Acta Univ. Sapientiae, Mathematica, **3**, 2 (2011) 135–143

Malmquist-Takenaka functions on local fields

Ilona Simon
University of Pécs
Institute of Mathematics and Informatics
Pécs, Hungary
email: simoni@gamma.ttk.pte.hu

Abstract. The complex variant of the discrete Malmquist-Takenaka system plays an important role in system identification. We introduce the analogue of these functions on two dyadic local fields using the analogue of the Blaschke-functions on these fields. This results a generalization of the discrete Laguerre system. Properties of these systems, Fourier expansion and summability questions are presented.

1 Introduction

The discrete Laguerre functions and their generalizations, the Malmquist-Takenaka and Kautz systems are often used in control theory to identify the transfer function. Let us recall, that the discrete Laguerre functions $L_n^{(a)}$ $(n \in \mathbb{N})$ contain a complex parameter $a \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and can be expressed by the Blaschke functions

$$B_a(z) := \frac{z-a}{1-\bar{a}z} \quad (z \in \mathbb{C}, a \in \mathbb{D}).$$

The discrete Laguerre functions $L_n^{(a)}$ associated to B_a on $\mathbb{C}$ are defined by

$$L_k^{(a)}(z) := m_a(z)B_a^k(z), \text{ where } m_a(z) := \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} \quad (z \in \mathbb{C}, k \in \mathbb{Z})$$

2010 Mathematics Subject Classification: 43A25,43A55,11F85,42C10,33C47
Key words and phrases: abstract harmonic analysis, p-adic theory, local fields, orthogonal functions, Fourier series, summability

for $a \in \mathbb{D}$. The boundary of $\mathbb{D}$ is denoted by $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

The discrete Malmquist-Takenaka functions $\Psi_n^{(p)}$ on $\mathbb{C}$ are defined by

$$\Psi_0^p(z) := \frac{\sqrt{1-|a_0|^2}}{1-\bar{a}_0 z}, \qquad \Psi_n^{(p)}(z) := \frac{\sqrt{1-|a_n|^2}}{1-\bar{a}_n z} \prod_{j=0}^{n-1} B_{a_j}(z), \quad (z \in \mathbb{C}, k \in \mathbb{Z})$$

for $(a_j \in \mathbb{D}, j \in \mathbb{N})$ and $p = (a_0, a_1, a_2, \dots)$.

The discrete Malmquist-Takenaka system is orthogonal with respect to the scalar product $\langle F, G \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{it}) \overline{G}(e^{it})\ dt$. Note, that using the same parameters $a_j = a$ $(j \in \mathbb{N})$, the $\Psi_n^{(p)}$ functions give the discrete Laguerre system $(L_n^{(a)}, n \in \mathbb{N})$. For more on these systems see [1].

The analogue of the discrete Laguerre function is constructed in [4] as a composition of the corresponding characters and the Blaschke functions, inspired by the fact, that if a belongs to $\mathbb{D}$, then B_a is a bijection on $\mathbb{T}$, and B_a can be written in the form (see [1])

$$B_a(e^{is}) = e^{i\beta_a(s)} (s \in \mathbb{R}, a \in \mathbb{D}) \tag{1}$$

with some bijection $\beta_a : [-\pi, \pi] \to [-\pi, \pi]$. Obviously $L_k^{(0)}(z) = z^k$ $(k \in \mathbb{Z})$ coincides with the trigonometric system on $\mathbb{T}$. Thus the discrete Laguerre system except the factor m_a can be obtained from the trigonometric system by an argument transformation $T(z) = B_a(z) (z \in \mathbb{T})$.

We will construct the analogue of the discrete Malmquist-Takenaka functions starting from the generator system of the characters of the dyadic and 2-adic group and using an argument transformation. This is a UDMD product system, thus also a complete orthonormal system, which gives the discrete Laguerre system for identical parameters $a_n = a$ $(n \in \mathbb{N})$. Fourier expansion with respect these systems and summability questions are examined.

