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Longest runs in coin tossing. Comparison of recursive formulae, asymptotic theorems, computer simulations

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Abstract. The coin tossing experiment is studied. The length of the longest head run can be studied by asymptotic theorems [3, 4], by recursive formulae [7, 11] or by computer simulations [1]. The aim of the paper is to compare numerically the asymptotic results, the recursive formulae, and the simulation results. Moreover, we consider also the longest run (i.e. the longest pure heads or pure tails). We compare the distribution of the longest head run and that of the longest run.

1 Introduction

The success-run in a sequence of Bernoulli trials has been studied in a large number of papers. Consider the well-known coin tossing experiment. Let R_n denote the length of the longest run of consecutive heads (longest head run). Moreover, let R'_n denote the longest run of consecutive heads or consecutive tails (longest run). The asymptotic distribution of R_n is studied in several

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papers (see, e.g. [3, 4, 5, 6, 9, 10]). However, these results give approximations being accurate for large enough n. Precise values of the distributions can be calculated by certain recursive formulae (see, e.g. [2, 7, 11]). However it is difficult and slow to calculate them numerically for large n. The distributions of R_n and $R_n^{'}$ can be calculated by simulations, as well. Simulations can be applied both for small and large values of n, but they offer only approximations (which can be improved by using large number of repetitions). The comparison of the asymptotic theorems and the simulations are given in [1].

In this paper we compare numerically the asymptotic theorems, the recursive formulae and the simulations. As the case of a fair coin is well-known, we focus on a biased coin (i.e. when $P(\text{head}) = p \neq \frac{1}{2}$). Moreover, as our aim is to obtain precise numerical results, we emphasize the importance of the recursive formulae. We give detailed proofs for the (known) recursive formulae. Finally, we remark that most results in the literature concern the longest head run (i.e. R_n) but in practice people are interested in the longest run (i.e. $R_n^{'}$). Therefore, we concentrate mainly on $R_n^{'}$.

The numerical results show that the asymptotic theorems give bad results for small n (i.e. $n \leq 250$) and give practically precise results for large n (i.e. $n \geq 3000$). It can also be seen that for large n the distribution of $R_n^{'}$ is close to that of R_n if $p > \frac{1}{2}$ (p is the probability of a head).

We present recursion formulae offering the exact distribution of the longest run of heads (Section 2), and the distribution of the longest whatever run (Section 3). We consider the situation in which the probability of a head can take any value in $(0, 1)$.

2 The longest head run

Consider n independent tosses of a (biased) coin, and let R_n denote the length of the longest head run. The (cumulative) distribution function of R_n is the following

$$F_n(x) = P(R_n \leq x) = \sum_{k=0}^{n} C_n^{(k)}(x) p^k q^{n-k}, \tag{1}$$

where $C_n^{(k)}(x)$ is the number of strings of length n where exactly k heads occur, but not more than x heads occur consecutively. We have the following recursive formula for $C_n^{(k)}(x)$.

Proposition 1 *(See [11])*

$$
C_n^{(k)}(x) = \begin{cases} \sum\limits_{j=0}^{x} C_{n-1-j}^{(k-j)}(x), & \textit{if} \quad x < k < n, \\ \binom{n}{k}, & \textit{if} \quad 0 \le k \le x, \\ 0, & \textit{if} \quad x < k = n. \end{cases} \tag{2}
$$

Proof. If $x < k = n$, then $C_n^k(x) = 0$, because in this case all elements (being more than x) are heads, so there is no series containing less than or equal to x heads consecutively.

If $0 \le k \le x$, then the value of $C_n^k(x)$ is equal to the binomial coefficient. In this case there are less than or equal to x heads among n elements and we have to count those cases when the length of the longest head run is less than or equal to x. All possible sequences have this property, therefore $C_n^{(k)}(x) = \binom{n}{k}$.

