefly MCRDSL matrix) of G as the square matrix of order n, $RQ_C(G) := (q_{i,j})$, where

$$q_{i,j} = \begin{cases} 1 + Tr'(v_i) & \text{if} \quad i = j \quad \text{and} \quad v_i \in C \\ Tr'(v_i) & \text{if} \quad i = j \quad \text{and} \quad v_i \notin C \\ \dfrac{1}{d(v_i, v_j)} & \text{otherwise.} \end{cases}$$

Let $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ be the eigenvalues of the RDSL matrix $RQ(G)$. The largest eigenvalue $\rho_1 = \rho(G)$ of $RQ(G)$ is called the RDSL *spectral radius* of G. By the Perron-Frobenius theorem, there is a unique normalized positive eigenvector of $RQ(G)$ corresponding to ρ_1, which is called the (RDSL) principal eigenvector of G. Since the matrices $RQ_C(G)$ is irreducible, non-negative, symmetric and positive semi-definite, all their eigenvalues are non-negative.

For MCRDSL matrix, *auxiliary energy* (briefly MCRDSL energy) is defined as sum of its eigenvalues and denoted by $E_{RQ_C}(G)$.

This paper is organized as follows. In the next section, RDSL spectrum of some classes of graphs that are constructed by graph operations, is determined. In section 3, MCRDSL spectrum of some standard graphs such as complete graph, complete bipartite graph and cocktail party graph are computed. Extremal graphs with respect to MCRDSL energy of graph is obtained in section 4. Finally, in section 5, more bounds are given for MCRDSL energy of graph and RDSL spectral radius in terms of the eigenvalues of RDSL matrix, Zagreb indices and Harary index.

2 RDSL spectrum of some classes of graphs

It is a well known fact that almost all graphs are of diameter 2. Therefore in this section, we get the RDSL spectrum of some classes of graphs of diameter 2 or 3 that are arisen from graph operations such as cartesian product, InduBala product, extended double cover graph and complement of a graph.

The following lemma will be helpful in the sequel.

Lemma 1 [11] *Let*

$$A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$$

be a symmetric 2×2 *block matrix. Then, the spectrum of* A *is the union of the spectra of* $A_0 + A_1$ *and* $A_0 - A_1$.

We begin first with cartesian product of K_2 and a graph of diameter at most 2. The cartesian product of two graphs G and H, $G \times H$ is the graph with vertex set $V(G) \times V(H)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(G)$.

Theorem 2 *Let* G *be an* r*-regular graph of diameter at most 2 with an adjacency matrix* A *and* $\mathrm{Spec}(G) = \begin{pmatrix} r & \lambda_i \\ 1 & n_i \end{pmatrix}, i = 2, 3, \dots, k$. *Then, the RDSL spectrum of* $H = G \times K_2$ *is as follows,* $\mathrm{Spec}(RQ(G)) =$

$$\begin{pmatrix} n + r - \frac{1}{6} & \frac{5n + 4r + 1}{3} & \frac{4\lambda_i + 5n + 4r + 2}{6} & \frac{2\lambda_i + 5n + 4r - 1}{6} \\ 1 & 1 & n_i & n_i \end{pmatrix}$$

for $i = 2, 3, \dots, k$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, $V(K_2) = \{w_1, w_2\}$. Let A and $\bar{A}$ be the adjacency matrix of G and $\bar{G}$ respectively and J denotes the $n \times n$ square matrix whose all entries are 1. From the fact $d_H((v_i, w_j), (v_s, w_t)) = d_G(v_i, v_s) + d_{K_2}(w_j, w_t) = d_G(v_i, v_s) + 1$, one can see that all vertices of H have a same reciprocal transmission and $Tr'_H(v_i, w_j) = \frac{1}{6}(5n + 4r + 1)$. Then $Tr'(G) = \frac{1}{6}(5n + 4r + 1)I$. Since G is a graph of diameter 1 or 2, diameter of H is 2 or 3 and H is $r + 1$ regular. Thus the $RD(H)$ is of the form

$$RD(H) = \begin{pmatrix} A + \frac{1}{2}\bar{A} & J - \frac{1}{2}A - \frac{2}{3}\bar{A} \\ J - \frac{1}{2}A - \frac{2}{3}\bar{A} & A + \frac{1}{2}\bar{A} \end{pmatrix}$$

and consequently the RDSL matrix of H is of the form

$$RQ(H) = \begin{pmatrix} A + \frac{1}{2}\bar{A} + (\frac{5}{6}n + \frac{2}{3}r + \frac{1}{6})I & J - \frac{1}{2}A - \frac{2}{3}\bar{A} \\ J - \frac{1}{2}A - \frac{2}{3}\bar{A} & A + \frac{1}{2}\bar{A} + (\frac{5}{6}n + \frac{2}{3}r + \frac{1}{6})I \end{pmatrix}.$$

Now, by Lemma 1 and the fact $\bar{A} = J - I - A$, the spectrum of $RQ(H)$ is the union of the spectra

$$\frac{1}{6}(4A + 5J + (5n + 4r + 2)I)$$

and

$$\frac{1}{6}(2A + J + (5n + 4r - 1)I).$$

□

The next considered graph operation is extended double cover graph of a graph that is introduced by N. Alon [3] to studying networks. Spectra of extended double cover graphs was investigated in [7]. Let G be a graph on the vertex set $\{v_1, \ldots, v_n\}$. The extended double cover graph of G, denoted by G^*, is the bipartite graph with partitions X and Y where $X = \{x_1, x_2, \ldots x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, in which x_i and y_j are adjacent if and only if $i = j$ or v_i and v_j are adjacent in G. Now, we obtain the RDSL spectrum of the G^* of a regular graph G with diameter 2.