2 The Blaschke functions on the 2-series and on the 2-adic field

We use the basic notations, definitions and the description of the algebraic structure of the handbooks [3] and [2]. Denote by

$$\mathbb{B} := \left\{ a = (a^{(j)}, j \in \mathbb{Z}) \mid a^{(j)} \in \{0,1\} \text{ and } \lim_{j \to -\infty} a^{(j)} = 0 \right\}$$

the set of bytes, and by $\mathbb{A} := \{0, 1\}$ the set of bits. The numbers $a^{(j)}$ are called the additive digits of $a \in \mathbb{B}$. Also use the notion: $\mathbb{P} := \mathbb{N}\setminus\{0\}$. The zero element of $\mathbb{B}$ is $\theta := (x^{(j)} \in \mathbb{Z})$ where $x^{(j)} = 0$ for $j \in \mathbb{Z}$, that is, $\theta = (\cdots, 0, 0, 0, \cdots)$. The order of a byte $x \in \mathbb{B}$ is defined in the following way: For $x \neq \theta$ let $\pi(x) := n$ if and only if $x^{(n)} = 1$, and $x^{(j)} = 0$ for all $j < n$, furthermore set $\pi(\theta) = +\infty$. The norm of a byte x is introduced by the following rule: $\|x\| := 2^{-\pi(x)}$ for $x \in \mathbb{B} \setminus \{\theta\}$, and $\|\theta\| := 0$.

The sets $I_n(x) := \{y \in \mathbb{B} : y^{(k)} = x^{(k)}$ for $k < n\}$ are the intervals in $\mathbb{B}$ of rank $n \in \mathbb{Z}$ and center $x \in \mathbb{B}$. Consider $\mathbb{I}_n := \{x \in \mathbb{B} : \|x\| \leqq 2^{-n}\}$ $(n \in \mathbb{Z})$. $\mathbb{I} := \mathbb{I}_0$ can be identified with the set of sequences $\mathbb{I} = \{a = (a^{(j)}, j \in \mathbb{N}) |\ a^{(j)} \in \mathbb{A}\}$ via the map $(\dots, 0, 0, a^{(0)}, a^{(1)}, \dots) \to (a^{(0)}, a^{(1)}, \dots)$.

The 2-series (or logical) sum $a \overset{\circ}{+} b$ and product $a \circ b$ of elements $a, b \in \mathbb{B}$ are defined by

$$a \overset{\circ}{+} b := \Big(a^{(n)} + b^{(n)} \pmod 2,\ n \in \mathbb{Z}\Big)$$
$$a \circ b := (c^{(n)}, n \in \mathbb{Z}),\ \text{where } c^{(n)} := \sum_{k\in\mathbb{Z}} a^{(k)} b^{(n-k)} \pmod 2 \quad (n \in \mathbb{Z}).$$

$(\mathbb{B}, \overset{\circ}{+}, \circ)$ is a non-Archimedian normed field, i.e. $\|a \overset{\circ}{+} b\| \leqq \max\{\|a\|, \|b\|\}$, $|a \circ b\| = \|a\|\,\|b\| \quad (a, b \in \mathbb{B})$. The multiplicative identity of $\mathbb{B}$ is the element $e := (\delta_{n0}, n \in \mathbb{N})$.

The (logical) Blaschke function with parameter $a \in \mathbb{I}_1$ is defined in [4] by:

$$B_a(x) := \frac{x \overset{\circ}{+} a}{e \overset{\circ}{+} a \circ x} \quad (x \in \mathbb{I}).$$

Set $y = B_a(x)$ with $x \in \mathbb{I}, a \in \mathbb{I}_1$. Then we have $y = x \overset{\circ}{+} a \overset{\circ}{+} y \circ a \circ x$ and consequently for the n-th digit of y we get

$$\begin{cases} y^{(n)} = 0, \text{ for } n < 0, \\ y^{(n)} = x^{(n)} + a^{(n)} + (y \circ a \circ x)^{(n)} \pmod 2, \text{ for } n \geqq 0. \end{cases}$$

This is recursion for the bits of $y = B_a(x)$, since to compute $(y \circ a \circ x)^{(n)}$ we only need $y^{(k)}$-s with $k < n$. The bits $y^{(n)} = (B_a(x))^{(n)}$ can be written in the form

$$y^{(n)} = x^{(n)} + a^{(n)} + f_n(x^{(0)}, \cdots, x^{(n-1)}) \pmod 2 \tag{2}$$

where the functions $f_n : \mathbb{A}^n \to \mathbb{A}$ $(n = 1, 2, \cdots)$ depend only on the bits of a. The definition of the logical Blaschke functions and details about the recursion are considered in [4].