If $x < k < n$, then we need to consider the following. Our series may start with $j = 0, 1, 2, \ldots, x$ heads, then must be one tail, then a sequence follows containing $k-j$ heads among the remaining $n-j-1$ objects. In this sequence the length of the longest head run must be less than or equal to x. The number of these sequences equals exactly $C_{n-1-j}^{(k-j)}(x)$.

$$
\underbrace{\text{H} \ \ldots \ \text{H}}_{j \text{ heads}} \quad \text{T} \qquad \underbrace{\ldots \quad \text{H} \quad \ldots \quad \text{T} \quad \ldots}_{\substack{n-j-1 \text{ elements, containing } k-j \text{ heads,} \\ \text{and the length of the longest head run is less than or equal to } x}} \qquad \square
$$

The following table displays the values of $C_n^k(3)$ for $n \le 8$.

8									0
7								0	0
6							0	1	10
5						0	**2**	**12**	40
4					0	**3**	12	31	65
3				1	**4**	10	20	35	56
2			1	**3**	6	10	15	21	28
1		1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1	1
k/n	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*

The first four rows of the table ($k = 0, 1, 2, 3$) are part of Pascal's triangle. Entries above that four rows are computed by taking diagonal sums of four entries from the rows and columns below and to the left. The 'hockey stick' (printed in boldface in the table) illustrates the case $C_7^{(5)}(3) = 2{+}3{+}4{+}3 = 12$. Tossing a biased coin 8 times, we now have the probability of obtaining not more than three consecutive heads: $F_8(3) = C_8^{(0)}(3)p^0q^8 + C_8^{(1)}(3)p^1q^7 + \ldots + C_8^{(7)}(3)p^7q^1 + C_8^{(8)}(3)p^8q^0 = 1q^8 + 8pq^7 + 28p^2q^6 + 56p^3q^5 + 65p^4q^4 + 40p^5q^3 + 10p^6q^2 + 0 + 0$. Knowing the value of p we can calculate the exact result.

The asymptotic behaviour is described by the following theorem.

Theorem 1 (See [5].) *Let* $\mu(n) = -\frac{\log n}{\log p}$, $q = 1 - p$ *and let* W *have a double exponential distribution (i.e.* $P(W \leq t) = \exp(-\exp(-t))$*), then uniformly in* t:

$$P\left(R_n - \mu(qn) \leq t\right) - P\left(\left[\frac{W}{-\log p} + \{\mu(qn)\}\right] - \{\mu(qn)\} \leq t\right) \to 0 \qquad (3)$$

as $n \to \infty$ *where* $[a]$ *denotes the integer part of* a *and* $\{a\} = a - [a]$.

We emphasize that the above theorem does not offer a limiting law for $R_n - \mu(qn)$ but it gives a sequence of accompanying laws. The distances of the laws in the two sequences converge to 0 (as $n \to \infty$). So the above theorem is a merge theorem. Observe, the periodic property in the sequence of the accompanying laws.

3 The longest run

For a coin with $p \neq 0.5$ the (cumulative) distribution function $F'_n(x)$ is complicated. Let R'_n denote the length of the longest run in the sequence of n coin tossings. That is the maximum of the longest head run and the longest tail run. Let F'_n be the distribution function of R'_n.

$$F'_n(x) = P(R'_n \leq x) = \sum_{k=0}^{n} \overline{C}_n^{(k)}(x)p^k q^{n-k} \qquad (4)$$

where $\overline{C}_n^{(k)}(x)$ is the number of strings of length n with exactly k heads, but not more than x of heads and not more than x of tails occur consecutively (p is the probability of a head and $q = 1 - p$). First consider

$$\overline{C}_{m+k}^{(k)}(x) = C_{x+1}(m, k). \qquad (5)$$

Here $C_t(m,k)$ denotes the number of strings of m indistinguishable objects of type A and k indistinguishable objects of type B in which no t-clump (run of length t) occurs. (A and B may interpret head and tail, respectively.) We have the following recursive formulas for $C_t(m,k)$.

Proposition 2 (See [2].)

$$C_t(m,k) = \sum_{i=0}^{t-1} C_t(m-1,k-i) - \sum_{i=1}^{t-1} C_t(m-t,k-i) + e_t(m,k), \quad (6)$$

where

$$e_t(m,k) = \begin{cases} 1, & \text{if} \quad m=0 \quad \text{and} \quad 0 \le k < t, \\ -1, & \text{if} \quad m=t \quad \text{and} \quad 0 \le k < t, \\ 0, & \text{in all other cases} \end{cases},$$

moreover if $m = k = 0$*, then* $C_t(0,0) = 1$*, if* m *or* k *is negative, then* $C_t(m,k) = 0$.