Theorem 3 *Let* G *be an* r*-regular graph on* n *vertices with diameter 2 and let* $\mathrm{Spec}(G) = \begin{pmatrix} r & \lambda_i \\ 1 & n_i \end{pmatrix}$, $i = 2, 3, \ldots, k$. *Then, the RDSL spectrum of* G^* *is* $\mathrm{Spec}(RQ(G^*)) =$

$$\begin{pmatrix} \frac{5n+4r+1}{3} & n-1 & \frac{4\lambda_i+5n+4r+2}{6} & \frac{-4\lambda_i+5n+4r-6}{6} \\ 1 & 1 & n_i & n_i \end{pmatrix},$$

$$i = 2, 3, \ldots, k.$$

Proof. First note that G^* is $r+1$ regular graph with diameter 3 and any vertex $v \in V(G^*)$ has reciprocal transmission $\frac{1}{6}(5n+4r+1)$. It is not difficult to see that $RD(G^*)$ has the form

$$RD(G^*) = \begin{pmatrix} \frac{1}{2}(J-I) & A + \frac{1}{3}\bar{A} + I \\ A + \frac{1}{3}\bar{A} + I & \frac{1}{2}(J-I) \end{pmatrix},$$

and then we have

$$RQ(G^*) = \begin{pmatrix} \frac{1}{6}(3J + (5n+4r-2)I) & A + \frac{1}{3}\bar{A} + I \\ A + \frac{1}{3}\bar{A} + I & \frac{1}{6}(3J + (5n+4r-2)I) \end{pmatrix}.$$

Then, by Lemma 1, the spectrum of $RQ(G)$ is the union of the spectra

$$\frac{1}{6}(4A + 5J + (5n+4r+2)I)$$

and

$$\frac{1}{6}(-4A + J + (5n+4r-6)I).$$

□

Next graph operation is InduBala product. InduBala product of graphs introduced in [16], where the distance spectrum of InduBala product of graphs is determined. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs on disjoint sets of n_1 and n_2 vertices, respectively, then their *union* is the graph $G_1 \cup G_2 =$

$(V_1 \cup V_2, E_1 \cup E_2)$. Their *join* is denoted by $G_1 \nabla G_2$ and consists of $G_1 \cup G_2$ and all lines joining V_1 and V_2. The InduBala product of graphs is defined as follows. Let $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$. Take a disjoint copy $G_1' \nabla G_2'$ of $G_1 \nabla G_2$ with vertex sets $V(G_1') = \{u_1', u_2', \ldots, u_{n_1}'\}$ and $V(G_2') = \{v_1', v_2', \ldots, v_{n_2}'\}$. Now make v_i adjacent with v_i' for each $i = 1, 2, \ldots, n_2$. Structure of InduBala product of two graphs P_4 and K_3 is illustrated in Figure 1. Occasionally, it so happens that for certain families of

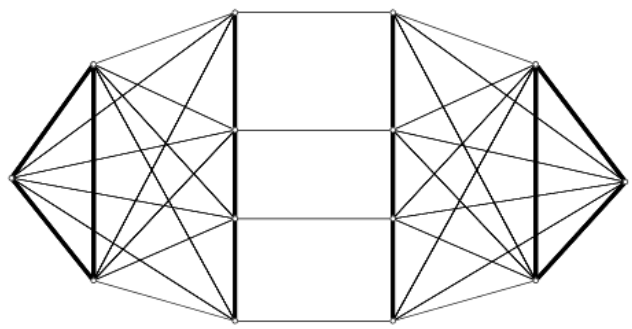

Figure 1: *The graph* $K_3 \blacktriangledown P_4$.

graphs it is possible to identify a graph by looking at the spectrum. Now, we describe the RDSL spectrum of the join of a regular graph with the union of two regular graphs of distinct vertex degrees.

Theorem 4 *For* $i = 0, 1, 2$, *let* G_i *be an* r_i*-regular graph of order* n_i *and eigenvalues* $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \ldots \geq \lambda_{i,n_i}$ *of the adjacency matrix* $A(G_i)$. *Then the RDSL spectrum of* $G_0 \nabla (G_1 \cup G_2)$ *consists of eigenvalues*

$$\frac{1}{2}(2m - n_0 + \lambda_{0,j} + r_0 - 2), \quad j = 2, \ldots, n_0,$$

and

$$\frac{1}{2}(m + n_0 + \lambda_{i,j} + r_i - 2), \quad i = 1, 2 \quad \textit{and} \quad j = 2, 3, \ldots, n_i,$$

where $m = \sum_{i=0}^{2} n_i$, *and three more eigenvalues which are the eigenvalues of the following matrix*

$$\begin{pmatrix} m + r_0 - 1 & n_1 & n_2 \\ n_0 & m - \frac{1}{2}n_2 + r_1 - 1 & \frac{1}{2}n_2 \\ n_0 & \frac{1}{2}n_1 & m - \frac{1}{2}n_1 + r_2 - 1 \end{pmatrix}. \qquad (1)$$

Proof. The reciprocal distance signless Laplacian matrix $F = G_0 \nabla (G_1 \cup G_2)$ has the form

$$\begin{pmatrix} S_0 & J & J \\ J & S_1 & \frac{1}{2}J \\ J & \frac{1}{2}J & S_2 \end{pmatrix},$$

where

$$S_0 = \frac{1}{2}\left((2m - n_0 + r_0 - 2)I + J + A(G_0)\right)$$

and for $i = 1, 2$

$$S_i = \frac{1}{2}\left((m + n_0 + r_i - 2)I + J + A(G_i)\right).$$

As a regular graph, G_0 has the all-one vector $\mathbf{1}$ as an eigenvector corresponding to the eigenvalue r_0, while all the other eigenvectors are orthogonal to $\mathbf{1}$. Let λ be an arbitrary eigenvalue of the adjacency matrix of G_0 with corresponding eigenvector X, such that $\mathbf{1}^T X = 0$, then $[X^T \quad 0 \quad 0]^T$ is an eigenvector of $RQ(F)$ corresponding to the eigenvalue $\frac{1}{2}(2m - n_0 + r_0 - 2 + \lambda)$. Now, let μ, ξ be arbitrary eigenvalues of the adjacency matrix of G_1 and G_2 with corresponding eigenvector Y and Z, respectively. In a similar way the vectors $[0 \quad X^T \quad 0]^T$ and $[0 \quad 0 \quad X^T]^T$ are eigenvectors of $RQ(F)$ with corresponding eigenvalues $\frac{1}{2}(m + n_0 + r_1 - 2 + \mu)$ and $\frac{1}{2}(m + n_0 + r_2 - 2 + \xi)$, respectively.