The 2-adic (or arithmetic) sum $a \overset{\bullet}{+} b$ of elements $a = (a^{(n)}, n \in \mathbb{Z}), b = (b^{(n)}, n \in \mathbb{Z}) \in \mathbb{B}$ is defined by $a \overset{\bullet}{+} b := (s_n, n \in \mathbb{Z})$, where the bits $q_n, s_n \in \mathbb{A}$ $(n \in \mathbb{Z})$ are obtained recursively as follows: $q_n = s_n = 0$ for $n < m := \min\{\pi(a), \pi(b)\}$, and

$$a^{(n)} + b^{(n)} + q_{n-1} = 2q_n + s_n \qquad \text{for } n \geq m.$$

The 2-adic (or arithmetic) product of $a, b \in \mathbb{B}$ is $a \bullet b := (p_n, n \in \mathbb{Z})$, where the sequences $q_n \in \mathbb{N}$ and $p_n \in \mathbb{A}$ $(n \in \mathbb{Z})$ are defined recursively by $q_n = p_n = 0$ $(n < m := \pi(a) + \pi(b))$ and

$$\sum_{j=-\infty}^{\infty} a^{(j)} b^{(n-j)} + q_{n-1} = 2q_n + p_n \qquad (n \geq m).$$

Note, that $\pi(a \bullet b) = \pi(a) + \pi(b)$ and $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ is a non-Archimedian normed field.

For $x \in \mathbb{I}$ and $a \in \mathbb{I}_1$ we have that $e \overset{\bullet}{-} a \bullet x \neq \theta$, thus $e \overset{\bullet}{-} a \bullet x$ has a multiplicative inverse in $\mathbb{B}$. The (arithmetical) Blaschke function with parameter $a \in \mathbb{I}_1$ is defined in [4] by:

$$B_a(x) := (x \overset{\bullet}{-} a) \bullet (e \overset{\bullet}{-} a \bullet x)^{-1} = \frac{x \overset{\bullet}{-} a}{e \overset{\bullet}{-} a \bullet x} \qquad (x \in \mathbb{I}). \tag{3}$$

The Blaschke function $B_a : \mathbb{I} \to \mathbb{I}$ is a bijection for any $a \in \mathbb{I}_1$ on $\mathbb{I}$ and on $\mathbb{S}_0$. The maps B_a $(a \in \mathbb{I}_1)$ form a commutative group with respect to the function composition. The byte $y = B_a(x)$ can be given in a recursive form (2) like on the logical field. The definition of the arithmetical Blaschke functions and details about the recursion are considered in [4].

Consider the Haar-measure μ on the fields $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ and $(\mathbb{B}, \overset{\circ}{+}, \circ)$. More details on the algebraic structure can be found in [3].

In the following we will present UDMD systems, which are considered in [3]. Denote with $\mathcal{A}$ the σ-algebra generated by the intervals $I_n(a)$ $(a \in \mathbb{I}, n \in \mathbb{N})$. Let $\lambda(I_n(a)) := 2^{-n}$ be the measure of $I_n(a)$. Extending this measure to $\mathcal{A}$ we get a probability measure space $(\mathbb{I}, \mathcal{A}, \lambda)$. Let $\mathcal{A}_n$ be the sub-σ-algebra of