We give a detailed proof which is not contained in [2].

Proof.
Case $m = 0$.

If $0 \le k < t$, then $C_t(0,k) = 1$, because this means that there is only one type of the elements but the number of objects is less than the length of the run. So there can not be any t run. As the elements are indistinguishable, this means only one order. In this case (6) means $1 = 0 - 0 + 1$. For example: $C_3(0,2) = C_3(-1,2) + C_3(-1,1) + C_3(-1,0) - [C_3(-3,1) + C_3(-3,0)] + 1 = 0 + 0 + 0 - [0 + 0] + 1 = 1$.

If $k \ge t$, then $C_t(0,k) = 0$, because there is only one type of the elements, but the number of objects is greater or equal to the length of the run. So there is no one sequence in which there is no t run. For example: $C_3(0,4) = C_3(-1,4) + C_3(-1,3) + C_3(-1,2) - [C_3(-3,3) + C_3(-3,2)] + 0 = 0 + 0 + 0 - [0 + 0] + 0 = 0$.

In case of $0 < m < t$ our formula (6) is the following.

$$C_t(m,k) = \sum_{i=0}^{t-1} C_t(m-1,k-i) - 0 + 0.$$

Because this case means the following

$$\underbrace{\text{B} \ldots \text{B}}_{i \text{ of B}} \quad \text{A} \quad \underbrace{\ldots \quad \text{B} \quad \ldots \quad \text{A} \quad \ldots}_{\substack{(k-i) \text{ of B, } (m-1) \text{ of A,} \\ \text{and there is no } t\text{-run among them}}}$$

The number of these sequences is: $C_t(m, -1, k-i)$. As $m < t$, so there can not be t run from A, so we do not need to subtract anything. For example: $C_3(2,2) = C_3(1,2) + C_3(1,1) + C_3(1,0) - [C_3(-1,1) + C_3(-1,0)] + 0 = 3 + 2 + 1 - [0 + 0] + 0 = 6$.

The case of $m = t$ and $0 \le k < t$.

This means that there are less than t of B elements, and the number of A elements is equal to t. In this case our formula is the following: $C_t(m,k) = \sum_{i=0}^{t-1} C_t(m-1, k-i) - \sum_{i=1}^{t-1} C_t(0, k-i) - 1$. The first sum consists of $k+1$ positive terms (not t), when the i-th term starts with i of B objects, then follows A, then follows a sequence consisting of $m-1$ A and $k-i$ B and not containing t run.

$$\underbrace{\mathrm{B}\ \dots\ \mathrm{B}}_{i \text{ of B}} \quad \mathrm{A} \quad \underbrace{\quad \dots \quad \mathrm{B} \quad \dots \quad \mathrm{A} \quad \dots \quad}_{\substack{(k-i) \text{ of B, } (m-1) \text{ of A,} \\ \text{and there is no } t\text{-run}}}$$

But there is a 'bad' term in each of them, when the $m = t$ A objects are consecutive. As the second sum consists of k terms, so the above $k+1$ bad cases are subtracted. For example: $C_3(3,2) = C_3(2,2) + C_3(2,1) + C_3(2,0) - [C_3(0,1) + C_3(0,0)] - 1 = 6 + 3 + 1 - [1 + 1] - 1 = 7$.

In the case of $m = t$ and $k \ge t$, our formula is the following

$$C_t(m,k) = \sum_{i=0}^{t-1} C_t(m-1, k-i) - \sum_{i=1}^{t-1} C_t(0, k-i) + 0.$$

If $i = 0$ in the first sum, then our possibility is the following

$$\mathrm{A} \quad \underbrace{\ \dots\ \mathrm{A}\ \dots\ \mathrm{B}\ \dots\ }_{\substack{k \text{ of B, } (m-1) \text{ of A} \\ \text{and there is no } t\text{-run}}}$$

The number of these sequences is $C_t(m-1, k)$. Seemingly there is one 'bad' event among them, when in the second part starts with $m-1$ A objects and they make a t run with the very first A object. But the k B objects are in the end of the second part and they would make a t run, so the above 'bad' situation is not included in $C_t(m-1, k)$.