In this way we obtain eigenvectors of the form $[X^T \quad 0 \quad 0]^T$, $[0 \quad X^T \quad 0]^T$ and $[0 \quad 0 \quad X^T]^T$ and these account for a total of $m-3$ eigenvectors. All these eigenvectors are orthogonal to $[\mathbf{1}^T \quad 0 \quad 0]^T$, $[0 \quad \mathbf{1}^T \quad 0]^T$ and $[0 \quad 0 \quad \mathbf{1}^T]^T$. Thus the remaining three eigenvectors of $RQ(F)$ are of the form $[\alpha\mathbf{1} \quad \beta\mathbf{1} \quad \gamma\mathbf{1}]^T$ for some $(\alpha, \beta, \gamma) \neq (0, 0, 0)$.

If ν is an eigenvalue of $RQ(F)$ with an corresponding eigenvector $(\alpha\mathbf{1}, \beta\mathbf{1}, \gamma\mathbf{1})^T$, then from $RQ(F)(\alpha\mathbf{1}, \beta\mathbf{1}, \gamma\mathbf{1})^T = \nu(\alpha\mathbf{1}, \beta\mathbf{1}, \gamma\mathbf{1})^T$, and $A(G_i)\mathbf{1} = r_i\mathbf{1}$ for $i = 0, 1, 2$, we get the system of equations:

$$\begin{aligned} &(m + r_0 - 1)\alpha + n_1\beta + n_2\gamma = \nu\alpha, \\ &n_0\alpha + (m - \frac{1}{2}n_2 + r_1 - 1)\beta + \frac{1}{2}n_2\gamma = \nu\beta, \\ &n_0\alpha + \frac{1}{2}n_1\beta + (m - \frac{1}{2}n_1 + r_2 - 1)\gamma = \nu\gamma, \end{aligned}$$

which have a nontrivial solution if and only if ν is an eigenvalue of (1). Further, it is obvious from above that any nontrivial solution of above system forms an eigenvector of $RQ(F)$ corresponding to eigenvalue ν. Since all 3 remaining eigenvectors of $RQ(F)$ must be formed in this way, we conclude that each eigenvalue of (1) is an eigenvalue of $RQ(F)$ as well. $\square$

Theorem 5 *For* $i = 1, 2$, *let* G_i *be an* r_i*-regular graph of order* n_i *and let* $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \ldots \geq \lambda_{i,n_i}$ *be the eigenvalues of the adjacency matrix* $A(G_i)$. *Then the RDSL spectrum of* $G_1 \blacktriangledown G_2$ *is the set consisting of eigenvalues*

$$\frac{1}{2}\left(\frac{5}{3}n_1 + 3n_2 + \lambda_{1,j} + r_1 - 2\right), \quad j = 2, 3, \ldots, n_1 \quad \textit{each with multiplicity 2},$$

and

$$\frac{1}{2}\left(3n_1 + \frac{5}{3}n_2 + \frac{4}{3}\lambda_{2,j} + \frac{4}{3}r_2 + \frac{2}{3}\right) \quad j = 2, 3, \ldots, n_2$$

and

$$\frac{1}{2}\left(3n_1 + \frac{5}{3}n_2 + \frac{2}{3}\lambda_{2,j} + \frac{4}{3}r_2 - 2\right), \quad j = 2, 3, \ldots, n_2,$$

also four more eigenvalues which are the eigenvalues of the matrix

$$\begin{pmatrix} m_1 & n_2 & \frac{1}{2}n_2 & \frac{1}{3}n_1 \\ n_1 & m_2 & \frac{1}{3}(n_2 + \frac{1}{6}r_2 + \frac{2}{3}) & \frac{1}{2}n_1 \\ \frac{1}{2}n_1 & \frac{1}{3}(n_2 + \frac{1}{6}r_2 + \frac{2}{3}) & m_2 & n_1 \\ \frac{1}{3}n_1 & \frac{1}{2}n_2 & n_2 & m_1 \end{pmatrix}, \tag{2}$$

where $m_1 = \frac{1}{2}\left(\frac{8}{3}n_1 + 3n_2 + 2r_1 - 2\right)$ *and* $m_2 = \frac{1}{2}\left(\frac{8}{3}n_2 + 3n_1 + \frac{7}{3}r_2 - \frac{2}{3}\right)$.

Proof. The RDSL matrix $H = G_1 \blacktriangledown G_2$ has the form

$$RQ(H) = \begin{pmatrix} S_1 & J & \frac{1}{2}J & \frac{1}{3}J \\ J & S_2 & \frac{1}{3}J + \frac{2}{3}I + \frac{1}{6}A(G_2) & \frac{1}{2}J \\ \frac{1}{2}J & \frac{1}{3}J + \frac{2}{3}I + \frac{1}{6}A(G_2) & S_3 & J \\ \frac{1}{3}J & \frac{1}{2}J & J & S_4 \end{pmatrix},$$

where

$$S_i = \frac{1}{2}\left(J + A(G_1) + (\frac{5}{3}n_1 + 3n_2 + r_1 - 2)I\right), \quad i = 1, 4$$

and

$$S_i = \frac{1}{2}\left(J + A(G_2) + (3n_1 + \frac{5}{3}n_2 + \frac{4}{3}r_2 - \frac{2}{3})I\right), \quad i = 2, 3.$$