$\mathcal{A}$ generated by the intervals $I_n(a)$ $(a \in \mathbb{I})$. Let $L(\mathcal{A}_n)$ denote the set of $\mathcal{A}_n$-measurable functions on $\mathbb{I}$. The conditional expectation of an $f \in L^1(\mathbb{I})$ with respect to $\mathcal{A}_n$ is of the form

$$(\mathcal{E}_n f)(x) := \frac{1}{\lambda(I_n(x))} \int_{I_n(x)} f d\lambda.$$

A sequence of functions $(f_n, n \in \mathbb{N})$ is called a dyadic martingale if each f_n is $\mathcal{A}_n$-measurable and

$$(\mathcal{E}_n f_{n+1}) = f_n \qquad (n \in \mathbb{N}).$$

The sequence of martingale differences of f_n $(n \in \mathbb{N})$ is the sequence

$$\phi_n := f_{n+1} - f_n \qquad (n \in \mathbb{N}).$$

We notice that every dyadic martingale difference sequence has the form $\phi_n = r_n g_n$ $(n \in \mathbb{N})$ where $(g_n, n \in \mathbb{N})$ is a sequence of functions such that each g_n is $\mathcal{A}_n$-measurable and $(r_n, n \in \mathbb{N})$ denotes the Rademacher system on $\mathbb{I}$:

$$r_n(x) := (-1)^{x^{(n)}} \ (n \in \mathbb{N}).$$

The martingale difference sequence $(\phi_n, n \in \mathbb{N})$ is called a unitary dyadic martingale difference sequence or a UDMD sequence if $|\phi_n(x)| = 1$ $(n \in \mathbb{N})$. Thus $(\phi_n, n \in \mathbb{N})$ is a UDMD sequence if and only if

$$\phi_n = r_n g_n, \ \ g_n \in L(\mathcal{A}_n), \ \ |g_n| = 1 \ \ (n \in \mathbb{N}). \tag{4}$$

A system $\psi = (\psi_m, m \in \mathbb{N})$ is said to be a UDMD product system if it is a product system generated by a UDMD system, i.e., there is a UDMD system $(\phi_n, n \in \mathbb{N})$ such that for each $m \in \mathbb{N}$, with binary expansion is given by $m = \sum_{j=0}^{\infty} m^{(j)} 2^j$ $(m^{(j)} \in \mathbb{A}, j \in \mathbb{N})$, the function ψ_m satisfies

$$\psi_m = \prod_{j=0}^{\infty} \phi_j^{m^{(j)}} \qquad (m \in \mathbb{N}).$$

The author constructed orthonormal systems in this way inspired by martingales in [4, 5].

3 The discrete Malmquist-Takenaka functions on the 2-series and 2-adic field

Let us define the discrete Malmquist-Takenaka functions with parameters $p = (a_0, a_1, \ldots)$ $(a_i \in \mathbb{I}_1, i \in \mathbb{N})$ on the 2-series field $(\mathbb{I}, \overset{\circ}{+}, \circ)$ in the following way: the system $\left(\Psi_k^{(p)}, k \in \mathbb{N}\right)$ is the product system generated by

$$(\Phi_{n,a_n} := r_n \circ B_{a_n}, n \in \mathbb{N}) \tag{5}$$

That is, $\Psi_k^{(p)}(x) = \prod_{j=0}^{\infty} \left[r_j(B_{a_j}(x))\right]^{k^{(j)}}$.

Theorem 1 *For every $a_n \in \mathbb{I}_1$ $(n \in \mathbb{N})$) the functions $\Phi_{n,a_n}(x) = r_n(B_{a_n}(x))$ $(x \in \mathbb{I},\ n \in \mathbb{N})$ form a UDMD system on $(\mathbb{I}, \overset{\circ}{+}, \circ)$.*

Proof. Using recursion form (2) of $y = B_{a_n}(x)$ we get

$$\Phi_{n,a_n}(x) = (-1)^{y^{(n)}} = (-1)^{x^{(n)}}(-1)^{(a_n)^{(n)} + f_n(x^{(0)}, \cdots, x^{(n-1)})} = r_n(x) g_n(x)$$

where $g_n(x) := (-1)^{(a_n)^{(n)} + f_n(x^{(0)}, \cdots, x^{(n-1)})}$ is $\mathcal{A}_n$-measurable, $g_n \in L(\mathcal{A}_n)$. Clearly, $|g_n(x)| = 1$ $(x \in \mathbb{I})$, thus $(\Phi_{n,a_n}, n \in \mathbb{N})$ is a UDMD sequence. $\square$