If $i = 1, 2, \dots, t-1$, then we have

$$\underbrace{\mathrm{B}\ \dots\ \mathrm{B}}_{i \text{ of B}} \quad \mathrm{A} \quad \underbrace{\quad \dots \quad \mathrm{B} \quad \dots \quad \mathrm{A} \quad \dots \quad}_{\substack{(k-i) \text{ of B, } (m-1) \text{ of A,} \\ \text{and there is no } t\text{-run}}}$$

The number of these sequences is $C_t(m-1, k-i)$. But there can be a 'bad' event in this situation, when all objects A ($m = t$) are next to each other, so we have to subtract $C_t(0, k-i)$ (it can be equal to 0 as well). For example: $C_3(3,4) = C_3(2,4) + C_3(2,3) + C_3(2,2) - [C_3(0,3) + C_3(0,2)] + 0 = 6 + 7 + 6 - [0 + 1] + 0 = 18$.

Case $m > 0$ and $m > t$.

Our sequence may start with i (i is less than t) same type objects (for example with B) then follows a different one (A) and ends with a string without t run.

$$\underbrace{B \ \dots \ B}_{i \text{ of } B} \quad A \quad \underbrace{\dots \quad B \quad \dots \quad A \quad \dots}_{\substack{(k-i) \text{ of } B,\ (m-1) \text{ of } A, \\ \text{and there is no } t\text{-run}}}$$

The number of these sequences is: $\sum_{i=1}^{t-1} C_t(m-1, k-i)$.

But among them there may be sequences when there are same A objects after the individual A, so that together there are t consecutive A objects and after them there is no t run

$$\underbrace{B \ \dots \ B}_{i \text{ of } B} \underbrace{A \ \dots \ A}_{t \text{ of } A} \underbrace{\dots \quad B \quad \dots \quad A \quad \dots}_{\substack{(k-i) \text{ of } B,\ (m-t) \text{ of } A \text{ and} \\ \text{there is no } t\text{-run}}}$$

The number of these strings is $\sum_{i=1}^{t-1} C_t(m-t, k-i)$, that we have to subtract from the previous sum. But in these there can be such sequences, when A object stands after the t run of A, so there can be another t run. The number of these can be denoted by $\sum_{i=1}^{t-1} C_t^*(m-t, k-i)$.

What happens is if $i = 0$, so our sequence starts with A? In this case the first object is A and then there is no t run

$$A \quad \underbrace{\dots \ A \ \dots \ B \ \dots}_{\substack{k \text{ of } B,\ (m-1) \text{ of } A \\ \text{and there is no } t\text{-run}}}$$

The number of these strings is $C_t(m-1, k)$. But in these strings there can be some sequences starting with t run and then there is no t run

$$\underbrace{A \ \dots \ A}_{t \text{ of } A,} \underbrace{B \ \dots \ B}_{i \text{ of } B} \underbrace{A \dots \ B \dots \ A \ \dots}_{\substack{(m-t) \text{ of } A,\ (k-i) \text{ of } B \\ \text{and there is no } t\text{-run} \\ (1 \leq i \leq (t-1))}}$$

The numbers of these strings is $\sum_{i=1}^{t-1} C_t^*(m-t, k-i)$, that we have to subtract from the previous sum.

Summarizing our results we get the following

$$C_t(m,k) = \sum_{i=1}^{t-1} C_t(m-1,k-i) - \left\{ \sum_{i=1}^{t-1} C_t(m-t,k-i) - \sum_{i=1}^{t-1} C_t^*(m-t,k-i) \right\} +$$

$$+ \left\{ C_t(m-1,k) - \sum_{i=1}^{t-1} C_t^*(m-t,k-i) \right\} + e_t(m,k).$$

For example: $C_3(5,2) = C_3(4,2) + C_3(4,1) + C_3(4,0) - [C_3(2,1) + C_3(2,0)] + 0 = 6 + 1 + 0 - [3+1] + 0 = 3$.