By analogy to the proof of Theorem 4, let λ be an arbitrary eigenvalue of the adjacency matrix of G_1 with corresponding eigenvector X, such that $\mathbf{1}^T X = 0$. Then $[X^T \ \ 0 \ \ 0 \ \ 0]^T$ is an eigenvector of $RQ(H)$ corresponding to the eigenvalue $\frac{1}{2}(\frac{5}{3}n_1 + 3n_2 + \lambda + r_1 - 2)$. In a similar way the vector $[0 \ \ 0 \ \ 0 \ \ X^T]^T$ is an eigenvector of $RQ(H)$ corresponding to the eigenvalue $\frac{1}{2}(\frac{5}{3}n_1 + 3n_2 + \lambda + r_1 - 2)$. Now let μ be an arbitrary eigenvalue of the adjacency matrix of G_2 with corresponding eigenvector Y, such that $\mathbf{1}^T Y = 0$. Then by a similar argument we see that the vectors $[0 \ \ Y^T \ \ Y^T \ \ 0]^T$ and $[0 \ \ -Y^T \ \ Y^T \ \ 0]^T$ are eigenvectors of $RQ(H)$ with corresponding eigenvalues $\frac{1}{2}\left(3n_1 + \frac{4}{3}\mu + \frac{5}{3}n_2 + \frac{4}{3}r_2 + \frac{2}{3}\right)$ and $\frac{1}{2}\left(3n_1 + \frac{2}{3}\mu + \frac{5}{3}n_2 + \frac{4}{3}r_2 - 2\right)$ respectively. In this way we obtain eigenvectors of the form $[X^T \ \ 0 \ \ 0 \ \ 0]^T$, $[0 \ \ 0 \ \ 0 \ \ X^T]^T$, $[0 \ \ Y^T \ \ Y^T \ \ 0]^T$ and $[0 \ \ -Y^T \ \ Y^T \ \ 0]^T$ and these account for a total of $2(n_1 + n_2) - 4$ eigenvectors. All these eigenvectors are orthogonal to $[\mathbf{1}^T \ \ 0 \ \ 0 \ \ 0]^T$, $[0 \ \ \mathbf{1}^T \ \ 0 \ \ 0]^T$, $[0 \ \ 0 \ \ \mathbf{1}^T \ \ 0]^T$ and $[0 \ \ 0 \ \ 0 \ \ \mathbf{1}^T]^T$. This means that these four vectors span the space spanned by the remaining four eigenvectors of $RQ(H)$. Thus the remaining four eigenvectors of $RQ(H)$ are of the form $[\alpha\mathbf{1} \ \ \beta\mathbf{1} \ \ \gamma\mathbf{1} \ \ \delta\mathbf{1}]^T$ for some $(\alpha, \beta, \gamma, \delta) \neq (0, 0, 0, 0)$. If ν is an eigenvalue of $RQ(H)$ with an eigenvector $(\alpha\mathbf{1} \ \ \beta\mathbf{1} \ \ \gamma\mathbf{1} \ \ \delta\mathbf{1})^T$, from

$RQ(H)(\alpha\mathbf{1} \quad \beta\mathbf{1} \quad \gamma\mathbf{1} \quad \delta\mathbf{1})^T = \nu(\alpha\mathbf{1} \quad \beta\mathbf{1} \quad \gamma\mathbf{1} \quad \delta\mathbf{1})^T$, and $A(G_i)\mathbf{1} = r_i\mathbf{1}$ for $i = 1, 2$, we get the system of equations:

$$\frac{1}{2}\left(\frac{8}{3}n_1 + 3n_2 + 2r_1 - 2\right)\alpha + n_2\beta + \frac{1}{2}n_2\gamma + \frac{1}{3}n_1\delta = \nu\alpha,$$
$$n_1\alpha + \frac{1}{2}\left(\frac{8}{3}n_2 + 3n_1 + \frac{7}{3}r_2 - \frac{2}{3}\right)\beta + \frac{1}{3}(n_2 + \frac{1}{6}r_2 + \frac{2}{3})\gamma + \frac{1}{2}n_1\delta = \nu\beta,$$
$$\frac{1}{2}n_1\alpha + \frac{1}{3}(n_2 + \frac{1}{6}r_2 + \frac{2}{3})\beta + \frac{1}{2}\left(\frac{8}{3}n_2 + 3n_1 + \frac{7}{3}r_2 - \frac{2}{3}\right)\gamma + n_1\delta = \nu\gamma$$
$$\frac{1}{3}n_1\alpha + \frac{1}{2}n_2\beta + n_2\gamma + \frac{1}{2}\left(\frac{8}{3}n_1 + 3n_2 + 2r_1 - 2\right)\delta = \nu\delta,$$

which have a nontrivial solution if and only if ν is an eigenvalue of (2). Further, it is obvious from above that any nontrivial solution of above system forms an eigenvector of $RQ(H)$ corresponding to eigenvalue ν. Since all four remaining eigenvectors of $RQ(H)$ must be formed in this way, we conclude that each eigenvalue of (2) is an eigenvalue of $RQ(H)$ as well. □

3 MCRDSL energy of some standard graphs

In this section, E_{RQ_C} is computed for some standard graphs such as complete graph, complete bipartite graph and cocktail party graph.

Example 6 Complete graph K_n.
For $n \geq 2$, the eigenvalues of the minimum covering Harary matrix of complete graph K_n was determined in [1, 22] as $\mathrm{Spec}(RD_C(K_n)) =$

$$\begin{pmatrix} 0 & \dfrac{n-1+\sqrt{(n+3)(n-1)}}{2} & \dfrac{n-1-\sqrt{(n+3)(n-1)}}{2} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Easily one can see that for complete graph K_n with vertex set $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and minimum covering set $C = \{v_1, v_2, \ldots, v_{n-1}\}$,

$$RQ_C(K_n) = (n-1)I + A_C(K_n).$$

Therefore, the eigenvalues of the matrix $RQ_C(K_n)$ *are as* $\mathrm{Spec}\big(RQ_C(K_n)\big) =$

$$\begin{pmatrix} n-1 & \frac{3(n-1)+\sqrt{(n+3)(n-1)}}{2} & \frac{3(n-1)-\sqrt{(n+3)(n-1)}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

and consequently $E_{RQ_C}(K_n) = n^2 - 1$. Another simpler way to compute the $E_{RQ_C}(K_n)$ is as follows

$$E_{RQ_C}(K_n) = trace(RQ_C(K_n)) = |C| + \sum_{i=1}^{n} Tr'(v_i) = n - 1 + n(n-1) = n^2 - 1.$$

Complete bipartite graph $K_{m,n}$.
Let $V(K_{m,n}) = \{v_1, v_2, \ldots, v_m\} \bigcup \{w_1, w_2, \ldots, w_n\}$ and $C = \{v_1, v_2, \ldots, v_m\}$ be the minimum covering set of $K_{m,n}$, $(m \leq n)$. Then,

$$E_{RQ_C}(K_{m,n}) = |C| + \sum_{i=1}^{m} Tr'(v_i) + \sum_{i=1}^{n} Tr'(w_i) = 2mn + \frac{1}{2}(m^2 + n^2 + m - n).$$

Cocktail party graph.
The Cocktail party graph of order n, $K_{n\times 2}$ is formed from the complete graph K_{2n} by removing n disjoint edges. Note that all vertices of $K_{n\times 2}$ have a same reciprocal transmission $\frac{3}{2}(n-1)$ and a minimum covering set is of order $2n-2$. Therefore,

$$E_{RQ_C}(K_{n\times 2}) = 2n - 2 + 2n\left(\frac{3}{2}(n-1)\right) = 3n^2 - n - 2.$$

4 Extremal graphs with respect to ERQ_C

In this section, we are concerned with the extremal graphs with respect to the minimum covering reciprocal distance signless Laplacian energy.