Corollary 1 *The logical Malmquist-Takenaka system, that is the product system $(\Psi_k^{(p)}, k \in \mathbb{N})$ generated by the system $(\Phi_{n,a_n}, n \in \mathbb{N})$ is a UDMD product system, consequently it is a complete orthonormal system on $(\mathbb{I}, \overset{\circ}{+}, \circ)$.*

We consider $\epsilon(t) := \exp(2\pi i t)$ $(t \in \mathbb{R})$. We will use the functions $(\nu_{2^n}(x), n \in \mathbb{N})$:

$$\nu_{2^n}(x) := \epsilon\left(\frac{x^{(n)}}{2} + \frac{x^{(n-1)}}{2^2} + \cdots + \frac{x^{(0)}}{2^{n+1}}\right) \qquad (x \in \mathbb{I}, n \in \mathbb{N}), \tag{6}$$

known as a generator system of the characters of the group $(\mathbb{I}, \overset{\bullet}{+})$. Let us define the arithmetical Malmquist-Takenaka functions with parameters $p = (a_0, a_1, \ldots)$ $(a_n \in \mathbb{I}_1, n \in \mathbb{N})$ on the 2-adic field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ in the following way: the system $\left(\Psi_k^{(p)}, k \in \mathbb{N}\right)$ is now the product system generated by

$$(\Phi_{n,a_n} := \nu_{2^n} \circ B_{a_n}, n \in \mathbb{N}) \tag{7}$$

That is, $\Psi_n^{(p)}(x) = \prod_{j=0}^{\infty} \left[\nu_{2^j}(B_{a_j}(x))\right]^{n^{(j)}}.(x \in (\mathbb{I}, \overset{\bullet}{+}, \bullet))$

Theorem 2 *For every* $a_n \in \mathbb{I}_1$ $(n \in \mathbb{N})$ *the functions* $\Phi_{n,a_n}(x) = \nu_{2^n}(B_{a_n}(x))$ $(x \in \mathbb{I},\ n \in \mathbb{N})$ *form a UDMD system on* $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$.

The proof is similar like on the 2-series field.

Corollary 2 *The arithmetical Malmquist-Takenaka functions, the product system* $(\Psi_k^{(p)}, k \in \mathbb{N})$ *generated by the system* $(\Phi_{n,a_n}, n \in \mathbb{N})$ *is a UDMD product system, consequently it is a complete orthonormal system on* $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$.

In the following we consider the corresponding Malmquist-Takenaka-systems on both fields $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ and $(\mathbb{I}, \overset{\circ}{+}, \circ)$.

4 Summability

The Malmquist-Takenaka-Fourier coefficients of an $f \in L^q(\mathbb{I})$ $(1 \le q \le \infty)$ are defined by

$$\widehat{f^{(p)}}(n) := \int_{\mathbb{I}} f(x)\Psi_n^{(p)}(x)d\mu(x). \quad (n \in \mathbb{N})$$

The n-th partial sums of the Malmquist-Takenaka-Fourier series $S^{(p)}f$ is now

$$S_n^{(p)}f := \sum_{k=0}^{n-1} \widehat{f^{(p)}}(k)\Psi_k^{(p)} \ (n \in \mathbb{N}^*).$$

Furthermore, the Malmquist-Takenaka-Cesaro (or $(MT - C, 1)$) means of $S^{(p)}f$ are defined by $\sigma_0^{(p)}f := 0$ and

$$\sigma_n^{(p)}f := \frac{1}{n}\sum_{k=1}^{n} S_k^{(p)}f, \qquad (n \in \mathbb{N}^*)$$

for $p = (a_0, a_1, \dots)$ with $a_n \in \mathbb{I}_1$ $(n \in \mathbb{N})$, $f \in L^q(\mathbb{I})$.