So recursive formula (6) is satisfied. □

Proposition 3 (See [2].) *Let* $t \geq 2$. *Then*

$$C_t(m,k) = C_t(m-1,k) + C_t(m,k-1) - C_t(m-t,k-1) - C_t(m-1,k-t) \quad (7)$$

$$+ C_t(m-t,k-t) + e_t^*(m,k),$$

where

$$e_t^*(m,k) = \begin{cases} 1, & \text{if} \quad (m,k) = (0,0) \quad \text{or} \quad (m,k) = (t,t), \\ -1, & \text{if} \quad (m,k) = (0,t) \quad \text{or} \quad (m,k) = (t,0), \\ 0, & \text{in all other cases}, \end{cases}$$

moreover if $m = k = 0$, *then* $C_t(0,0) = 1$, *if* m *or* k *is negative, then* $C_t(m,k) = 0$.

Here we give a proof being different from the one in [2].

Proof. Our sequence may start either with A or B

$$\left. \text{A} \quad \underbrace{\ldots \text{A} \ldots \text{B} \ldots}_{(m-1) \text{ of A and } k \text{ of B}} \right\} \text{The number of these sequences is } C_t(m-1,k).$$

$$\left. \text{B} \quad \underbrace{\ldots \text{A} \ldots \text{B} \ldots}_{m \text{ of A and } (k-1) \text{ of B}} \right\} \text{The number of these sequences is } C_t(m,k-1).$$

We have to subtract the number of those sequences in which after the first A element there are $t-1$ consecutive A's (so there is a t-clump) and then there is a different element and there is a string with no t-clump

$$\left.\underbrace{\text{A}\ldots\ \text{A}}_{\text{t of A}}\quad \text{B}\quad \underbrace{\ldots\ \text{A}\ldots\text{B}\ldots}_{(m-t)\text{ of A and }(k-1)\text{ of B}}\right\}\ \text{The number of these sequences is } C_t(m-t,k-1).$$

$$\left.\underbrace{\text{B}\ldots\ \text{B}}_{\text{t of B}}\quad \text{A}\quad \underbrace{\ldots\ \text{A}\ldots\text{B}\ldots}_{(m-1)\text{ of A and }(k-t)\text{ of B}}\right\}\ \text{The number of these sequences is } C_t(m-1,k-t).$$

But these cases contain the following sequences as well.

The sequence starts with t consecutive A's followed with t consecutive B's and ends with a string containing $m-t$ A and $k-t$ B elements and not containing t clump but starting with A. The number of these sequences is $C_t^{(A)}(m-t,k-t)$. Changing the role of A and B we get again $C_t^{(B)}(m-t,k-t)$ sequences. But for the sum of them we have $C_t^{(A)}(m-t,k-t)+C_t^{(B)}(m-t,k-t)=C_t(m-t,k-t)$.

Summarizing the above statements we can get our formula

$$C_t(m,k)=C_t(m-1,k)+C_t(m,k-1)-$$

$$-\{C_t(m-t,k-1)+C_t(m-1,k-t)-C_t(m-t,k-t)\}+e_t^*(m,k).$$

□

To see how these work, let us calculate some data in case where $t=3$ and m and k are less than 10:

$m\diagdown k$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
0	1	1	1	0	0	0	0	0	0	0
1	1	2	3	2	1	0	0	0	0	0
2	1	3	6	7	6	3	1	0	0	0
3	0	2	7	**14**	**18**	16	10	4	1	0
4	0	1	6	18	34	45	43	30	15	5
5	0	0	3	**16**	**45**	**84**	113	114	87	50
6	0	0	1	10	43	**113**	208	285	300	246
7	0	0	0	4	30	114	285	518	720	786
8	0	0	0	1	15	87	300	720	1296	1823
9	0	0	0	0	5	50	246	786	1823	3254

For example $C_3(6,5) = (84+45+16)-(18+14) = 113$. (See the numbers in bold style in the above table.)

The number of terms on the right hand side of (6), increases with t, but in formula (7), the right hand side has only six terms no matter how large t is.