Theorem 7 *Let* G *be a simple graph with* n *vertices and* m *edges. If* C *is the minimum covering set of* G*, then*

$$ERQ_C(G) = \tau(G) + 2H(G).$$

Proof. Let $\rho_1, \rho_2, \ldots, \rho_n$ be the eigenvalues of the matrix $RQ_C(G)$. The result follows from the well known fact that $\sum_{i=1}^{n} \rho_i = \mathrm{Trace}(RQ_C(G))$ and the eigenvalues of $RQ_C(G)$ are non-negative. □

Corollary 8 *Let G be a graph of order n, then $E_{RQ_C}(G) \leq n^2 - 1$. The equality holds if and only if $G \cong K_n$.*

Proof. It is a well known fact that adding any edge to graph G, increase the Harary index of G and do not decrease the vertex covering number of G. Consequently, K_n has the maximum Harary index among all graphs of order n. Clearly, $\tau(G) \leq n-1$. Therefore

$$E_{RQ_C}(G) = \tau(G) + 2H(G) \leq n - 1 + n(n-1) = n^2 - 1,$$

and the equality holds if and only if $G \cong K_n$. □

Corollary 9 *Let $G \neq K_n$ be a graph of order n. Then*

$$ERQ_C(G) \leq n^2 - 3,$$

with equality holds if and only if $G \cong K_n - e$, where e is an edge of K_n.

Let $G(n, \beta)$ denotes the constructed graph by join of K_β and $\overline{K_{n-\beta}}$. Note that $\tau(G(n, \beta)) = \beta$. Let $T(n, \beta)$ be a tree obtained from $K_{1,n-\beta}$ by attaching a pendant vertex to its $\beta - 1$ pendant vertices. In [15] and [9] lower and upper bounds on Harary index were obtained in terms of independence number and matching number. It was proved that graphs $G(n, \beta)$ and $T(n, \beta)$ have the maximum value of Harary index among all graphs and trees of a same order n and same independence number $n - \beta$, respectively. Consequently, $G(n, \beta)$ and $T(n, \beta)$ get the maximum value of ERQ_C among graphs of order n and vertex covering number β as well. Hence we conclude that:

Theorem 10 *Let G be a graph of order n and vertex cover number β. Then,*

$$ERQ_C(G) \leq 2\binom{\beta}{2} + \binom{n-\beta}{2} + \beta(2n - 2\beta + 1).$$

The equality holds if and only if $G \cong G(n, \beta)$.

In [10], it is proved that of all trees of order n, star graph S_n is the unique graph of maximum value of Harary index. But it is not true for ERQ_C. For example, see the figure 2, two graphs S_5 and $T(5, 2)$ where $ERQ_C(T(5, 2)) > ERQ_C(S_5)$.

In the following, an upper bound is given for trees of order n and vertex cover β.

Figure 2: *Graphs* S_5 *and* $T(5,2)$, $ERQ_C(T(5,2)) > ERQ_C(S_5)$

Theorem 11 *Let* T *be a tree of order* n *and vertex cover number* β. *Then,*

$$ERQ_C(T) \leq \frac{1}{12}\left(6n^2 + (10-4\beta)n + \beta^2 + 21\beta - 22\right).$$

The equality holds if and only if $T \cong T(n,\beta)$.

It is a well known fact that for any bipartite graph G of order n, $\alpha(G) + \beta(G) = n$, (see [5]). Therefore, the following corollary is immediate.

Corollary 12 *Let* T *be a tree of order* n. *If* T *has perfect matching, then*

$$ERQ_C(T) \leq \frac{1}{48}(15n^2 + 82n - 88),$$

with equality holding if and only if $T \cong T(n, \frac{n}{2})$.

Lemma 13 *Let* T *be a tree of order* n *and diameter* d. *Then* $\lceil \frac{d+1}{2} \rceil \leq \alpha(T) \leq \lceil n - \frac{d}{2} \rceil$.

Proof. Notice that T has P_{d+1} as subgraph. The proof follows from the fact that $\alpha(T) \geq \alpha(P_{d+1}) = \lceil \frac{d+1}{2} \rceil$ and $\tau(T) \geq \tau(P_{d+1}) = \lfloor \frac{d}{2} \rfloor$. □

A lower bound for Harary index among trees of diameter d and order n is obtained by Xu et al. [9] as follow.

Lemma 14 *Let* T *be a tree of order* n *and diameter* d. *Then*

$$H(T) \leq \frac{1}{24}(M_1(G) + 2M_2(G) + 3n^2 + 11n - 24),$$

with equality holds if and only if T *is of diameter at most* 4.

Now, an upper bound for ERQ_C of trees is obtained by using Lemmas 13 and 14 as:

Corollary 15 *Let* T *be a tree of order* n *and diameter* d*. Then*

$$ERQ_C(T) \leq \frac{1}{12}(M_1 + 2M_2 + 3n^2 + 23n - 24) - \lfloor \frac{d+1}{2} \rfloor,$$

with equality holds if and only if $T = P_n$ *where* $2 \leq n \leq 5$ *or* T *is a graph constructed by* P_5 *and attaching a vertex to the central vertex of* P_5*.*

Proof. From Lemma 13, we get $\tau(T) \leq n - \lfloor \frac{d+1}{2} \rfloor$. Among trees of diameter $d \leq 4$, trees P_n, $2 \leq n \leq 5$ and a graph constructed by P_5 and attaching a vertex to the central vertex of P_5, have vertex cover $\tau(T) = n - \lfloor \frac{d+1}{2} \rfloor$. □

Let $\Gamma(n, d)$ be the set of all graphs of order n and diameter d, obtained from a path P_{d+1} and a complete graph K_{n-d-1} that each vertex of K_{n-d-1} is connected to a central vertex in P_{d+1} and its two neighbors. In [10], some upper and lower bounds were obtained for graphs of given diameter and number of edges. In the following, we show that graphs of $\Gamma(n, d)$ get the maximum value of Harary index and ERQ_C among graphs of given order n and diameter d. Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denotes the n-th harmonic number. It is easy to see that $H(P_n) = nH_{n-1} - n + 1$.