Properties of UDMD product systems are valid for the Malmquist-Takenaka system $(\Psi_k^{(p)}, k \in \mathbb{N})$, thus applying the general theorem on convergence presented in [3], holds the following:

Theorem 3 *For any* $f \in L^q(\mathbb{I})$ $(1 \le q < \infty)$ *we have*

$$\begin{aligned} &\lim_{n\to\infty} \|S_{2^n}^{(p)}f - f\|_q = 0, \quad \textit{and} \\ &\lim_{n\to\infty} \|\sigma_n^{(p)}f - f\|_q = 0. \end{aligned} \tag{8}$$

Moreover, (8) holds for $q = \infty$ *when* f *is continuous on* $\mathbb{I}$.

Clearly, a.e. convergence holds for $S_{2^n}^{(p)}f$ for any integrable f and for $S_m^{(p)}f$, ($m \in \mathbb{P}$) if $f \in L^q(\mathbb{I})$ and $q > 1$. This is a consequence of a general theorem in [3], pp.101-105 or [2]. This holds for $q = 1$ with identical parameters $a_n = a \in \mathbb{I}_1$ ($n \in \mathbb{N}$), that is, in the case of the discrete Laguerre system $L_n^{(a)}(x)$. See [4].

We will see in the next proposition, that the Malmquist-Takenaka system is a generalization of the discrete Laguerre system on both fields defined in [4] as follows.

The functions corresponding the trigonometric system mentioned in the Introduction, are the characters of the corresponding groups. Namely, the Walsh-Paley functions ($w_k, k \in \mathbb{N}$) defined by

$$w_k(x) = \prod_{n=0}^{\infty} r_n(x)^{k^{(n)}} = (-1)^{\sum_{j=0}^{+\infty} k^{(j)} x^{(j)}} \ (x \in \mathbb{I},\ k = \sum_{j=0}^{\infty} k^{(j)} 2^j \in \mathbb{N}\ (k^{(j)} \in \mathbb{A})), \tag{9}$$

are the characters of $(\mathbb{I}, \overset{\circ}{+})$. In particular, the Walsh-Paley functions form a product system generated by the Rademacher system ($r_n, n \in \mathbb{N}$).

And the functions ($v_k, k \in \mathbb{N}$) are the characters of $(\mathbb{I}, \overset{\bullet}{+})$ defined as the product system generated by the functions ($v_{2^n}(x), n \in \mathbb{N}$) defined in (6).

The discrete Laguerre functions associated to B_a are defined in the following way:

$$L_k^{(a)}(x) := w_k(B_a(x)) \ (k \in \mathbb{N}, x \in (\mathbb{I}, \overset{\circ}{+}))$$

and

$$L_k^{(a)}(x) := v_k(B_a(x)) \ (k \in \mathbb{N}, x \in (\mathbb{I}, \overset{\bullet}{+}))$$

for any $a \in \mathbb{I}_1$.

Proposition 1 *Using identical parameters* $a_n = a \in \mathbb{I}_1$ ($n \in \mathbb{N}$) *the Malmquist-Takenaka functions* $\Psi_n^{(p)}(x)$ *give the discrete Laguerre system* $L_n^{(a)}(x)$ *on both fields* $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ *and* $(\mathbb{I}, \overset{\circ}{+}, \circ)$.

Clearly, with the special identical parameters $a_n = \theta$ ($n \in \mathbb{N}$) this method gives the characters of the corresponding field. Thus the Malmquist-Takenaka system is also a generalization of the character system of the corresponding additive group.

References

[1] J. Bokor, F. Schipp, Approximate linear H^∞ identification in Laguerre and Kautz basis, *Automatica J. IFAC*, **34** (1998), 463–468.

[2] F. Schipp, W. R. Wade, P. Simon, J. Pál, *Walsh Series, An Introduction to Dyadic Harmonic Analysis*, Adam Hilger Ltd., Bristol and New York, (1990).

[3] F. Schipp, W. R. Wade, *Transforms on normed fields*, Leaflets in Mathematics, Janus Pannonius University Pécs, 1995.

[4] I. Simon, Discrete Laguerre functions on the dyadic fields, Pure Math. Appl. (PU.M.A.), **17** (2006), 459–468.

[5] I. Simon, The characters of the Blaschke-group, Stud. Univ. Babes-Bolyai Math., **54** (2009), 149–160.

Received: July 23, 2010