Let p denote the probability of a head. To find the asymptotic behaviour of R'_n, denote by $V_n(p)$ the probability that the longest run in n trials is formed by heads. Then, by Theorem 5 of [8],

$$\lim_{n\to\infty} V_n(p) = \begin{cases} 0, & \text{if} \quad 0 \le p < 1/2, \\ 1, & \text{if} \quad 1/2 < p \le 1. \end{cases} \tag{8}$$

Therefore, if $p > 1/2$, the asymptotic behaviour of R'_n is the same as that of R_n.

It means that "the one with lower chances" will not intervene in the formation of the longest run. When n is sufficiently large, the values that $F'_n(x)$ are well approximated by the values of $F_n(x)$ calculated for the case of $P(\text{head}) = \max\{p, 1-p\}$. The longest run will almost certainly be composed of whichever is more likely between heads and tails.

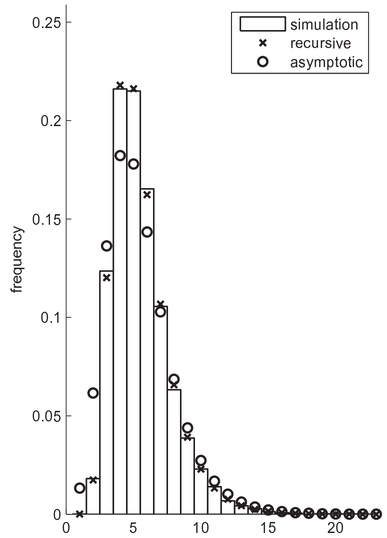
4 Numerical results, simulations

For numerical calculation we used MATLAB software. The data of the computer are INTEL Core2 Quad Q9550 processor, 4Gb, memory DDR3. The following table shows some running times

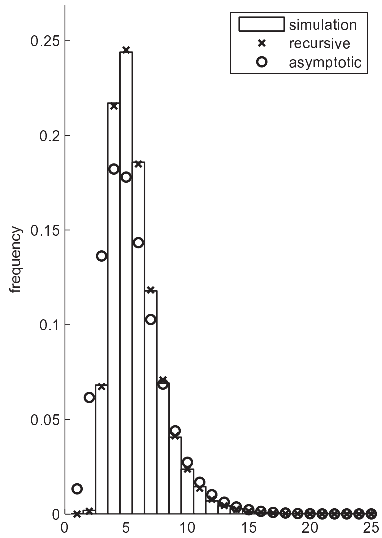
	n	repetition	running time
$p = 0.6$	3,100	20.000	172.6209 sec
	1,000	20.000	15.4258 sec
	250	20.000	2.8452 sec
	30	20.000	2.0678 sec

We calculated the distributions of R_n and R'_n. We considered the precise values obtained by recursion, the asymptotic values offered by asymptotic theorems, and used simulation with 20.000 repetitions. On the figures below $\times$ denotes the result of the recursion, o belongs to the asymptotic result, while the histogram shows the relative frequencies calculated by simulation. If n is small, the recursive algorithm is fast, but it slows down if n increases. For biased coin we used $p = 0.6$. We show the results for short trials ($n = 30$), medium trials ($n = 250$), and long trials ($n = 1000$ and $n = 3100$). We can see from the results that the asymptotic theorem does not give good (close to the recursive) results for small n. But we should say that if $n > 3000$, then the results of the recursion and the results of the asymptotic theorem are almost the same. As the algorithm is slowing down, we offer to use the asymptotic theorem instead of the recursion in case of large n. The asymptotic value is a good approximation if $n \geq 1000$. The figures below show that the distribution of R'_n is far from that of R_n for small n ($n = 30$). However, they are practically the same if n is large.

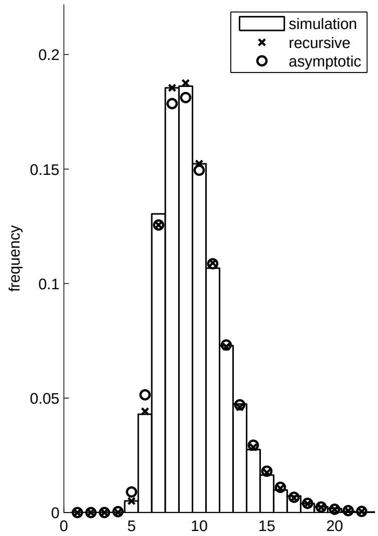
If p is much larger than $1/2$, the distribution of R'_n is quite close to R_n for moderate values of n as well. These facts give numerical evidence of (8).