Theorem 16 *Let* G *be a graph on* n *vertices and diameter* d*. Then*

$$H(G) \leq (d+1)H_d - d + \binom{n-d-1}{2} + (n-d-1)(H_{\lfloor \frac{d}{2} \rfloor} + H_{\lfloor \frac{d+1}{2} \rfloor} + 1),$$

with equality holds if and only if $G \in \Gamma(n, d)$*.*

Proof. Let P_{d+1} be a path connecting two vertices of distance d. Let $W_1 = V(P_{d+1})$ and $W_2 = V(G) - V(P_{d+1})$. Note that each vertex of W_2 is connected to at most 3 vertices of W_1. It is not difficult to see that in a path P_m, a central vertex x has maximum reciprocal transmission $Tr'(x) = H_{\lfloor \frac{m-1}{2} \rfloor} + H_{\lfloor \frac{m}{2} \rfloor}$.

Let x be a central vertex of P_{d+1}. Then

$$\begin{aligned} H(G) &= H(P_{d+1}) + \sum_{\{u,v\} \subseteq W_2} \frac{1}{d(u,v)} + \sum_{u \in W_1} \sum_{v \in W_2} \frac{1}{d(u,v)} \\ &\leq H(P_{d+1}) + \binom{n-d-1}{2} + (n-d-1)(Tr'_{P_{d+1}}(x) + 1). \end{aligned}$$

The equality holds if and only if all vertices in W_2 are adjacent and for each vertex $v \in W_2$, the equality $\sum_{u \in W_1} \frac{1}{d(u,v)} = Tr'_{P_{d+1}}(x) + 1$ holds if and only if v is adjacent to x and its two neighbors in P_{d+1}. Thus $G \in \Gamma(n, d)$. □

5 More bounds on ERQ_C and largest eigenvalue of RQ_C matrix

The following lemmas refer to the real non-negative numbers, and will be helpful in the sequel.

Lemma 17 [18] *If a_i and $b_i, 1 \leq i \leq n$, are non-negative real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \Big(\sum_{i=1}^{n} a_i b_i \Big)^2 \leq \frac{n^2}{4} (M_1 M_2 - m_1 m_2)^2,$$

where $M_1 = \max_{1 \leq i \leq n} a_i$, $M_2 = \max_{1 \leq i \leq n} b_i$, $m_1 = \min_{1 \leq i \leq n} a_i$ *and* $m_2 = \min_{1 \leq i \leq n} b_i$.

Lemma 18 [20] *If a_i and $b_i, 1 \leq i \leq n$, are positive real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{1}{4} \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 \left(\sum_{i=1}^{n} a_i b_i \right)^2,$$

where $M_1 = \max_{1 \leq i \leq n} a_i$, $M_2 = \max_{1 \leq i \leq n} b_i$, $m_1 = \min_{1 \leq i \leq n} a_i$ *and* $m_2 = \min_{1 \leq i \leq n} b_i$.

Lemma 19 [8] *If a_i and $b_i, 1 \leq i \leq n$, are non-negative real numbers for which there exist real numbers r and R, so that $r \leq \frac{b_i}{a_i} \leq R, a_i \neq 0$, for each $i = 1, 2, \ldots, n$. Then*

$$\sum_{i=1}^{n} b_i^2 + rR \sum_{i=1}^{n} a_i^2 \leq (r + R) \sum_{i=1}^{n} a_i b_i.$$

Equality holds if and only if $b_i = a_i r$ or $b_i = a_i R$ for at least one i, where $1 \leq i \leq n$.

Lemma 20 *Let G be a graph of order n and C be a minimum vertex covering set. If $\rho_1, \rho_2, \ldots, \rho_n$ are the eigenvalues of $RQ_C(G)$, then*

$$\sum_{i=1}^{n} \rho_i^2 = \sigma_2(G) + 2H_2(G) + \tau(G) + 2 \sum_{v \in C} Tr'(v).$$

Proof. We have

$$\begin{aligned}\sum_{i=1}^{n}\rho_i^2 &= \sum_{j=1}^{n}\sum_{i=1}^{n}q_{ij}q_{ji}=\sum_{i=1}^{n}(q_{ii})^2+2\sum_{1\le i<j\le n}(q_{ij})^2\\ &= \sum_{v\notin C}(Tr'(v))^2+\sum_{v\in C}(1+Tr'(v))^2+2\sum_{1\le i<j\le n}(q_{ij})^2\\ &= \sigma_2(G)+\tau(G)+2H_2(G)+2\sum_{v\in C}Tr'(v).\end{aligned}$$

□

Corollary 21 *Let G be an (n,m) graph with diameter at most 2 and C be a minimum covering set. If $\rho_1,\rho_2,\ldots,\rho_n$ are the eigenvalues of $RQ_C(G)$, then*

$$\sum_{i=1}^{n}\rho_i^2=\frac{1}{2}(n+1)\binom{n}{2}+\frac{1}{4}M_1(G)+nm+n\tau(G)+\sum_{v\in C}\deg(v),$$

where $M_1(G)=\sum_{i=1}^{n}\deg(v_i)^2$ is known as the first Zagreb index.

Proof. Let $V(G)=\{v_1,v_2,\ldots,v_n\}$. Since $\mathrm{diam}(G)\le 2$, hence we get $Tr'(v)=\frac{1}{2}(n+\deg(v)-1)$. Let $RQ_C(G)=(q_{ij})$. From the fact $\sum_{i=1}^{n}\rho_i^2=trace(RQ_C(G))^2$, we get

$$\begin{aligned}\sum_{i=1}^{n}\rho_i^2 &= \sigma_2(G)+\tau(G)+2H_2(G)+2\sum_{v\in C}Tr'(v)\\ &= \sum_{v\in V(G)}(Tr'(v))^2+\tau(G)+2\sum_{v\in C}Tr'(v)+m+\binom{n}{2}\\ &= \frac{1}{4}\left(n(n-1)^2+M_1(G)+4(n-1)m\right)+n\tau(G)\\ &\quad+\sum_{v\in C}\deg(v)+m+\binom{n}{2}\\ &= \frac{1}{2}(n+1)\binom{n}{2}+\frac{1}{4}M_1(G)+nm+n\tau(G)+\sum_{v\in C}\deg(v),\end{aligned}$$

and the proof is complete. □

In the following, some bounds are presented for $E_{RQ_C}(G)$.

Theorem 22 *Let* G *be a simple graph of order* n*. If* C *is the minimum vertex covering set and* $\Delta = \det(RQ_C(G))$*, then*

$$\sqrt{\sigma_2(G) + 2H_2(G) + \tau(G) + 2\sum_{v\in C} Tr'(v) + n(n-1)\Delta^{\frac{2}{n}}}$$
$$\leq E_{RQ_C}(G)$$
$$\leq \sqrt{n\left(\sigma_2(G) + 2H_2(G) + \tau(G) + 2\sum_{v\in C} Tr'(v)\right)}.$$

Proof. Let $\rho_1, \rho_2, \ldots, \rho_n$ be the eigenvalues of $RQ_C(G)$. First, we show the right-hand side inequality. Setting $a_i = 1$ and $b_i = \rho_i$ in the Cauchy Schwarz inequality, $\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)$ and using Lemma 20, we get

$$\left(\sum_{i=1}^n \rho_i\right)^2 \leq \left(\sum_{i=1}^n 1\right)\left(\sum_{i=1}^n \rho_i^2\right)$$
$$\left(E_{RQ_C}(G)\right)^2 \leq n\left(\sigma_2(G) + 2H_2(G) + \tau(G) + 2\sum_{v\in C} Tr'(v)\right).$$

For the left inequality, consider the AM-GM inequality (which says that arithmetic mean of a set of non-negative real number is greater than or equal to geometric mean of them), on the set of $\{\rho_i\rho_j | 1 \leq i < j \leq n\}$, then

$$\begin{aligned}
\frac{1}{\binom{n}{2}} \sum_{1\leq i<j\leq j} \rho_i\rho_j &\geq \left(\prod_{1\leq i<j\leq n} \rho_i\rho_j\right)^{\frac{1}{\binom{n}{2}}} \\
&= \left(\prod_{i=1}^n \rho_i\right)^{\frac{n-1}{\binom{n}{2}}} = \left(\prod_{i=1}^n \rho_i\right)^{\frac{2}{n}} \\
&= \Delta^{\frac{2}{n}}.
\end{aligned}$$

Now, we get

$$\begin{aligned}
E_{RQ_C}^2(G) &= \left(\sum_{i=1}^n \rho_i\right)^2 = \sum_{i=1}^n \rho_i^2 + 2\sum_{1\leq i<j\leq n} \rho_i\rho_j \\
&\geq \sigma_2(G) + 2H_2(G) + \tau(G) + 2\sum_{v\in C} Tr'(v) + n(n-1)\Delta^{\frac{2}{n}}.
\end{aligned}$$

Theorem 23 *If* ρ_1 *is the largest eigenvalue of* $RQ_C(G)$, *then* $\rho_1 \geq \dfrac{4H(G)+\tau(G)}{n}$.

Proof. Let $X = \underbrace{(1,1,\ldots,1)}_{n}^{T}$ be the all one vector. Then, by the Rayleigh Principle (see [4]),

$$\begin{aligned}\rho_1 &\geq \frac{X^T RQ_C(G) X}{X^T X} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}}{n}\\ &= \frac{2\sum_{1\leq i<j\leq n}\dfrac{1}{d(v_i,v_j)} + \displaystyle\sum_{i=1, v_i\notin C}^{n} Tr'_{v_i} + \sum_{i=1, v_i\in C}^{n}(1+Tr'_i)}{n}\\ &= \frac{4H(G)+\tau(G)}{n}.\end{aligned}$$

□

Using Lemmas 17, 18 and setting $a_i = 1$ and $b_i = \rho_i$, we get the following two lower bounds for E_{RQ_C} of a graph G.

Theorem 24 *Let* G *be a connected graph which* $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n$ *are the eigenvalues of MCRDSL matrix of* G. *Then*

$$E_{RQ_C}(G) \geq \sqrt{n\left(\sigma_2(G) + 2H_2(G) + \tau(G) + 2\sum_{v\in C} Tr'(v)\right) - \frac{n^2}{4}(\rho_1-\rho_n)^2}, \quad (3)$$

and

$$E_{RQ_C}(G) \geq \frac{2\sqrt{\rho_1\rho_n}}{\rho_1+\rho_n}\sqrt{n\left(\sigma_2(G) + 2H_2(G) + \tau(G) + 2\sum_{v\in C} Tr'(v)\right)}. \quad (4)$$

Lemma 25 [6] *If* a_i *and* $b_i, 1 \leq i \leq n$, *are non-negative real numbers for which there exist real numbers* a, b, A *and* B, *so that for each* $i = 1, \ldots, n$, *we have* $a \leq a_i \leq A$ *and* $b \leq b_i \leq B$. *Then*

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \alpha(n)(A-a)(B-b),$$

where $\alpha(n) = n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$, *while* $[x]$ *denotes integer part of a real number* x. *Equality holds if and only if* $a_1 = a_2 = \cdots = a_n$ *and* $b_1 = b_2 = \cdots = b_n$.

Another lower bound is obtained for E_{RQ_C} of a graph by applying Lemma 25 and setting $a_i = b_i = \rho_i$, $a = b = \rho_n$ and $A = B = \rho_1$.