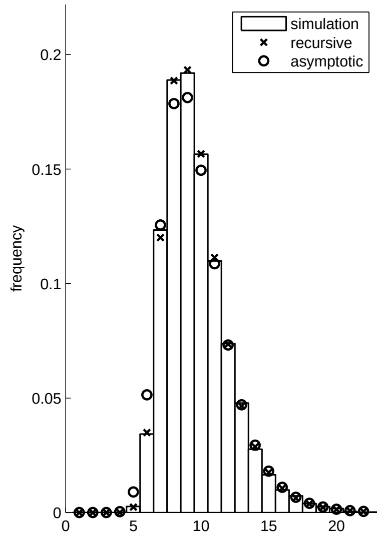
Distribution of the longest head run
$p = 0.6, n = 30.$



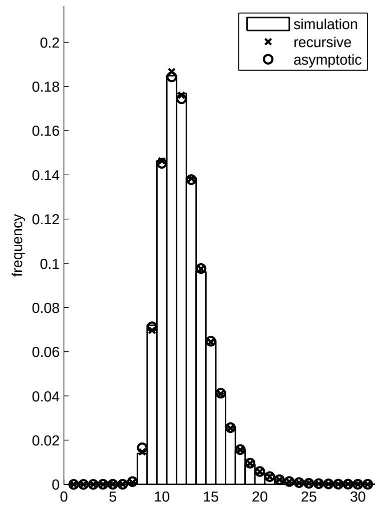
Distribution of the longest run
$p = 0.6, n = 30.$



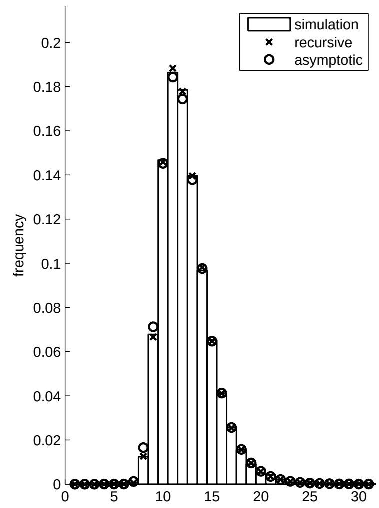
Distribution of the longest head run
$p = 0.6, n = 250.$



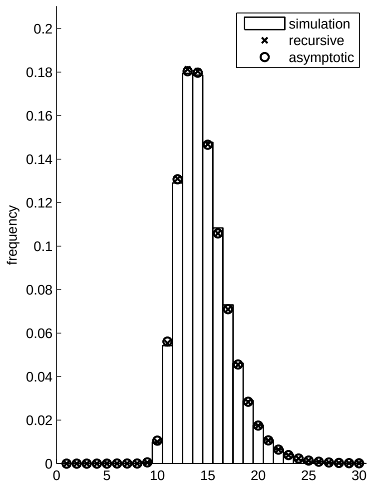
Distribution of the longest run
$p = 0.6, n = 250.$



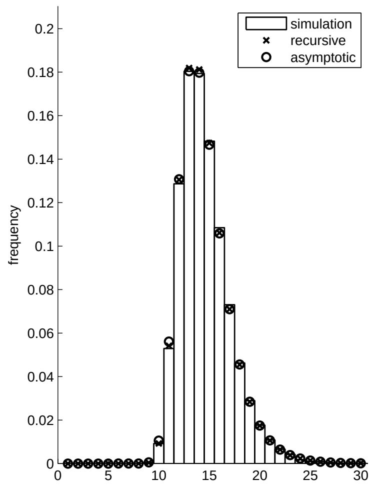
Distribution of the longest head run
$p = 0.6, n = 1000.$



Distribution of the longest run
$p = 0.6, n = 1000.$



Distribution of the longest head run
$p = 0.6, n = 3100.$



Distribution of the longest run
$p = 0.6, n = 3100.$

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