Theorem 26 *Let* G *be a connected graph and* $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n$ *be the eigenvalues of* $RQ_C(G)$*. Then*

$$E_{RQ_C}(G) \geq \sqrt{n\left(\sigma_2(G) + 2H_2(G) + \tau(G) + 2\sum_{v \in C} Tr'(v)\right) - \alpha(n)(\rho_1 - \rho_n)^2}. \quad (5)$$

Corollary 27 *Since* $\alpha(n) = n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}]) \leq \frac{n^2}{4}$*, then according to (5), we have that*

$$\begin{aligned} E_{RQ_C}(G) &\geq \sqrt{n\left(\sigma_2(G) + 2H_2(G) + \tau(G) + 2\sum_{v \in C} Tr'(v)\right) - \alpha(n)(\rho_1 - \rho_n)^2} \\ &\geq \sqrt{n\left(\sigma_2(G) + 2H_2(G) + \tau(G) + 2\sum_{v \in C} Tr'(v)\right) - \frac{n^2}{4}(\rho_1 - \rho_n)^2}. \end{aligned}$$

This means that inequality (5) is stronger than inequality (3).

Theorem 28 *Let* G *be a connected graph and* $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n$ *be the eigenvalues of* $RQ_C(G)$*. Then*

$$E_{RQ_C}(G) \geq \frac{n\rho_1\rho_n + \sigma_2(G) + 2H_2(G) + \tau(G) + 2\sum_{v \in C} Tr'(v)}{\rho_1 + \rho_n}. \quad (6)$$

Proof. The result follows by setting $a_i = 1$, $b_i = \rho_i$ $R = \rho_1$ and $r = \rho_n$ in the Lemma 19. □

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References

[1] C. Adiga, A. Bayad, I. Gutman, S. A. Srinivas, The minimum covering energy of a graph, *Kragujevac J. Sci.*, **34** (2012), 39–56. ⇒219, 220, 229

[2] A. Alhevaz, M. Baghipur, H. S. Ramane, Computing the reciprocal distance signless Laplacian eigenvalues and energy of graphs, *Le Matematiche*, (2018), in press. ⇒221

[3] N. Alon, Eigenvalues and expanders, *Combinatorica*, **6** (1986), 83–96. ⇒223

[4] G. E. Backus, J. F. Gilbert, Numerical applications of a formalism for geophysical inverse problems, *Geophys. J. R. Ustr. Soc.*, **13** (1967), 247–276. ⇒237

[5] J. A. Bondy, U. S. R. Murty, *Graph Theory with Applications*, Springer, Macmillan, New York, 1976. 1976. ⇒232

[6] M. Biernacki, H. Pidek, C. Ryll-Nardzewski, Sur une inégalité entre des intégrales définies. (French) *Ann. Univ. Mariae Curie-Sklodowska. Sect. A.* **4** (1950), 1–4. ⇒237

[7] Z. Chen, Spectra of extended double cover graphs, *Czechoslovak J. Math.*, **54** (2004), 1077–1082. ⇒223

[8] J. B. Diaz, F. T. Metcalf, Stronger forms of a class of inequalities of G. Pólya–G. Szegö, and L.V. Kantorovich, *Bull. Amer. Math. Soc.*, **69** (1963), 415–418. ⇒ 234

[9] K. Ch. Das, K. Xu, I. Gutman, On Zagreb and Harary Indices, *MATCH Commun. Math. Comput. Chem.*, **70** (2013), 301–314 ⇒231, 232

[10] K. Ch. Das, B. Zhou, N. Trinajstić, Bounds on Harary index, *J. Math. Chem.*, **46** (2009), 1377–1393. ⇒231, 233

[11] P. J. Davis, *Circulant Matrices*, John Wiley and Sons, New York, 1979. ⇒222

[12] I. Gutman, The energy of a graph, *Ber. Math-Statist. Sekt. Forschungsz. Graz* **103** (1978), 1–22. ⇒219

[13] I. Gutman, *The energy of a graph: old and new results*, Algebraic Combinatorics and Applications, A. Betten, A. Kohnert, R. Laue and A. Wassermann, eds., Springer, Berlin, (2001), pp. 196–211. ⇒219

[14] I. Gutman, B. Zhou, Laplacian energy of a graph, *Linear Algebra Appl.*, **414** (2006), 29–37. ⇒219

[15] A. Ilić, G. Yu, L. Feng, The Harary index of trees, *Utilitas Math.*, **87** (2012), 21–32. ⇒231

[16] G. Indulal, R. Balakrishnan, Distance spectrum of Indu-Bala product of graphs, *AKCE Int. J. Graphs Combin.*, **13** (2016), 230–234. ⇒224

[17] O. Ivanciuć, T.S. Balaban, A.T. Balaban, Reciprocal distance matrix, related local vertex invariants and topological indices, *J. Math. Chem.* **12** (1993), 309–318. ⇒220, 221

[18] N. Ozeki, On the estimation of inequalities by maximum and minimum values, *J. College Arts Sci. Chiba Univ.*, **5**(2) (1968), 199–203. ⇒234

[19] D. Plavšić, S. Nikolić, N. Trinajstić, Z. Mihalić, On the Harary index for the characterization of chemical graphs, *J. Math. Chem.*, **12** (1993), 235–250. ⇒ 221

[20] G. Pólya, G. Szegö, *Problems and Theorems in Analysis, Series, Integral Calculus, Theory of Functions* (*Classics in Mathematics*), Springer, Berlin, 1972. ⇒ 234

[21] M. R. Rajesh Kanna, B. N. Dharmendra, G. Sridhara, Minimum covering distance energy of a graph, *Appl. Math. Sci.*, **7**(11) (2013), 5525–5536. ⇒ 219

[22] M. R. Rajesh Kanna, B. N. Dharmendra, R. Pradeep Kumar, Minimum covering Harary energy of a graph, *Int. J. Pure. Appl. Math.*, **90**(3) (2014), 371–385. ⇒ 219, 220, 229

[23] H. Wang, H. Hua, Note on Laplacian energy of graphs, *MATCH Commun. Math. Comput. Chem.*, **59** (2008), 373–380. ⇒ 219

[24] B. Zhou, I. Gutman, On Laplacian energy of graphs, *MATCH Commun. Math. Comput. Chem.*, **57** (2007), 211–220. ⇒ 219

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